Random processes with separable covariance functions: Construction of dynamical model and its application for simulation and estimation

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***Abstract*—This paper demonstrates that for a given random process, having a canonical form, there exists a dynamical system equivalent in the sense that its output has the same covariance function. It is shown that the dynamical approach is more effective for simulating a Markovian and non-Markovian random processes, computationally is less demanding, especially with increasing of the dimension of simulated processes. Some numerical examples and experiments are presented to show the advantages of the proposed method for simulating random processes as well as for solving the filtering problems.**

***Index Terms*—Canonical representation, data assimilation, dynamical system, Markovian and non-Markovian processes, statistical simulation, stochastic process.**

I. Introduction

It is well known that generating pseudo random object, process ... is one of the key issues in mathematical-statistical modeling of complex systems. Mathematical-statistical modeling plays an important role in almost every area of science and technology, especially in the field of experimental research and development. A straightforward experiment on complex stochastic systems is the biggest drain on time and resources, too expensive; or, as in the case of meteorology and oceanography modeling ... physical experiments may be simply impossible [18]. That is why in practice, a more popular method to study complex systems is simulation. Statistical simulation using repeated random sampling to determine the properties of some phenomenon, is hence a very useful and efficient tool to test and select alternatives based on some criteria, to analyze and interpret data from simulated results, to understand behavior prediction of the underlying systems ...

This paper addresses the problem of construction of models for random processes (RP), with the objective to well approximate their covariance functions (CovFs), to better generate their samples or to produce the estimates of high quality on the basis of available observations. We will restrict our attention to the class of RPs having CovF with separable variables (for details, see Section II). Note that this class of RPs, in some sense, is equivalent to the class of RPs which can be represented in the canonical form (CF) [17]. Namely, let $\zeta(t)$ be an p-dimensional zero-mean random vector process. Given its CovF $K_\zeta(t,\tau)$, we are interested in solving two following problems : The first is to generate a zero-mean RP $\zeta_a(t)$ whose CovF matches the given $K_\zeta(t,\tau)$, either exactly or approximately. The second quantity of interest is a low-rank approximation to the covariance. That is, one is interested in computing, for a given number of components, such representation that fits best the covariance $K_\zeta(t,\tau)$. As will become clear in the future, the proposed algorithm (called Dynamical Systems - DS) allows to obtain a recursive procedure for simulating different RPs such as Markovian or non-Markovian, stationary or non-stationary, discrete or continuous ... Compared to the CF, the DS approach achieves a more significant speedup and requires less memory. As in practice, only a limited number of components of CF can be used, it will be shown that the DS is capable of producing samples more close to the "true" ones, compared to that obtained from the CF. Moreover, the DS approach is easy to apply to the problem of filter' design with correlated noises and appears to be more efficient in terms of Root Mean Square (RMS) error.

The paper is organized as follows. In Section II the CF theory is presented briefly. In fact the CF theory is closely related to differential eigenvalue/eigenvector problems. To avoid solving this type of problems, one of practical widely used algorithms for computing the coefficients and coordinate functions of the CF based on LU (lower upper) factorization is given in this section (see [17]). Section III summarizes the procedures for construction of DSs for Markov and non-Markov RPs conditioned that their CovFs are separable and known. Computational complexity of two approaches CF and DS is clearly seen from examining simple numerical examples in Section IV. The performances of CF and DS approaches are compared in Section V by simulation studies. Here two particular problems, one is a simulation of samples for an RP, another is related to filtering problem, are considered. It is shown that applying the algorithms CF and DS for solving these two problems offers the superior performance of the DS approach with respect to the CF approach. As potential applications of the DS approach, the time series prediction problem as well as data assimilation in very high dimensional oceanic models are also exposed here. It is worth of mentioning that due to different types of approximations, non-linearities of system dynamics, model reduction ... the residual sequence (RS) in a sub-optimal filter forms a time-correlated process and an efficient whitening process can be

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performed successfully by modeling the RS as a Markovian or non-Markovian. The conclusions are given in Section VI.

II. Canonical form of random process

A. Eigen-structure of covariance and canonical form

In this paper, otherwise stated, all random variables are assumed to have zero mean value. For simplicity, let (Ω, A, P) be some probability space, $L_2 = L_2(\Omega, A, P)$ be a Hilbert space of real variables induced by the inner product $< ., . >$, i.e. if $\zeta \in L_2$ then $E\zeta^2 < \infty$ and

$$< \zeta, \eta >= E(\zeta, \eta), \forall \zeta, \eta \in L_2.$$

Consequently, the distance between two random variables ζ and η is

$$d(\zeta, \eta) = [E(\zeta - \eta)^2]^{1/2}$$

We have then $\zeta = \eta$ implies that ζ and η may be different in a set of probability 0.

Let $\zeta(t)$ be a process with $E[\zeta^2(t)] < \infty$. For each t, $\zeta(t) \in L_2$. Parametrically, $\zeta(t)$ is therefore regarded as a "curve" in L_2. This curve is continuous at t_0 if

$$\lim_{t \to t_0} E[\zeta(t) - \zeta(t_0)]^2 = 0$$

The RPs satisfying this condition for all $t \in [0, T]$ are said to be continuous in the quadratic mean. Consider the collection of real-valued RPs $\zeta(t)$ for which

$$\int_0^T E[\zeta^2(t)]dt < \infty.$$

Stochastic processes in this set are verified to constitute a linear vector space denoted as $L_2[0, T]$. Define the inner product in this space as

$$< \zeta(t), \eta(t) >_L = E[< \zeta(t), \eta(t) >_t], < \zeta(t), \eta(t) >_t := \int_0^T \zeta(t)\eta(t)dt$$

One of the most interesting results of the theory of RPs is that the normed vector space for processes previously defined is separable. Consequently, there exists a complete (and, by assumption, orthonormal) set $\varphi_i(t), i = 1, 2, ...$ of deterministic (non-random) functions which constitutes a basis. A process in this space can be represented as

$$\zeta(t) = \sum_{k=1}^{\infty} \xi_k \varphi_k(t), \xi_k =< \zeta(t), \varphi_k(t) >_t . \quad (1)$$

More precisely the following result holds

Lemma 1: (Karhunen-Loève theorem [14]) A measurable continuous in quadratic mean RP defined over a probability space (Ω, A, P) can be represented in the form (1) (for $E\zeta(t) = 0$). In (1) ξ_k is a sequence of uncorrelated random variables, $E(\xi_k^2) = \lambda_k$, where λ_k and $\varphi_k(t)$ are the eigenvalue and eigenvector of the CovF $K_\zeta(t, \tau)$,

$$\int_{[0,T]} K_\zeta(t, \tau)\varphi_k(\tau)d\tau = \lambda_k \varphi_k(t),$$

$$K_\zeta(t, \tau) = E[\zeta(t)\zeta(\tau)] = \sum_k \lambda_k \varphi_k(t)\varphi_k(\tau). \quad (2)$$

Return to the definition of the CF for the RP $\zeta(t)$. According to [17], canonical form of $\zeta(t)$ is called any representation for $\zeta(t)$ in the form of its mathematical expectation and the sum of mutually uncorrelated elementary random processes $Y_k(t) = v_k x_k(t)$ where v_k is a random variable, $x_k(t)$ is a deterministic function, i.e.

$$\zeta(t) = \bar{\zeta}(t) + \sum_{k=1}^{n} v_k x_k(t), \bar{\zeta}(t) = E[\zeta(t)] \quad (3)$$

In (3) v_k are the coefficients of CF, $x_k(t)$ - coordinate functions. Thus the n-truncated approximation of (1) $\zeta_n(t) = \sum_{k=1}^{n} \xi_k \varphi_k(t)$ is an CF for $\zeta(t)$.

Representation of the RP in the form of CF is a very convenient method for performing different operations with random functions, especially for linear random functions. In the CF only its coefficients are random variables. The dependence of RP on temporal variable t is expressed only through the coordinate functions $x_k(t)$ which are deterministic. As a consequence, performing different operations like differentiation, integration ... reduces to corresponding operations with deterministic coordinate functions. The fact that the members of the CF are uncorrelated considerably simplifies the formulas for correlation functions of considered RP. However the CF, as will be shown later, is a less efficient tool for generating realizations of RP as well as for estimating the RP compared to the dynamical systems-based method.

B. One practical method for construction of canonical form

Lemma 1 gives us an overview of how the RP can be represented through eigenvalues and eigenvectors of its covariance matrix. Since computing eigenvalues and eigenvectors is time and memory consuming, another, computationally more attractive methods, are proposed for practical applications of the CF.

Let $\{v_k\}$ be a set of uncorrelated random variables,

$$E(v_i) = 0, E(v_i v_j) = \sigma_i^2 \delta_{ij} \quad (4)$$

where δ_{ij} be the Kronecker symbol. Suppose we would like to approximate the process $\zeta(t)$ by

$$\zeta(t) \approx \zeta_n(t) \approx \sum_{k=1}^{n} v_k x_k(t) \quad (5)$$

By minimizing the mean square distance

$$J(x) = E[\zeta(t) - \sum_{k=1}^{n} v_k x_k(t)]^2 \to \min_{x_1,...,x_n} \quad (6)$$

one obtains

$$x_k^0(t) = E[\zeta(t)v_k]/\sigma_k^2 \quad (7)$$

It remains to determine the variables v_k. Generally speaking, v_k should be such that the sum (5) must be convergent as $n \to \infty$ (see Lemma 1 for ξ_k). To obtain a more attractive algorithm, let $v_1^n := (v_1, ..., v_n)^T$, $A = |a_{ij}|_{i,j=1}^n$, $y_1^n := (\zeta(t_1), ..., \zeta(t_n))^T$. Then we want to find v_1^n as a function of y_1^n,

$$v_1^n = A y_1^n \quad (8)$$

To satisfy (4), the coefficients a_{ij} must be determined as a solution of the system

$$AK(n)A^T = D, D := diag(\sigma_2^1, ..., \sigma_2^n), \quad (9)$$

where $K_\zeta(n) := |K_\zeta(t_i, t_j)|_{i,j=1}^n$. It is seen, for example, Eq. (9) is satisfied if the columns of A are the eigenvectors of $K_\zeta(n)$ and $\sigma_k^2, k = 1, ...$ are the eigenvalues of $K_\zeta(n)$. One great disadvantage of this choice is that the system (9) must be solved each time when we want to involve more values of ζ to construct the CF. A recursive algorithm can be obtained by applying the LDU (Low-Diagonal-Upper) algorithm [6]. In this case, the matrix A is assumed to be, for $i \leq j$,

$$a_{ii} = 1, a_{ij} = 0, j > i \quad (10)$$

The elements $a_{ij}, j < i$ are defined now from (9). We have then

$$v_1 = \zeta_1, v_i = \sum_{j=1}^{i-1} a_{ij}\zeta_j + \zeta_i, i = 2, 3, ... \quad (11)$$

$$\sigma_k^2 = \sum_{i,j=1}^{k} a_{ki}a_{kj}K_\zeta(t_i, t_j), \quad (12)$$

$$x_k^0(t) = \frac{1}{\sigma_k^2}\sum_{j=1}^{k} a_{kj}K_\zeta(t, t_j), \sum_{j=1}^{k} a_{kj}x_l^0(t_j) = \delta_{kl}. \quad (13)$$

The algorithm (10)-(13) produces the process $\zeta_n(t)$ which is the same as $\zeta(t)$ only at $t = t_i, i = 1, ..., n$ (in the sense of their covariance structure). Thus in spite of the fact that $\zeta_n(t) = \sum_{k=1}^n v_k x_k^0(t)$ is not exactly the CF for $\zeta(t)$, this method, without requiring to solve an eigenvalue problem, allows to approximate the initial process, when needed, to approximate the process $\zeta(t)$ at other points by different techniques like interpolation theory (based on different coordinate functions $x_k(t)$ such as orthogonal polynomial, splines ...), variational-difference method ...[5]-[15]. This method is of importance for the problem of order reduction of dynamical systems and in some sense is closely related to the Principal Components Analysis (or Empirical Orthogonal Function in geodynamics, [12]). The latter creates a new set of orthogonal variables that contain the same information as the original set. It rotates the axes of variation to give a new set of orthogonal axes, ordered so that they summarize decreasing proportions of the variance (see also the Singular Value Decomposition which can be thought of as decomposing a matrix into a weighted, ordered sum of separable matrices [6]).

From (10)-(13) one can prove

Lemma 2: [17] Consider the RP $\zeta(t)$. Then the optimal in mean square CF $\zeta^0(t)$ is given by

$$v_1 = \zeta_1, \sigma_1^2 = K_\zeta(t_1, t_1), x_1^0(t) = \frac{1}{\sigma_1^2}K_\zeta(t, t_1),$$

$$v_k = \zeta_k - \sum_{j=1}^{k-1} v_j x_j^0(t_k),$$

$$\sigma_k^2 = K_\zeta(t_k, t_k) - \sum_{j=1}^{k-1} \sigma_j^2[x_j^0(t_k)]^2,$$

$$x_k^0(t) = \frac{1}{\sigma_k^2}[K_\zeta(t, t_k) - \sum_{j=1}^{k-1} \sigma_j^2 x_j^0(t)x_j^0(t_k)],$$

$$k = 2, 3, ... \quad (14)$$

with the mean square error

$$P(t) = E[\zeta(t) - \sum_{k=1}^{n} v_k x_k^0(t)]^2 = K_\zeta(t, t) - \sum_{k=1}^{n} \sigma_k^2[x_k^0(t)]^2 \quad (15)$$

C. Application of a canonical form for a random process in control systems

In the control systems engineering, the following class of RPs is frequently encountered

$$\zeta(t) = \sum_{k=1}^{n} \xi_k f_k(t) \quad (16)$$

here $f_k(t)$ are known deterministic functions, ξ_k are random variables. Considering $\xi_k, k = 1, 2, ...$ as a sequence of zero mean random variables, applying Lemma 2 yields the representation for ξ_k in the form of a linear combination of uncorrelated variables v_k,

$$\xi_k = \sum_{j=1}^{k} a_{kj}v_j. \quad (17)$$

Substituting (17) into (16) gives the CF for $\zeta(t)$

$$\zeta(t) = \sum_{k=1}^{n} v_k x_k(t), x_k(t) = \sum_{j=k}^{n} a_{jk}f_j(t). \quad (18)$$

Note that the representation (16) is widely used to approximate the RP by a linear combination of known functions with random coefficients (regression models). Many applications with this class of RPs can be found in [11]-[13].

D. Canonical form and random process with separable variables

The fact that the RP $\zeta(t)$ has a CF implies that its CovF has the following form

$$K_\zeta(t, \tau) = E[\sum_{i,j} v_i v_j x_i(t)x_j(\tau)] = \sum_{i=1}^{n} \sigma_i^2 x_i(t)x_j(\tau) \quad (19)$$

With the notations $K_i(t) = x_i(t), K_i(\tau) = \sigma_i^2 x_i(\tau)$ it implies

$$K_\zeta(t, \tau) = \sum_{i=1}^{n} K_i(t)K_i(\tau) \quad (20)$$

Thus the CovF $K_\zeta(t, \tau)$ as a function of t and τ is a function of separable variables. The RPs with CovF of the form (20) is investigated in [16]. We shall call (20) an CovF with separable variables (Cov-SV). Thus when the RP $\zeta(t)$ can be represented in the CF, it has the CovF belonging to the class of CovF-SVs. We note that all the results related to the CFs, CovFs of the scalar RPs can be extended to the vector RPs [17]. General conditions related to the class of discontinuous CovFs $K_\zeta(t, \tau)$ can be found in [17], [5].

III. Dynamical system (DS) theory for random processes with CovF-SV

It turns out that the RP $\zeta(t)$ having an CF, can be generated as an output of some DS. This fact is shown in this section. Parallelly a numerical procedure for determining all parameters of the corresponding DS will be obtained which allows to efficiently generate RPs considered in Section II based on the results of [16]. We note that the DS representation exists in the discrete and continuous form. The discrete DS yields the RP having the same CovF as $K_\zeta(t,\tau)$ at the desired discrete instants $t = t_i, i = 1, ..., n$ whereas the continuous DS produces the RP with $K_\zeta(t,\tau)$ at all time-instants t. This method can serve as an efficient tool for simulating samples of the considered RP as well as for solving the estimation problems with correlated noises as studied in [7].

A. DS representation for Gauss-Markovian RP

Theorem 1: (Discrete case) [16]. Let $\{w_i^{(1)}\}$ be an n_1-dimensional RP with zero mean and CF

$$Q_1(i,j) = K_1^{(1)}(i)P_1^{(1)}(j) \tag{21}$$

where $K_1^{(1)}(i)$ is non-singular. Then the condition (21) is necessary and sufficient for $\{w_i^{(1)}\}$ to be generated as an output of the DS described by the difference equation

$$w_{i+1}^{(1)} = C_i^{(1)} w_i^{(1)} + w_i^{(10)} \tag{22}$$

where $\{w_i^{(10)}\}$ is a white RP uncorrelated with $w_0^{(1)}$ such that

$$E(w_i^{(10))} = 0, E[w_i^{(10)}(w_i^{(10)})^T] = Q_{10}(i)\delta_{ij} \tag{23}$$

For a given (21), the fundamental matrix $C_i^{(1)}$ in (23) and the covariance matrix $Q_{10}(i)$ are determined by

$$C_i^{(1)} = K_1^{(1)}(i+1)[K_1^{(1)}(i)]^{-1} \tag{24}$$

$$Q_{10}(i) = K_1^{(1)}(i+1)\{P_1^{(1)}(i+1)[(K_1^{(1),T}(i+1)]^{-1} - P_1^{(1)}(i)[K_1^{(1),T}(i+1)]^{-1}\}K_1^{(1),T}(i+1) \tag{25}$$

where $K_1^{(1)}(0) = I$ - the unit matrix.

Theorem 2: (Continuous case [13]) Let $w_1(t)$ be an n_1-dimentional RP with zero mean and CovF

$$Q_1(t,\tau) = K_1^{(1)}(t)P_1^{(1)}(\tau) \tag{26}$$

where $K_1^{(1)}(t), P_1^{(1)}(t)$ are $(n_1 \times n_1)$ matrices having continuous derivatives, $K_1^{(1)}(t)$ is non-singular, $K_1^{(1)}(0) = I$. Then (26) is a necessary and sufficient condition for $w_1(t)$ to be represented as

$$\dot{w}_1(t) = C_1(t)w_1(t) + w_{10}(t) \tag{27}$$

where $w_{10}(t)$ is a white RP unccorelated with $w_1(0)$ such that

$$E[w_{10}(t)] = 0, E[w_{10}(t)[w_{10}^T(\tau)] = Q_{10}(t)\delta(t-\tau) \tag{28}$$

$\delta(t-\tau)$ is a Diract function. For $Q_1(t,\tau)$ of the form (26),

$$C_1(t) = [\frac{d}{dt}K_1^{(1)}(t)][K_1^{(1)}(t)]^{-1} \tag{29}$$

$$Q_{10}(t) = K_1^{(1)}(t)\frac{d}{dt}[P_1^{(1)}(t)(K_1^{(1),T}(t))^{-1}]K_1^{(1),T}(t) \tag{30}$$

Comment 3.1. The condition $K_1^{(1)}(0) = I$ can be removed.

B. DS representation for a non-Markovian RP

Theorem 3: (Discrete case, [16]) Let $\zeta_i := w_i^2$ be an n_2-dimensional RP with zero mean and covariance matrix

$$K_\zeta(i,j) = Q_2(i,j) := \sum_{l=1}^{2} K_l^{(2)}(i)P_l^{(2)}(j) \tag{31}$$

where $K_1^{(2)}(i), P_1^{(2)}(i)$ are $(n_2 \times n_2)$ matrices, $K_1^{(2)}(i)$ is non-singular; $K_2^{(2)}(i), P_2^{(2)}(i)$ - $(n_2 \times n_1)$ matrices. Let $w_i^{(1)}$ be an output of the DS (22) which has $K_1^{(1)}(i), P_1^{(1)}(i)$ satisfying the relationship

$$K_2^{(2)}(i)P_1^{(1)}(i) = P_2^{(2),T}(i)K_1^{(1),T}(i) \tag{32}$$

Then (31) is necessary and together with (32) are sufficient for $w_i^{(2)}$ to be presented as

$$w_{i+1}^{(2)} = C_i^{(2)} w_i^{(2)} + D_i^{(2)} w_i^{(1)} + w_i^{(20)} \tag{33}$$

where it is supposed that

$$E[w_i^{(20)}] = 0, E[w_i^{(20)} w_j^{(20),T}] = Q_{20}(i)\delta_{ij} \tag{34}$$

$C_i^{(2)}, D_i^{(2)}$ are the matrices of dimensions $(n_2 \times n_2)$ and $(n_2 \times n_1)$ respectively, $w_0^{(2)}$ is a random vector such that

$$E[w_0^{(1)} w_0^{(2),T}] = 0, E[w_i^{(20)} w_0^{(2),T}] = 0,$$
$$E[w_i^{(20)} w_0^{(2),T}] = 0, E[w_i^{(20)} w_0^{(1),T}] = 0$$
$$E[w_i^{(k0)} w_j^{(l0),T}] = W_{kl}^{(i),T}\delta_{ij}, K_1^{(2)}(0) = I, K_2^{(2)}(0) = 0. \tag{35}$$

For the given (31) and (32), the matrices $C_i^{(2)}, C_i^{(1)}, D_i^{(2)}$ and the RPs $w_i^{(10)}, w_i^{(20)}$ are determined by

$$C_i^{(2)} = K_1^{(2)}(i+1)[K_1^{(2)}(i)]^{-1},$$
$$D_i^{(2)} = -C_1^{(2)}K_2^{(2)}(i)[K_1^{(1)}(i)]^{-1} + K_2^{(2)}(i+1)[K_1^{(1)}(i)]^{-1},$$
$$C_i^{(1)} = K_1^{(1)}(i+1)[K_1^{(1)}(i)]^{-1},$$
$$Q_{20}(i) = K_1^{(2)}(i+1)\{P_1^{(2)}(i+1)[(K_1^{(2),T}(i+1)]^{-1} - P_1^{(2)}(i)[K_1^{(2),T}(i+1)]^{-1}\}K_1^{(2),T}(i+1)$$
$$Q_{10}(i) = K_1^{(1)}(i+1)\{P_1^{(1)}(i+1)[(K_1^{(1),T}(i+1)]^{-1} - P_1^{(1)}(i)[K_1^{(1),T}(i+1)]^{-1}\}K_1^{(1),T}(i+1)$$
$$W_{11}(i) = W_{11}^T(i) = K_1^{(2)}(i+1)\{P_1^{(1)}(i+1)[(K_1^{(1),T}(i+1)]^{-1} - P_1^{(1)}(i)[K_1^{(1),T}(i+1)]^{-1}\}K_1^{(1),T}(i+1) \tag{36}$$

The similar results can be obtained for the continuous RPs and the reader is referred to [16] for more details.

Comment 3.2 As shown in [16],

(i) The conditions in (35) are not the constraints for $K_\zeta(i,j)$.

(ii) The case when $K_1^{(2)}(i)$ is singular can be overcome by introducing two matrices $K_{01}^{(2)}(i), P_{01}^{(2)}(i)$ of dimensions $(n_2 \times n_2)$, with $K_{01}^{(2)}(i)$ being non-singular. Then one can write

$$\begin{aligned} Q_2(i,j) &= \sum_{l=1}^{2} K_{0l}^{(2)}(i) P_{0l}^{(2)}(i) \\ K_{02}^{(2)}(i) &= [-K_{01}^{(2)}(i), K_1^{(2)}(i), K_2^{(2)}(i)] \\ P_{02}^{(2)}(i) &= [P_{01}^{(2),T}(i), P_1^{(2),T}(i), P_2^{(2),T}(i)] \end{aligned} \tag{37}$$

By this way one can construct the DS for $\zeta(t)$. We note the interesting fact that from (37) it is true $R[P_{02}^{(2),T}(i)] \subset R[K_{02}^{(2),T}(i)], \forall i$ which is equivalent to the requirement (32) where $R[A]$ denotes a linear space spanned by the columns of the matrix A.

(iii) Let

$$K_\zeta(i,j) = Q_2(i,j) = \sum_{l=1}^{N} K_l^{(N)}(i) P_l^{(N)}(j) \tag{38}$$

(compared (38) with (20)), where for simplicity $K_1^{(N)}$ is non-singular. Putting

$$\begin{aligned} K_1^{(2)}(i) = K_1^{(N)}(i), P_1^{(2)}(i) = P_1^{(N)}(i), \\ K_2^{(2)}(i) = [K_2^{(N)}(i), ..., K_N^{(N)}(i)], \\ P_2^{(2)}(i) = [P_2^{(N),T}(i), ..., P_N^{(N),T}(i)], \end{aligned}$$

we obtain again the DS (31) or (33) for the RP $\zeta(t)$.

IV. Comparison between CF and DS approaches: Examples

A. Stationary process

1) Application of Lemma 1: Consider the RP with zero mean and CovF

$$K_\zeta(t,\tau) = K_\zeta(t-\tau) = c^2 e^{-\beta|t-\tau|}, t, \tau \in [0,T] \tag{39}$$

Generally speaking, it is impossible to find analytically the system of eigenvalues $\{\lambda_k\}$ and eigenvectors $\{x_k(t)\}$ of $K_\zeta(t,\tau)$. However, in this particular case one can prove that [17]

$$\begin{aligned} \zeta(t) &= \sum_k v_k x_k(t), \\ x_k(t) &= \frac{2}{T+\lambda_k} sin[\omega_k(t-\frac{T}{2}) + \frac{k\pi}{2}], \\ \sigma_k^2 &= \frac{c^2}{2}\lambda_k(T+\lambda_k), \lambda_k = c^2 \frac{2\beta}{\beta^2+\omega_k^2}, \end{aligned} \tag{40}$$

where ω_k are the positive roots of the equation, $tan\omega T = -\frac{2\beta\omega}{\beta^2-\omega^2}$. Thus one can simulate the realizations for $\zeta(t)$ using the formula (40) subject to v_k having zero mean and variance σ_k^2.

2) Application of Lemma 2: Suppose it is impossible to obtain the system of eigenvalues and eigenvectors of $K_\zeta(t,\tau)$. To construct a model for $\zeta(t)$ to generate its samples, let us follow Lemma 2 to obtain $\zeta_a(t)$ whose CovF matches exactly the CovF of $\zeta(t)$ at the points $t_k = (k-1)\Delta t, k = 1, ..., n$. For $\rho := e^{-\beta\Delta t}$ we have

$$\begin{aligned} \zeta_a(t) &= \sum_{k=1}^{n} v_k x_k^0(t), \\ \sigma_1^2 &= c^2, x_1^0(t) = e^{-\beta|t|}, \\ \sigma_2^2 = c^2(1-\rho^2), x_2^0(t) &= \frac{1}{1-\rho^2}[e^{-\beta|t-\Delta t|} - \rho e^{-\beta|t|}], \\ \sigma_3^2 &= c^2 - \sum_{k=1}^{2} \sigma_k^2 [x_k^0(t_3)]^2, \\ x_3^0(t) &= \frac{1}{\sigma_3^2}[e^{-\beta|t-2\Delta t|} - \sum_{k=1}^{2} \sigma_k^2 x_k^0(t) x_k^0(t_3)^2], \\ x_1^0(t_3) &= e^{-2\beta\Delta t} = \rho^2, \\ x_2^0(t_3) &= \frac{1}{1-\rho^2}[e^{-\beta\Delta t} - \rho e^{-\beta 2\Delta t}] = \rho. \end{aligned} \tag{41}$$

The coefficients σ_k^2 and coordinate functions $x_k^0(t)$ can be computed in the similar manner for $k = 4, 5, ...$ The coefficients σ_k^2 are the covariances of v_k which are of zero mean and uncorrelated. The algorithm (41) is written out here to see its complexity when dealing with an arbitrary RP. If the values of $\zeta(t)$ are given at t_k, it is possible to write out the formula for v_k as a function of $\zeta(t_k)$ as done in Lemma 2. Note that in this particular example, it is possible to present these elements in a more compact form as

$$\begin{aligned} \sigma_k^2 &= c(1-\rho^2), \\ x_k^0(t) &= \frac{1}{1-\rho^2}[e^{-\beta|t-(k-1)\Delta t|} - \rho e^{t-\beta|t-(k-2)\Delta t|}]. \end{aligned} \tag{42}$$

Similarly one can write out the formulas (15) for estimation errors.

3) Application of Theorem 1: Consider the problem in (ii) and let us apply Theorem 1 to obtain the algorithm for simulating the process $\zeta(t)$ at $t_k, k = 1, 2, ...$ It is easy to see that (39) is equivalent to (21) subject to

$$\begin{aligned} t \geq \tau : K_1(t) = e^{-\beta t}, P_1(\tau) = c^2 e^{\beta\tau}, \\ t < \tau : K_1(t) = e^{\beta t}, P_1(\tau) = c^2 e^{-\beta\tau} \end{aligned} \tag{43}$$

This choice ensures $K_1(0) = 1$. It is seen that it is sufficient to consider the case $t \geq \tau$. We have then from Theorem 1,

$$\begin{aligned} C_k^{(1)} &= e^{-\beta(t_{k+1}-t_k)} = \rho, \\ Q_{10}(k) &= c^2(1-e^{-\beta\Delta t}) = c^2(1-\rho^2). \end{aligned} \tag{44}$$

Thus we have the following recursive equation for simulating $\zeta_k = \zeta(t_k)$,

$$\zeta_{k+1} = \rho\zeta_k + w_k, \tag{45}$$

w_k is an uncorrelated sequence of zero mean and variance $Q_{10}(k)$ defined in (44).

4) Application of Theorem 2:

$$\dot{\zeta}(t) = -\beta\zeta(t) + w_t,$$
$$C_1(t) = -\beta, Q_{10}(t) = 2c^2\beta, \quad (46)$$

where $w_(t)$ is a noise sequence with zero mean and intensity $Q_{10}(t)$.

One sees here that the DS-representations (43)-(45) and (46) are very simple compared to the CFs expressed by (40) and (41). The result (42) says that all coordinate functions $x_k(t)$ participate in the CF with the same weight v_k since $\sigma_k^2 = c(1-\rho^2)$ is constant for all k. Thus generally speaking a large amount of coordinate functions $x_k(t)$ must be computed to well approximate the required RP in the CF (see (41)).

B. Non-stationary random process

Consider the non-stationary RP with zero mean and covariance

$$K_\zeta(t,\tau) = c^2 e^{-\beta_t t} e^{-\beta_\tau \tau} = K_1(t)P_1(\tau),$$
$$K_1(t) = e^{-\beta_t t}, P_1(\tau) = c^2 e^{-\beta_\tau \tau}, \quad (47)$$

which ensures $K_1(0) = 1$.

1) Application of Lemma 2: The optimal in mean square CF based on the values of $\zeta(t)$ at the points $t_i = (i-1)\Delta t, i = 1, ..., n$ is expressed by ($\rho(t) := e^{-\beta_t \Delta t}$),

$$\sigma_1^2 = c^2, x_1^0(t) = \frac{e^{-\beta_t t}}{c^2},$$
$$\sigma_2^2 = (c^2-1)\rho^2(t_1), x_2^0(t) = \frac{(c^2+1)\rho(t_2)e^{-\beta_t t}}{\rho^2(t_1)}$$
$$etc \quad (48)$$

For this simple example, write out here the formulas for $\sigma_k, x_k^0(t)$ for all $k = 3, 4, ...$ is a hard task, not to say on a possible inability to realize this procedure.

2) Application of Theorem 1: From Theorem 1 one obtains the following simple difference equation for simulation of $\zeta(t_k) := w_k^{(1)}$,

$$w_{k+1}^{(1)} = C_k w_k^{(1)}, k = 0, 1, 2, ...$$
$$C_k = e^{-\alpha_k \Delta t}, \alpha_k = k\Delta\beta_k + \beta_i,$$
$$\Delta\beta_k := \beta_{k+1} - \beta_k, \beta_k := \beta_{t_k}, Q_1(0,0) = c^2 \quad (49)$$

C. Non-Markovian random process

Consider the RP $\zeta(t)$

$$\zeta(t) = \sum_{l=1}^{2} \xi_l f_l(t),$$
$$\xi = (\xi_1, \xi_2)^T, E(\xi) = 0, E(\xi\xi^T) = \Xi,$$
$$\Xi = [\Xi_{ij}], i, j = 1, 2, \Xi_{11} = \Xi_{22} = 2, \Xi_{12} = \Xi_{21} = 1,$$
$$f_1(t) = \frac{1}{1+t}, f_2(t) = \frac{1}{1+t^2}. \quad (50)$$

Thus the process $\zeta(t)$ has the covariance

$$K_\zeta(t,\tau) =$$
$$f_1(t)[2f_1(\tau) + f_2(\tau)] + f_2(t)[2f_2(\tau) + f_1(\tau)] \quad (51)$$

1) Application of Lemma 2: For $t_k = (k-1)\Delta t$, application of Lemma 2 subject to (51) yields

$$\sigma_1^2 = 6, x_1^0(t) = 2[\frac{1}{1+t} + \frac{1}{1+t^2}],$$
$$\sigma_2^2 = K_\zeta(t_2, t_2) - \sigma_1^2[x_1^0(t_2)]^2,$$
$$K_\zeta(t_2, t_2) = 2[\frac{1}{(1+\Delta t)^2} + \frac{1}{(1+\Delta t^2)^2} + \frac{1}{(1+\Delta t)(1+\Delta t^2)}],$$
$$x_2^0(t) = \frac{1}{\sigma_2^2}[K_\zeta(t, t_2) - 4x_1^0(t)x_1^0(t_2)],$$
$$x_1^0(t_2) = 3[\frac{1}{1+\Delta t} + \frac{1}{1+\Delta t^2}], etc \quad (52)$$

2) Application of Theorem 3: To make the choice of $K_1^{(1)}(t), P_1^{(1)}(t)$ easier (since $K(t,\tau) = K_1^{(1)}(t)P_1^{(1)}(\tau)$ must be a covariance function and satisfy (32)), represent (50) in the equivalent form

$$K_\zeta(t,\tau) = \sum_{l=1}^{2} K_l^{(2)}(t)P_l^{(2)}(\tau),$$
$$K_l^{(2)}(t) = P_l^{(2)}(t) = x_l(t), l = 1, 2,$$
$$x_1(t) = \frac{a_{11}}{1+t} + \frac{a_{21}}{1+t^2}, x_2(t) = \frac{a_{22}}{1+t^2},$$
$$a_{11} = 1,414, a_{12} = 0, a_{21} = 0,707, a_{22} = 1.225. \quad (53)$$

The representation (53) is obtained by applying the Cholesky algorithm to Ξ which results in $\Xi = AA^T$ (see Section II.C). Then from Theorem 3 one obtains the following equations for simulating of $\zeta(t) := w_i^{(2)}$, noticing that by the choice $K_1^{(1)}(t) = P_1^{(1)} = 1$ the condition (32) is automatically satisfied,

$$w_{k+1}^{(2)} = C_k^{(2)} w_k^{(2)} + D_k^{(2)} w_k^{(1)},$$
$$C_k^{(2)} = \frac{x_1(k+1)}{x_1(k)}, D_k^{(2)} = -C_k^{(2)} x_2(k) + x_2(k+1),$$
$$x_1(k) := \frac{a_{11}}{(k-1)\Delta t} + \frac{a_{21}}{1+[(k-1)\Delta t]^2},$$
$$x_2(k) := \frac{a_{22}}{1+[(k-1)\Delta t]^2},$$
$$w_1^{(1)}(k+1) = w_1^{(1)}(k), k = 1, 2, ...,$$
$$E[w_0^{(2)} w_0^{(1)}] = 0, Q_1(0,0) = 1, Q_2(0,0) = 6. \quad (54)$$

It is seen that $\zeta(t)$ is not Markovian since $w_k^{(1)} = w_0^{(1)}, \forall k$ is not an uncorelated sequence.

V. Simulation studies

A. Simulation of process (50)

Let us consider the process (50) and assume that ξ is normally distributed with the statistics given in (50). In this experiment the following three algorithms will be applied to simulate the realizations of (50):

(i) Simulate the realizations for ξ and hence $\zeta_1(t_k) = \sum_{l=1}^{2} \xi_l f_l^0(t_k)$, $t_k = (k-1)\Delta k, k = 1, ..., 100; \Delta = 0.1$. Concretely, we simulate $\xi = Av$ where A is given in (53). As to $v = (v_1, v_2)^T$, it is normally distributed with zero mean and

the unit covariance (Algorithm 1 - A1). These realizations can serve as "true" and will be used as references to be compared with realizations produced by other algorithms.

(ii) Apply the algorithm (52) to construct the basis functions $x_k^0(t)$ and to simulate the process as (Lemma 2)

$$\zeta_2(t) = \sum_{k=1}^{n} \epsilon_k x_k^0(t), \epsilon_k = \sigma_k v_k, \quad (55)$$

subject to covariance function (51). Mention that the realizations of $v = (v_1, v_2)^T$ are taken to be that obtained in (i) (Algorithm 2 - A2).

(iii) Apply the algorithm (53)-(54) (Dynamical approach). The resulting realizations are denoted as $\zeta_3(t_i)$. Here the realizations of $(w_0^1, w_0^2)^T$ are taken as $(w_0^1, w_0^2)^T = v$, v is given in (i) (Algorithm 3 - A3).

The simulation results are shown partly in Fig. 1. Here 8 realizations produced by each of three algorithms A1, A2 and A3 are shown. To facilitate visualization, the i^{th} realization is displayed at the points $[100(i-1)+1, ..., 100i]$ on the horizontal axis. All three algorithms have the same initial realization $\zeta_k(1), k = 1, 2, 3$. One sees from Fig. 2 that the algorithms A1, A3 have produced almost the same realizations whereas the realizations 4^{th}, 5^{th}, 8^{th} of A2 in Fig. 1 are far away from that of A1, especially at the beginning. In A2, only two coordinate functions are involved. Notice that the variance $\sigma_2^2 = 0.00328$ hence it has no sense to involve more coordinate functions in simulation of RP. As $\sigma_1^2 = 6$, the first component accounts for 99 % of the variability in the data. To be convinced that σ_2^2 is small in general, as σ_2^2 is a function of only Δt, in Fig. 3 the values $\sigma_2^2(\Delta t)$ are displayed against Δt. We have also computed the (average) difference (in absolute value) between the true $K_\zeta(t_i, t_i)$ (51) and that resulting from $K_{\zeta_o}(t_i, t_i), \zeta_o(t) = v_1 x_1^0(t)$ in (52) (v_1 is a random variable with zero mean and variance $\sigma_1^2 = 6$) and found that this difference is almost the same as that presented in Fig. 3. It justifies once more that the first component $v_1 x_1^0(t)$ has the covariance structure almost identical to that of $\zeta(t)$. On average, the difference between two covariance values $|K_\zeta(t_i, t_j) - K_{\zeta_o}(t_i, t_j)|, i, j = 1, ..., 100$ is equal to 0.1706.

To have the idea on how the algorithms A2, A3 well simulate the "true" realizations, Figs 4-5 show two curves "RMS-A2" and "RMS-A3" corresponding to the sample RMS between the "true" realizations and simulated by A2, A3.

$$RMS - Ak = \frac{1}{NrN} \sum_{r=1}^{Nr} \sum_{i=1}^{N} (\zeta_k^r(t_i) - \zeta_1^r(t_i))^2$$

where $\zeta_k^r(t_i)$ is the r^{th} realization of $\zeta_k(t_i)$. In Figs 4-5, $Nr = 1000, N = 100$. It is evident that in average, the realizations of $\zeta_3^r(t_i)$ are much closer to that of $\zeta_1^r(t_i)$ than those of $\zeta_2^r(t_i)$.

B. Estimation of random process (50)

Let the following observation model be available

$$z(t_{io}) = \zeta(t_{io}) + \nu(t_{io}), io = 1, 2, ...,$$
$$t_{io} = 1 + (io-1)\Delta T, \Delta T = 10\delta t, \delta t = 0.1, \quad (56)$$

with $\zeta(t_{io})$ being given by (50) and $\nu(t_{io})$ is an uncorrelated sequence of observational error with zero mean and variance

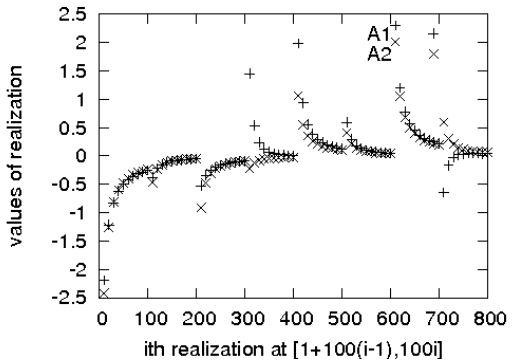

Fig. 1. Realizations produced by A1 and A2. The realizations 4^{th}, 5^{th}, 8^{th} of A2 are far too much from the true A1.

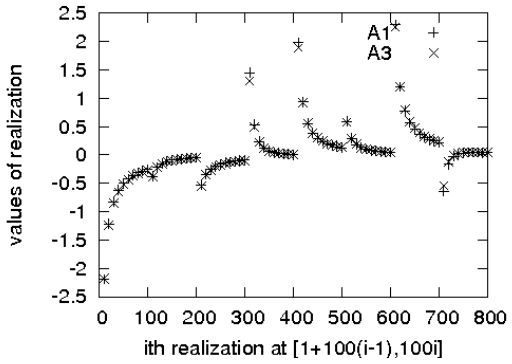

Fig. 2. Realizations produced by A1 and A3 : All realizations of the A3 are close to that of A1.

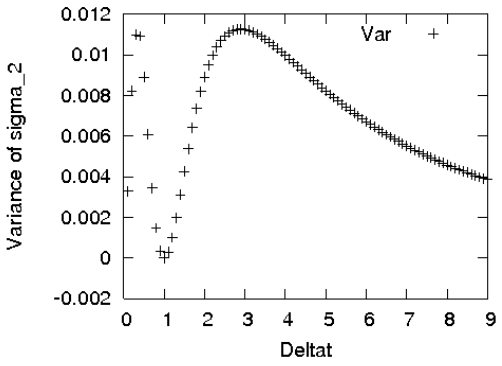

Fig. 3. The variance of the second coefficient in the CF: it is small enough, the proof that there is no need to involve more components in the CF.

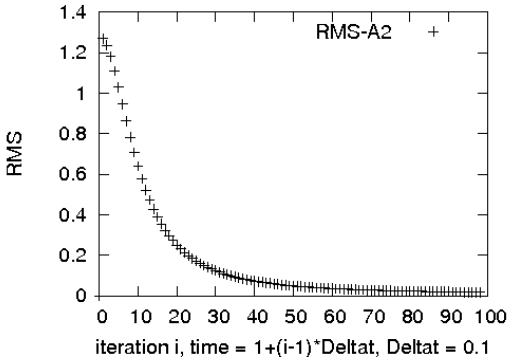

Fig. 4. Sample RMS between realizations produced by A1 and A2.

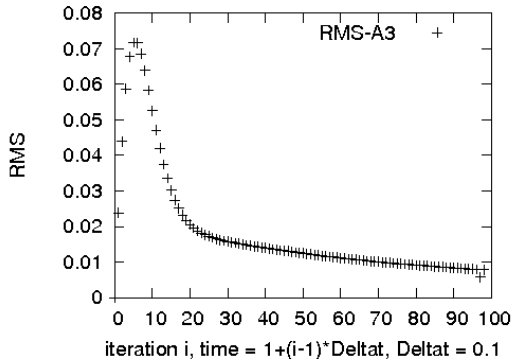

Fig. 5. Sample RMS between realizations produced by A1 and A3. The curve RMS-A3 shows that compared to RMS-A2 in Fig. 4, the realizations of the A3 are much closer to the "true" A1.

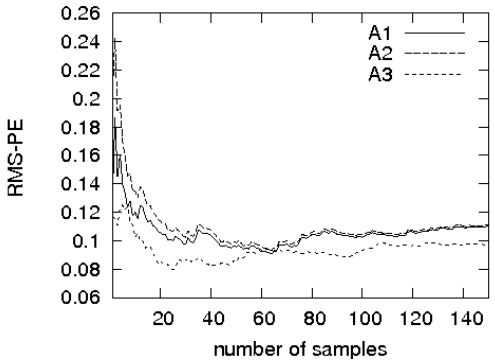

Fig. 6. Ensemble averaged RMSs of the PE produced by A1, A2 and A3. The prediction is made over $\Delta t = 1$. The errors are stabilized after about 100 realizations. The algorithm A3 has produced the better estimates. A1 and A2 are of nearly the same performance.

$\sigma_\nu^2 = 1$. Assume further that $\nu(t_{io})$ and ξ in (50) is uncorrelated. The problem we want to solve here is to estimate the value of $\zeta(t_i)$. It is evident that if we desire to obtain the estimate of $\zeta(t_{io=io'})$ given by $z(t_{io}), io = 1, ..., io'$ then we have the filtering problem. Analogously one can consider the smoothing or prediction problems. For example, the prediction concerns the estimation of $\zeta(t_i), t_i > t_{io}$ ... Note that the above estimation problem is encountered frequently in data assimilation where δt represents a model time step in the numerical model integration (for example, in minutes) and ΔT symbolizes the interval between two observations (in hours, days).

1) Filtering problem: .

(i.1) Algorithm A1. From (50)(56) one has the following observation model

$$\begin{aligned} z(t_{io}) = H(t_{io})\theta + \nu(t_{io}), io = 1, 2, ..., \\ H(t_{io}) = [f_1(t_{io}), f_2(t_{io})], \theta := (\theta_1, \theta_2)^T, \\ E(\theta) = 0, E(\theta\theta^T) = \Xi. \end{aligned} \quad (57)$$

The optimal in minimum mean square (MMS) estimation procedure is

$$\begin{aligned} \theta(t_{io+1}) = \theta(t_{io}) + K(t_{io+1})[z(t_{io+1}) - H(t_{io+1})\theta(t_{io})], \\ \theta(t_1) = 0, io = 1, 2, ..., \\ K(t_{io+1}) = P(t_{io+1}) \\ H^T(t_{io+1})[H(t_{io+1})P(t_{io+1})H^T(t_{io+1}) + I\sigma_\nu^2]^{-1}, \\ P(t_{io+1}) = [I - K(t_{io+1})H^T(t_{io+1})]P(t_{io}), P(t_1) = \Xi \end{aligned} \quad (58)$$

Thus the MMS filtered estimate $\hat{\zeta}(t_{io})$ can be obtained as $\hat{\zeta}(t_{io}) = H(t_{io})\hat{\theta}(t_{io})$.

(i.2) Algorithm A2. For the model (52) the algorithm remains the same as A1 with the differences

$$H(t_{io}) = [x_1^0(t_{io}), x_2^0(t_{io})], P(t_1) = diag[\sigma_1^2, \sigma_2^2]. \quad (59)$$

(i.3) Algorithm A3. Direct application of the Kalman filter (KF) to the DS (54) subject to the observation system (57) is impossible since the sequence $w_i^{(1)}$ is not uncorrelated. Introducing $y(t_i) = (w_i^{(2)}, w_i^{(1)})^T$ leads to the filtering problem in state-space form

$$\begin{aligned} y(t_{io+1}) = C_{io}^e y(t_{io}), y(t_1) := (w_1^{(2)}, w_1^{(1)})^T, \\ C^e(t_{io}) = |c_{kl}^e|_{k,l=1}^2, \\ c_{11}^e = C_{t_{io}}^{(1)}, c_{12}^e = D_{t_{io}}^{(1)}, c_{21}^e = 0, c_{22}^e = 1, \\ z(t_{io}) = H^e y(t_{io}) + \nu(t_{io}), H^e = [I, 0], io = 1, 2, ..., \end{aligned} \quad (60)$$

The MMS filter can be written out as

$$\begin{aligned} \hat{y}(t_{io+1}) = C_{io}^e \hat{y}(t_{io}) + \\ K(t_{io+1})[z(t_{io+1}) - H^e C_{io}^e \hat{y}_p(t_{io})], \hat{y}(t_1)(t_1) = 0, \\ K(t_{io+1}) = M(t_{io+1})H^{e,T}[H^e M(t_{io+1})H^{e,T}]^+, \\ M(t_{io+1}) = C_{io}^e P(t_{io}) C_{io}^{e,T}, P(t_1) = diag[6, 1], \\ P(t_{io+1}) = [I - K(t_{io+1})H^e]M(t_{io+1}), \\ \hat{\zeta}(t_{io}) = [I, 0]\hat{y}(t_{io}). \end{aligned} \quad (61)$$

2) Numerical results: In Fig. 6 we show the ensemble averaged RMSs of the prediction error (PE) produced by three algorithms A1, A2, A3. The same RMSs but for the filtered error (FE) are displayed in Fig. 7. It is undoubted that the DS approach (A3) produces the best estimates. The algorithm A1 is slightly better than A2 especially at the beginning, but in general they behave in the same way. If A3 is capable of improving considerably the filtered estimate compared to the predicted estimate (see Fig. 8), only non-significant improvement of the filtered estimates compared to their predicted estimates is observed for A1 and A2. Noticing that all three algorithms have filtered well the observation noise since its variance is equal 1. It means that the algorithms reduce about 90 % noise level in the estimates. The reason of the better performance of A3 may be explained by the fact that the filter A3 estimates directly the process $\zeta(t)$ whereas two filters A1, A2 estimate indirectly $\zeta(t)$ (through the estimation of the coefficients in decompositions).

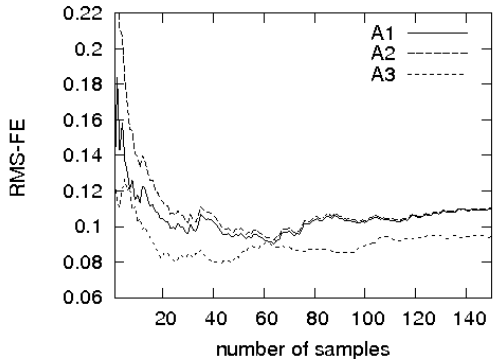

Fig. 7. The same as in Fig. 6 but for the FE.

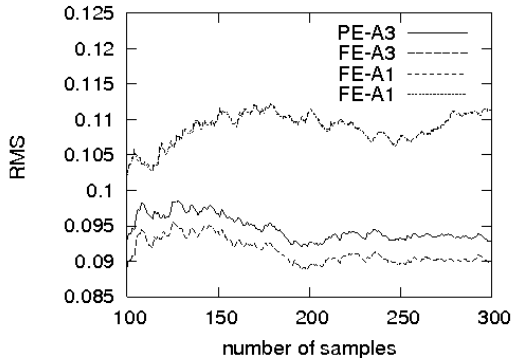

Fig. 8. Ensemble averaged RMSs of the PE and FE produced by A1 and A3. One sees here there is a significant decrease of FE compared to that of the PE in A3. In contrast, only a non-significant improvement of the FE is observed (compared to PE) in A1. The same performance is produced by A2 and it is not shown here.

C. Prediction of time series

1) Construction of DS: Let $x(t)$ be a random process and consider the problem of optimal linear prediction $x(t)$. This happens, for example, when one wants to estimate $x(t)$ as the daily rainfall, wind speed, temperature ... at instant t given a series of historical data $x(t-1), ..., x(t-n)$. Then the problem of construction of a dynamical model for $x(t)$ is of the first importance.

Let us introduce the following dynamical model

$$x(t) = a_1 x(t-1) + ... + a_n x(t-n) + w(t) \tag{62}$$

First for simplicity, let $w(t) = 0$. Introduce $X(t) = [x(t), ..., x(t-n+1)]^T$. We have then a state-space representation

$$X(t) = AX(t-1),$$
$$A = \begin{pmatrix} a_1 & a_2 & ... & a_n \\ 1 & 0 & 0 & 0 \\ 0 & 1 & ... & 0 \\ 0 & ... & 1 & 0 \end{pmatrix} = \begin{pmatrix} a^T \\ ------ \\ I_{n-1}, \quad 0 \end{pmatrix} \tag{63}$$

Thus if A is given, an optimal predictor $\hat{X}(t)$ can be obtained by integration of (63) from $X(t-1)$. In practice, usually we are given a series $z(t-1), ...z(t-n), ...$ of observations (contaminated with noise) for the process $x(t)$, hence estimating the matrix A is required first before being able to make the prediction. One way to obtain an optimal A_o is to require that A_o will minimize the mean prediction error, i.e.

$$J(A_o) = \min_A J(A),$$
$$J(A) := E[(X(t) - AX(t-1))^T (X(t) - AX(t-1))] \tag{64}$$

Taking a derivative of $J(A)$ with respect to A leads to $E[(X(t) - A_o X(t-1))X^T(t-1)] = 0$ (orthogonal principle) hence

$$A_o = K_X(t, t-1) K_X^{-1}(t-1),$$
$$K_X(t, t-1) := E[X(t)X^T(t-1)],$$
$$K_X(t-1) := E[X(t-1)X^T(t-1)]. \tag{65}$$

Mention that (65) constitutes a basis for the derivation of the fundamental matrix $C_i^{(1)}$ in Theorem 1.

Inversely, suppose we are given a DS

$$X(t) = AX(t-1), t = 1, 2, ...,$$
$$X(t) = [x(t), ..., x(t-n+1)]^T, \tag{66}$$

where A may be any unknown matrix. We will show now that the optimal estimate A_o, determined by (65), will have the structure (63) with

$$a_o^T = K_{xX}(t, t-1) K_X^{-1}(t-1). \tag{67}$$

Really from the definition of $X(t)$, if we represent $K_X^T(t, t-1) = [b, B^T], b \in R^n, B \in R^{(n-1)\times n}$ then $K_X^T(t-1)$ has the structure $K_X^T(t-1) = [B^T, c^T], c \in R^n$. If we denote by $K_X^{-1}(t-1) = [\tilde{B}, \tilde{c}]$ then from $K_X^T(t, t-1) K_X^{-1}(t-1) = I$ it follows

$$B\tilde{B} = I, c^T \tilde{B} = 0, \tag{68}$$

Consider (67) using the introduced definitions for $K_{xX}(t, t-1), K_X(t-1)$ one comes to

$$A_o = K_{xX}(t, t-1) K_X^{-1}(t-1) = [b, B^T]^T [\tilde{B}, \tilde{c}]$$
$$= [d, D^T]^T, d := b^T [\tilde{B}, \tilde{c}], D = [I, 0] \tag{69}$$

which proves that A_o has the structure (63) with $a = a_o = d$. Thus if we are given a series of data $z(1), ..., z(t-1)$ then the optimal MMS prediction for $x(t)$ can be obtained by first estimating a_o and next to integrate the model (63).

2) Numerical experiment: Fig. 9 displays the values of a typical time series $x(t)$ at 100 first time instants (the curve "true") whose observations $z(t) = x(t) + v(t)$ (the curve "obs") are obtained by adding the noise $v(t)$ with zero mean and unit variance σ. In the experiment we will assume, however, that σ is unknown. Let us be given the observations at 300 time instants and the experiment consists in using the set of 100 first observations $Z_{[1:100]} := [z(1), ..., z(100)]$ to estimate the system dynamics; the 200 last observations will be used

TABLE I
ESTIMATED DYNAMICS PARAMETERS

dimension	a_1	a_2	a_3	a_4	a_5
$n=1$	-0.415	-	-	-	-
$n=2$	-0.508	-0.226	-	-	-
$n=3$	-0.53	-0.273	-0.091	-	-
$n=4$	-0.543	-0.305	-0.148	-0.103	-
$n=5$	-0.546	-0.308	-0.1457	-0.093	0.0207

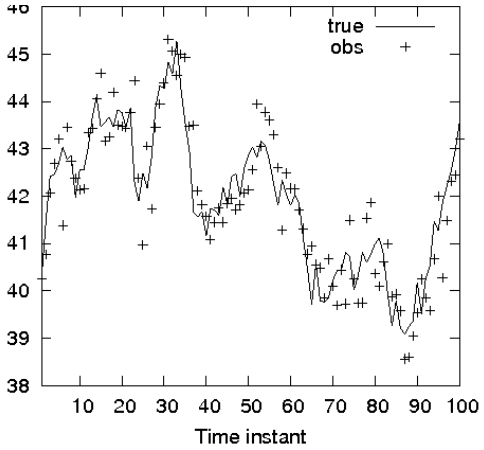

Fig. 9. Typical time series "true" and its observations "obs" contaminated with an uncorrelated noise sequence of zero mean and unknown variance

to validate the performance of the estimation algorithm. The matrices $K_{xX}(t-1,t)$ and $K_X(t-1)$ in (67) are approximated from $Z_{[1:100]}$. Applying the described above algorithm for estimating a_o subject to $n=2$ yields $a_1=-0.508$, $a_2=-0.226$ (see also Table I). In Fig. 10 we show the RMS of PE produced by the DS approach and that of one-step persistence error ($PER(1)$). Recall that the p-ahead persistence error $PER(p)$ is defined as $PER(p)=z(t+p)-z(t)$. The Fig. 10 can be considered as a validation test which justifies that the model largely outperforms persistence. Using the models of lower and higher dimensions results in the estimates for the dynamics parameters displayed in Table I. Experiments carried out by models with $n>2$ cannot lead to a noticeable decrease of RMS, compared to the case $n=2$. The worse performance is observed for $n=1$ (its RMS is higher about 3% compared to the case $n=2$).

D. Data assimilation in oceanic model

1) Adaptive filter based on DS representation for PE system output: The objective of oceanic data assimilation is to estimate the ocean state and to produce its best forecast for the period of interest (10 days, for example) using a numerical model (NM) and available observations. Last years the satellite sea surface height (SSH) is one of the most important sources of observations. Due to very high dimension of the NM (order of 10^{6-7}) and large set of observations (order of 10^{4-5}), its non-linearities ... at the present only approximate filters of simplified structure can be implemented. As a consequence, the resulting PE for the system output (a residual sequence - RS) $\zeta(t)$ remains usually time-correlated. It is worth of noticing that under standard conditions of

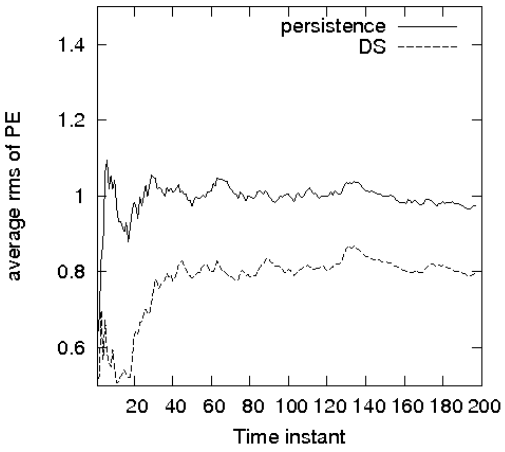

Fig. 10. Time average RMS of the prediction error (PE) resulting from the DS approach subject to $n=2$ and that of $PER(1)$. It is seen that the DS has produced the performance much better than the persistence.

validity of the Kalman Filter (KF) (the underlying system is a linear dynamical system, all error terms and measurements have a Gaussian distribution ...), the RS represents a white sequence (hence it is an innovation sequence). Thus, a time-correlation of the RS signifies that the designed filter is non-optimal.

In this situation, one possible and efficient way (see below) to improve the filter' performance is to impose a Markovian or non-Markov structure (see Theorems 1,2 for example) for the RS in order to reduce as much as possible its time correlation.

More concretely, consider the filtering problem

$$x(t+1)=\Phi x(t)+w(t),$$
$$z(t+1)=Hx(t+1)+v(t+1), t=0,1,2,... \qquad (70)$$

where $x(t)\in R^n, z(t)\in R^p, w(t), v(t)$ are the system and observational noise. Suppose the filter used in the assimilation is of the form

$$\hat{x}(t+1)=\Phi\hat{x}(t)+K(t+1)\zeta(t+1),$$
$$\zeta(t+1):=z(t+1)-H(t+1)\Phi\hat{x}(t), \qquad (71)$$

where $\zeta(t)$ is an RS. Under the standard conditions related to the distributions of $x(0), w(t), v(t)$, the gain $K(t)$ can be computed using the KF formalism and the resulting $\zeta(t)$ represents a time uncorrelated sequence. However when the gain $K(t)$ is obtained using different approximations (from physical considerations, order reduction of system state, wrong noise statistics, without involving a time-consuming Ricatti equation ...), the filter is not optimal and the sequence $\zeta(t)$ is time correlated.

One simple approximation for better representing $\zeta(t)$ is to assume that $\zeta(t)$ is a Markov RP. This assumption leads to the following equation for $\zeta(t)$ (see Eq. (22)),

$$\zeta(t)=\Psi_1(t)\zeta(t-1)+\Psi_2(t)\zeta(t-2)+...+$$
$$\Psi_d(t)\zeta(t-d)+\nu(t) \qquad (72)$$

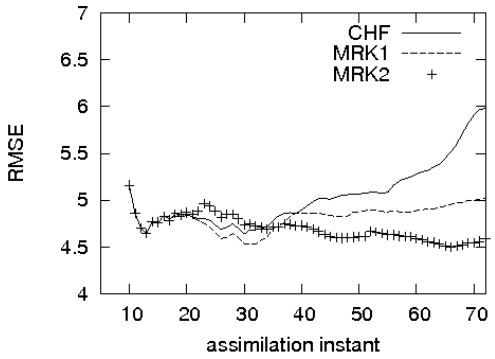

Fig. 11. The time average RMS of PE for the SSH resulting from the filter CHF (the curve "CHF"), CHF-MRK1 (the curve "MRK1") and CHF-MRK2 (the curve "MRK2").

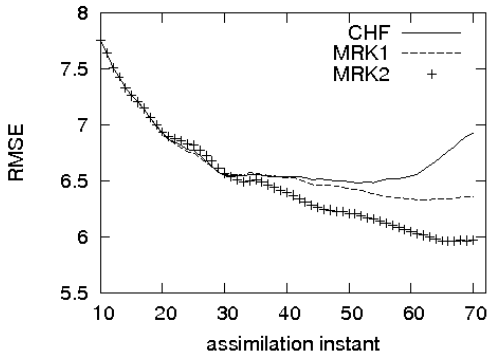

Fig. 12. The time average RMS of PE for the total velocity (u, v) resulting from the filter CHF (the curve "CHF"), CHF-MRK1 (the curve "MRK1") and CHF-MRK2 (the curve "MRK2").

where $\nu(t)$ is a white sequence. However in this equation the matrices $\Psi_k(t), k = 1, ..., d$ are unknown; That is true also for the statistics of $\nu(t)$. Notice that Eq. (72) is of the form (62) hence we can apply the results in Section 5.D to estimate the elements of $\Psi_k(t), k = 1, ..., d$. In the experiment to follow, for simplicity it is assumed that $\Psi_k(t) = \psi_k, k = 1, ..., d$ where ψ_k are time-invariant scalar parameter. The equation for estimating ψ_k is given by (67).

2) Experiment: We have tested the described above approach in the experiment with the Cooper-Haines Filter (CHF) which is developed by Cooper and Haines [4] for assimilating the SSH observations in the oceanic models. The objective of this test is to verify whether it is possible to obtain better results by imposing new DS structures for the RS. Remember that the gain in the CHF is constructed on the basis of the principle of conservation of potential velocity. The experiment has been carried out using the noise-free SSH data (3 years) and the NM Micom [9] to model the circulation in the North Atlantic. The observations are simulated as along-track altimetric observations hence are sparsly located. An optimal interpolation has been applied to generate a smooth set of observations, making possible the calculation of the horizontal velocity from the layer thickness estimates. This model has the horizontal grid (140 $\times$ 180), 4 vertical layers with three variables: SSH, layer thickness $h(i, j, k)$ and two velocity components $[u(i, j, k), v(i, j, k)]$. The system state consisting of (h, u, v) has the dimension $n = 302400$.

In Fig. 11 we show the time averaged RMS of the SSH PE, which are produced by three filters: CHF, CHF-MRK1 in which the RS is modelled by the DS (72) subject to $d = 1$ and CHF-MRK2 (subject to $d = 2$). It is seen that after about 30 iterations while the error becomes to grow in the CHF, the CHF-MRK1 allows to stabilize the estimation error. Moreover, the CHF-MRK2 is capable of decreasing the PE in a continuous way. The PEs produced by these three filters for the total velocity (u, v) are also displayed in Fig. 12 which exhibits a great advantage of two filters CHF-MRK1, CHF-MRK2 over the CHF.

It is seen that by introducing a simple Markov or non-Markov structure for the RS, it is possible to improve significantly the filter' performance if the original filter is still far from an optimal one. We have applied this approach, with nearly the same success, to a so called Prediction Error Filter (PEF) (developed in [8]) for the MICOM model in the North Atlantic domain as well as for a much more complex ocean model HYCOM (HYbrid Coordinate Ocean Model) with the coastal Bay of Biscay configuration [1]. Notice that the PEF outperforms largely the CHF in terms of its (much) lower estimation error. It is hoped that the presented approach will certainly find practitioners who wish to adopt it for yielding more accurate solutions to practical engineering problems.

VI. Conclusions

In this paper an algorithm for the construction of a DS model for a random process given CovF-SV is described. It is shown that this approach is applicable to a wide class of RPs having canonical forms. As seen from numerical examples and simulation studies, compared to the CF approach, the proposed algorithm is simpler to implement and generates samples closer to the references (samples of the "true" RP). Moreover, application of DS model is proved to be beneficial in solving the estimation problems like filtering, smoothing ... of Markov or non-Markov RPs yielding better performance compared to the CF approach. When solving estimation problems with correlated noises (see [2], [10], [7] ...), the DS models, formulated in Theorems 1-3, are usually assumed to be given. That is why it is of primary importance to be aware of how one can construct the DS models for Markov and non-Markov RPs, from the knowledge on statistics of the correlated noise sequences. Time correlation is present in a majority of engineering problems, in particular for the atmospheric and oceanographic observations [3]. Even with noise-free observations, the models in Theorems 1-3 can still serve as an useful tool for improving the filter' performance. This fact has been demonstrated in Section V.D where the experiment has been carried out on assimilation of noise-free SSH observations in the ocean model MICOM for the North Atlantic. The major difficulties we have in solving oceanic assimilation problems are due to very high dimension of numerical models, its non-linearities, sparse observations ... which do not allow to apply an optimal filter like the Kalman filter. Consequently only simplified, sub-optimal filters are

feasible in practice. Whitening the RS in the filter is a natural and logic way to improve the filter' performance. This can be done efficiently, as shown in this paper, by means of introducing a Markov or non-Markov model structure for the RS and estimating the unknown system parameters based on realizations of the RS during filtering process. Numerical results from the experiment with SSH data assimilation in the oceanic MICOM model show that by this way it is possible to reduce the estimation error up to about 15% compared to the level of the initial error.

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