The error of the Galerkin method for a nonhomogeneous Kirchhoff type wave equation

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Abstract**—**The paper deals with the boundary value problem for a nonlinear integro-differential equation describing the dynamic state of a beam. To approximate the solution with respect to a spatial variable, the Galerkin method is used, the error of which is estimated. At the end of the paper a difference-iteration technique of solving the Galerkin system is presented.

Keywords**—**Nonlinear beam equation, approximate algorithm, Galerkin method, error estimate.

I. Problem Formulation

Let us consider the nonlinear differential equation

$$\frac{\partial^2 w}{\partial t^2}(x,t)+\frac{\partial^4 w}{\partial x^4}(x,t) -\left(\alpha+\beta\int_0^L\left(\frac{\partial w}{d\xi}(\xi,t)\right)^2 d\xi\right)\frac{\partial^2 w}{\partial x^2}(x,t) = f(x,t),\quad 0<x<L,\quad 0<t\le T, \tag{1}$$

with the initial boundary conditions

$$\begin{gathered} w(x,0)=w^0(x),\quad \frac{\partial w}{\partial t}(x,0)=w^1(x),\\ w(0,t)=w(L,t)=0,\\ \frac{\partial^2 w}{\partial x^2}(0,t)=\frac{\partial^2 w}{\partial x^2}(L,t)=0,\\ 0\le x\le L,\quad 0\le t\le T, \end{gathered} \tag{2}$$

where α, β, L and T are some positive constants, $f(x,t)$, $w^0(x)$, $w^1(x)$ are the given functions and $w(x,t)$ is the function we want to define.

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II. Background of the Problem

Equation (1) describes the oscillation of a beam. The corresponding homogeneous equation was obtained by Woinowsky-Krieger [27] in 1950.

The nonlinear term in the brackets is the correction to the classical Euler-Bernoulli equation

$$w_{tt}+c^2 w_{xxxx}=0,$$

where the tension changes induced by the vibration of the beam during deflection are not taken into account. This nonlinear term was for the first time proposed by Kirchhoff [13] who generalized d'Alembert's classical model. Therefore equation (1) is often called a Kirchhoff type equation for a dynamic beam. Note that Arosio [1] calls the function of the integral $\int_0^L w_x^2\,dx$ the Kirchhoff correction (briefly, the *K*-correction) and makes a reasonable statement that the *K*-correction is inherent in a lot of physical phenomena.

The works dealing with the mathematical aspects of equation (1) when $f(x,t)=0$ and its generalization

$$w_{tt}+w_{xxxx}-M\left(\int_0^L w_\xi^2\,d\xi\right)w_{xx}=f(x,t,w),$$
$$M(\lambda)\ge const>0,$$

as well as some modifications of the above equations belong to Ball [2, 3], Biler [5], Brito [6], Dickey [10], Guo and Guo [12], Kouemou-Patcheu [14], Medeiros [17], Menezes et al. [18], Panizzi [20], Pereira [25] and to others. The subject of investigation concerned the questions of the existence and uniqueness of a solution [2, 3, 12, 14, 17, 18, 20, 25], its asymptotic behaviour [5, 6, 10, 14], stabilization and control problems [12] and so on.

The topic of an approximate solution of Kirchhoff equations, which the present paper is concerned with, was treated by Choo and Chung [7], Choo et al. [8], Clark et al. [9], Geveci and Christie [11]. Speaking more exactly, the finite difference and finite element approximate solutions are investigated and the corresponding error estimates are derived in [7, 8]. Numerical analysis of solutions for a beam with moving boundary is carried out in [9]. The question of the stability and convergence of a semidiscrete and fully discrete approximation is dealt with in [11]. The problem of an approximate solution of a static Kirchhoff equation was studied by Ma [16] and Tsai [26].

Approximate methods for other equations containing the *K*-correction or being reduced to equations with it are investigated in [22, 23, 24].

III. Assumptions

Suppose that the initial functions are represented in the form

$$w^l(x)=\sum_{i=1}^{\infty} a_i^{(l)}\sin\frac{i\pi}{L}x,\qquad (3)$$
$$l=0,1,\quad 0\le x\le L,$$

and

$$a_i^{(0)2}\le\frac{\omega_0}{i^{p+4}},\quad a_i^{(1)2}\le\frac{\omega_1}{i^{p}},\quad i=1,2,\ldots,\qquad (4)$$

where p, ω_0, ω_1 are some positive numbers and also $p>1$.

Assume that

$$f(x,t)\in C(0,T;L_2(0,L)).\qquad (5)$$

Suppose that there exists a solution of problem (1), (2) which is represented in the form

$$w(x,t)=\sum_{i=1}^{\infty} w_i(t)\sin\frac{i\pi}{L}x,\qquad (6)$$

where the coefficients $w_i(t)$ satisfy the following infinite system of differential equations

$$w_i''(t)+\left(\frac{\pi i}{L}\right)^4 w_i(t)$$
$$+\left(\frac{\pi i}{L}\right)^2\left(\alpha+\beta\frac{\pi^2}{2L}\sum_{j=1}^{\infty} j^2 w_j^2(t)\right)w_i(t)$$
$$=f_i(t),\qquad (7)$$
$$f_i(t)=\frac{2}{L}\int_0^L f(x,t)\sin\frac{i\pi}{L}x\,dx,$$
$$i=1,2,\ldots,\quad 0<t\le T,$$

with the initial conditions

$$w_i(0)=a_i^{(0)},\quad w_i'(0)=a_i^{(1)},\quad i=1,2,\ldots.\qquad (8)$$

Assume also that

$$\text{the series}\quad \sum_{i=1}^{\infty} w_i'^2(t)\quad\text{and}\quad\sum_{i=1}^{\infty} i^4 w_i^2(t)\qquad (9)$$

converge.

IV. The Galerkin Approximation

Let us perform approximation of the solution with respect to the variable x. For this we use the Galerkin method. A solution will be sought in the form of a finite series

$$w_n(x,t)=\sum_{i=1}^{n} w_{ni}(t)\sin\frac{i\pi}{L}x,\qquad (10)$$

where the coefficients $w_{ni}(t)$ are solutions of the system of differential equations

$$w_{ni}''(t)+\left(\frac{\pi i}{L}\right)^4 w_{ni}(t)$$
$$+\left(\frac{\pi i}{L}\right)^4\left(\alpha+\beta\frac{\pi^2}{2L}\sum_{j=1}^{n} j^2 w_{nj}^2(t)\right)w_{ni}(t)\qquad (11)$$
$$=f_i(t),\quad i=1,2,\ldots,\quad 0<t\le T,$$

with the initial conditions

$$w_{ni}(0)=a_i^{(0)},\quad w_{ni}'(0)=a_i^{(1)},\qquad (12)$$
$$i=1,2,\ldots,n.$$

Now we are going to estimate the error of the Galerkin method. To achieve this aim it is necessary to introduce several notions and to prove some auxiliary statements. Let λ and μ be n-dimensional vectors, $\lambda=(\lambda_i)_{i=1}^n$, $\mu=(\mu_i)_{i=1}^n$. In the first place, we define respectively the scalar product and the norm

$$(\lambda,\mu)_n=\sum_{i=1}^{n}\lambda_i\mu_i,\quad \|\lambda\|_n=(\lambda,\lambda)_n^{\frac{1}{2}}.\qquad (13.1)$$

Next, using the functions $w_{ni}(t)$, $f_i(t)$ and the coefficients $a_i^{(l)}$, $i=1,2,\ldots,n$, $l=0,1$, from (10), (7) and (3) we form the vectors

$$\boldsymbol{w}_n(t)=(w_{ni}(t))_{i=1}^n,\quad \boldsymbol{f}_n(t)=(f_i(t))_{i=1}^n,$$
$$\boldsymbol{a}_n^l=(a_i^{(l)}(t))_{i=1}^n,\quad l=0,1.\qquad (13.2)$$

We also define the matrix and the energetic norm

$$Q_n = \frac{\pi}{L} diag(1,2,\dots,n),$$
$$\|\lambda\|_{Q_n^{2l}} = \left(Q_n^{2l}\lambda,\lambda\right)_n^{\frac{1}{2}}, \quad l = 1,2. \tag{13.3}$$

Using this notation, (11), (12) can be written in the vector form

$$\boldsymbol{w}_n''(t) + Q_n^4 \boldsymbol{w}_n(t) + \left(\alpha + \beta\frac{L}{2}\|\boldsymbol{w}_n(t)\|_{Q_n^2}^2\right) Q_n^2 \boldsymbol{w}_n(t) = \boldsymbol{f}_n(t), \tag{14}$$
$0 < t \le T$,
$$\boldsymbol{w}_n(0) = \boldsymbol{a}_n^0, \quad \boldsymbol{w}_n'(0) = \boldsymbol{a}_n^1. \tag{15}$$

V. The Error of the Galerkin Method

By the coefficients of decomposition (6) we form the vector

$$p_n \boldsymbol{w}(t) = \left(w_i(t)\right)_{i=1}^n. \tag{16}$$

By the error of the Galerkin method we understand the difference between the vectors $\boldsymbol{w}_n(t)$ and $p_n\boldsymbol{w}(t)$

$$\Delta w_n(t) = \boldsymbol{w}_n(t) - p_n \boldsymbol{w}(t). \tag{17}$$

Let us derive an equation for the error.

Using (16) and (13), the first *n* equations of system in (7) and the first *n* equalities from each of the initial conditions (8) are written in the form

$$\left(p_n\boldsymbol{w}(t)\right)'' + Q_n^4 p_n \boldsymbol{w}(t) + \left(\alpha + \beta\frac{L}{2}\|p_n\boldsymbol{w}(t)\|_{Q_n^2}^2\right) Q_n^2 p_n \boldsymbol{w}(t) + \boldsymbol{z}_n(t) = \boldsymbol{f}_n(t), \tag{18}$$
$0 < t \le T$,
$$p_n\boldsymbol{w}(0) = \boldsymbol{a}_n^0, \quad \left(p_n\boldsymbol{w}\right)'(0) = \boldsymbol{a}_n^1, \tag{19}$$

where $\boldsymbol{z}_n(t)$ is the vector defined by the formula

$$\boldsymbol{z}_n(t) = \beta\frac{\pi^2}{2L}\left(\sum_{\boldsymbol{i=n+1}}^{\infty} i^2 w_i^2(t)\right) Q_n^2 p_n \boldsymbol{w}(t). \tag{20}$$

Subtracting (18) and (19) from (14) and (15), respectively, and taking into account (17), we write the equation for the error

$$\left(\Delta\boldsymbol{w}_n(t)\right)'' + Q_n^4 \Delta\boldsymbol{w}_n(t) + \left(\alpha + \beta\frac{L}{2}\|\boldsymbol{w}_n(t)\|_{Q_n^2}^2\right) Q_n^2 \Delta\boldsymbol{w}_n(t) - \beta\frac{L}{2}\left(\|p_n\boldsymbol{w}(t)\|_{Q_n^2}^2 - \|\boldsymbol{w}_n(t)\|_{Q_n^2}^2\right) Q_n^2 p_n \boldsymbol{w}(t) = z_n(t) \tag{21}$$

with the boundary conditions

$$\Delta\boldsymbol{w}_n(0) = 0, \quad \left(\Delta\boldsymbol{w}_n\right)'(0) = 0. \tag{22}$$

Equation (21) and conditions (22) are the starting point of the investigation of the problem of Galerkin method accuracy estimation.

Lemma 1. The estimate

$$\|p_n\boldsymbol{w}(t)\|_{Q_n^{2l}}^2 \le c_{2-l}, \quad l = 1,2, \tag{23}$$

where c_0 and c_1 do not depend on *n* and *t*, is valid.

Proof. We multiply the equation in (7) by $2w_i'(t)$ and sum the obtained expression over $i = 1,2,\dots$. If we use (5) and (9) and denote

$$\Phi(t) = \sum_{i=1}^{\infty} w_i'^2(t) + \left(\frac{\pi}{L}\right)^4 \sum_{i=1}^{\infty} i^4 w_i^2(t) + \frac{1}{\beta L}\left(\alpha + \beta\frac{\pi^2}{2L}\sum_{i=1}^{\infty} i^2 w_i^2(t)\right)^2, \tag{24}$$

then the result is written as $\Phi'(t) = 2\sum_{i=1}^{\infty} f_i(t) w_t'(t)$, which means that for $0 < t \le T$ we have

$$\Phi(t) \le \Phi(0) + 2\int_0^t \left(\sum_{i=1}^{\infty} f_i^2(\tau)\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} w_i'^2(\tau)\right)^{\frac{1}{2}} d\tau.$$

Taking (24) into account we infer that

$$\Phi(t) \le \Phi(0) + 2\sup_{0<t\le T}\sum_{i=1}^{\infty} f_i^2(t) \int_0^t \Phi^{\frac{1}{2}}(\tau) d\tau. \tag{25}$$

We need to use in (25) the following Bellman and Bihari generalization of Gronwall's inequality [4].

Let $y : [0,\infty) \to [0,\infty)$ be a continuous function and

$z:(0,\infty)\to(0,\infty)$ be a nondecreasing continuous function. Then the inequality $y(t)\le c+\int_0^t z(y(\tau))d\tau$, $0\le t<\infty$, where c is a positive constant, implies $y(t)\le Z^{-1}(Z_0)<\infty$, $0\le t<Z_0$, for a positive number Z_0 smaller than $Z(\infty)$. Here

$$Z(t)=\int_c^t \frac{d\tau}{z(\tau)},\quad t\ge c.$$

In the case under consideration

$$y(t)=\Phi(t),\quad c=\Phi(0),\quad z(\tau)=m\tau^{\frac{1}{2}},$$
$$m=2\left(\max_{0\le t\le T}\sum_{i=1}^{\infty} f_i^2(t)\right)^{\frac{1}{2}},\quad Z_0=T.$$

Thus

$$Z(t)=\frac{2}{m}\left(t^{\frac{1}{2}}-c^{\frac{1}{2}}\right),\quad Z^{-1}(t)=\left(c^{\frac{1}{2}}+\frac{m}{2}t\right)^2.$$

As a result we obtain

$$\Phi(t)\le\left(\Phi^{\frac{1}{2}}(0)+T\left(\sup_{0<t\le T}\sum_{i=1}^{\infty} f_i^2(t)\right)^{\frac{1}{2}}\right)^2. \tag{26}$$

By (26), (24) and the relations

$$\left\|\left(p_n\boldsymbol{w}(t)\right)'\right\|_n^2\le\sum_{i=1}^{\infty} w_i'^2(t),$$
$$\left\|\left(p_n\boldsymbol{w}(t)\right)'\right\|_{Q_n^{2l}}^2\le\sum_{i=1}^{\infty}\left(\frac{\pi i}{L}\right)^{2l} w_i^2(t),\quad l=1,2,$$
$$\int_0^L f^2(x,t)dx=\frac{L}{2}\sum_{i=1}^{\infty} f_i^2(t)$$

which follow from (16), (13), (7) and (5), we see that

$$\left\|\left(p_n\boldsymbol{w}(t)\right)'\right\|_n^2+\left\|p_n\boldsymbol{w}(t)\right\|_{Q_n^4}^2$$
$$+\frac{1}{\beta L}\left(\alpha+\frac{1}{2}\beta L\left\|p_n\boldsymbol{w}(t)\right\|_{Q_n^2}^2\right)^2\le c_0, \tag{27}$$

where

$$c_0=\left(\Phi^{\frac{1}{2}}(0)+T\left(\frac{2}{L}\sup_{0<t\le T}\int_0^L f^2(x,t)dx\right)^{\frac{1}{2}}\right)^2. \tag{28}$$

Let us calculate $\Phi(0)$. Using (24), (8), (3) and (4) we get

$$\begin{aligned}\Phi(0)&=\sum_{i=1}^{\infty} a_i^{(1)2}+\left(\frac{\pi}{L}\right)^4\sum_{i=1}^{\infty} i^4 a_i^{(0)2}\\&+\frac{1}{\beta L}\left(\alpha+\beta\frac{\pi^2}{2L}\sum_{i=1}^{\infty} i^2 a_i^{(0)2}\right)^2\\&=\frac{2}{L}\int_0^L\left[\left(w^1(x)\right)^2+\left(w^{0''}(x)\right)^2\right]dx\\&+\frac{1}{\beta L}\left(\alpha+\beta\int_0^L\left(w^{0'}(x)\right)^2dx\right)^2.\end{aligned} \tag{29}$$

From (27), first taking into account that by virtue of (13) $\left\|p_n\boldsymbol{w}(t)\right\|_{Q_n^4}\ge\frac{\pi}{L}\left\|p_n\boldsymbol{w}(t)\right\|_{Q_n^2}$ we obtain (23) for $l=1$, where

$$\begin{aligned}c_1=2\frac{1}{\beta L}&\left[\left(\left(\frac{\pi}{L}\right)^4+2\alpha\left(\frac{\pi}{L}\right)^2+c_0\beta L\right)^{\frac{1}{2}}\right.\\&\left.-\left(\left(\frac{\pi}{L}\right)^2+\alpha\right)\right],\end{aligned} \tag{30}$$

and then verify the fulfillment of (23) for $l=2$, where c_0 is defined by (28). □

Lemma 2. The inequality

$$\left\|\boldsymbol{w}_n(t)\right\|_{Q_n^2}^2\le c_2, \tag{31}$$

where the value c_2 does not depend on t, is valid.

Proof. Multiplying (14) scalarly by $2\boldsymbol{w}_n'(t)$, we obtain $\Phi_n'(t)=2\left(\boldsymbol{f}_n(t),\boldsymbol{w}_n'(t)\right)_n$, where

$$\Phi_n(t)=\|\boldsymbol{w}'_n(t)\|_n^2+\|\boldsymbol{w}_n(t)\|_{Q_n^4}^2$$
$$+\frac{1}{\beta L}\left(\alpha+\frac{1}{2}\beta L\|\boldsymbol{w}_n(t)\|_{Q_n^2}^2\right)^2. \tag{32}$$

Therefore we get the relation

$$\Phi_n(t)\le\Phi_n(0)+2\int_0^t\|\boldsymbol{f}_n(\tau)\|_n\|\boldsymbol{w}'_n(\tau)\|_n\,d\tau\,. \tag{33}$$

Let us apply the Bellman-Bihari inequality and definition (32) to (23). We have

$$y(t)=\Phi_n(0),\quad c=\Phi_n(0),\quad z(\tau)=m\tau^{\frac{1}{2}},$$
$$m=2\sup_{0<t\le T}\|\boldsymbol{f}_n(t)\|_n,\quad Z_0=T.$$

Therefore as above

$$Z(t)=\frac{2}{m}\left(t^{\frac{1}{2}}-c^{\frac{1}{2}}\right)^2,\quad Z^{-1}(t)=\left(c^{\frac{1}{2}}+\frac{m}{2}t\right)^2.$$

Hence we conclude that

$$\Phi_n(t)\le\left(\Phi_n^{\frac{1}{2}}(0)+T\sup_{0<t\le T}\|\boldsymbol{f}_n(t)\|_n\right)^{\frac{1}{2}}.$$

This relation which together with (32), (13) and (7) imply the fulfillment of (31) with

$$c_2=2\frac{1}{\beta L}\left[\left(\left(\frac{\pi}{L}\right)^4+2\alpha\left(\frac{\pi}{L}\right)^2+c_3\beta L\right)^{\frac{1}{2}}\right.$$
$$\left.-\left(\left(\frac{\pi}{L}\right)^2+\alpha\right)\right], \tag{34}$$

where

$$c_3=\left(\Phi_n^{\frac{1}{2}}(0)+T\left(\frac{2}{L}\sup_{0<t\le T}\sum_{i=1}^n f_i^2(t)\right)^{\frac{1}{2}}\right)^2, \tag{35}$$

and give the inequality

$$\Phi_n(t)\le c_3 \tag{36}$$

to be used below. □

If it is required to calculate or estimate c_2, we may use the following formulas for $\Phi_n(0)$

$$\Phi_n(0)=\sum_{i=1}^n a_i^{(1)2}+\left(\frac{\pi}{L}\right)^4\sum_{i=1}^n i^4 a_i^{(0)2}$$
$$+\frac{1}{\beta L}\left(\alpha+\beta\frac{\pi^2}{2L}\sum_{i=1}^n i^2a_i^{(0)2}\right)^2\le\Phi(0),$$
$$\Phi_n(0)\le\left(\omega_1+\omega_0\left(\frac{\pi}{L}\right)^4\right)\left(1+\frac{1}{p-1}\left(1-\frac{1}{n^{p-1}}\right)\right)$$
$$+\frac{1}{\beta L}\left(\alpha+\omega_0\beta\frac{\pi^2}{2L}\right)^2\left(1+\frac{1}{p+1}\left(1-\frac{1}{n^{p+1}}\right)\right)^2, \tag{37}$$

which are the result of the application of (32), (15), (13) together with (4), (3), (29). Besides the integral test for the convergence of series is used, by which

$$\sum_{i=1}^n\frac{1}{i^{p+l}}\le 1+\int_1^n\frac{1}{x^{p+l}}dx,\quad l=0,2\,.$$

Applying (30), (34)-(36), (28) and (7), we observe that

$$c_2\le c_1\,. \tag{38}$$

Lemma 3. The inequality

$$\|\boldsymbol{z}_n(t)\|_n\le\frac{c_4}{n^{p-1}}, \tag{39}$$

where the value c_4 does not depend on *t*, is valid.

Proof. From (20) and (13) it follows that

$$\|\boldsymbol{z}_n(t)\|_n=\beta\frac{\pi^2}{2L}\sum_{i=n+1}^\infty i^2w_i^2(t)\|p_n\boldsymbol{w}(t)\|_{Q_n^4}\,. \tag{40}$$

Using (9), let us introduce into consideration the function

$$\Psi_n(t)=\sum_{i=n+1}^\infty w_i'^2(t)+\left(\frac{\pi}{L}\right)^4\sum_{i=n+1}^\infty i^4w_i^2(t)$$
$$+\left(\frac{\pi}{L}\right)^2\left(\alpha+\beta\frac{\pi^2}{2L}\sum_{j=1}^\infty j^2w_j^2(t)\right)\sum_{i=n+1}^\infty i^2w_i^2(t). \tag{41}$$

After multiplying the equation in (7) by $2w_i'(t)$ and summing the resulting equality over $i = n+1, n+2, \ldots$, we obtain

$$\Psi_n'(t) = \beta\pi\left(\frac{\pi}{L}\right)^3 \sum_{j=1}^{\infty} j^2 w_j(t) w_j'(t) \sum_{i=n+1}^{\infty} i^2 w_i^2(t). \qquad (42)$$

By (24), (26), (28) and (7) we have

$$\left|\sum_{j=1}^{\infty} j^2 w_j(t) w_j'(t)\right|$$
$$\leq \frac{1}{2}\left(\frac{L}{\pi}\right)^2 \left(\sum_{j=1}^{\infty} w_j'^2(t) + \left(\frac{\pi}{L}\right)^4 \sum_{j=1}^{\infty} j^4 w_j^2(t)\right) \qquad (43)$$
$$\leq \frac{1}{2}\left(\frac{L}{\pi}\right)^2 \Phi(t) \leq \frac{1}{2} c_0 \left(\frac{L}{\pi}\right)^2.$$

Further, comparing the sum $\sum_{i=n+1}^{\infty} i^2 w_i^2(t)$ from (40) with the function $\Psi_n(t)$ from (41), we infer

$$\sum_{i=n+1}^{\infty} i^2 w_i^2(t) \leq \left(\frac{L}{\pi}\right)^2 \left(\alpha + \left(\frac{\pi}{L}\right)^2\right)^{-1} \Psi_n(t). \qquad (44)$$

By virtue of (42)-(44) and the Gronwall inequality

$$\Psi_n(t) \leq \Psi_n(0) \exp\left[\frac{1}{2} c_0 \beta L \left(\alpha + \left(\frac{\pi}{L}\right)^2\right)^{-1} t\right]. \qquad (45)$$

We need to estimate $\Psi_n(0)$. This estimate is obtained by using (41), (8), (4), (3) and the formula

$$\sum_{i=n+1}^{\infty} \frac{1}{i^{p+l}} \leq \int_n^{\infty} \frac{1}{x^{p+l}}, \quad l = 0,2,$$

which follows from the integral test for the convergence of series. As a result we have

$$\Psi_n(0) = \sum_{i=n+1}^{\infty} a_i^{(1)2} + \left(\frac{\pi}{L}\right)^4 \sum_{i=n+1}^{\infty} i^4 a_i^{(0)2}$$
$$+ \left(\frac{\pi}{L}\right)^2 \left(\alpha + \beta \frac{\pi^2}{2L} \sum_{j=1}^{\infty} j^2 a_j^{(0)2}\right) \sum_{i=n+1}^{\infty} i^2 a_i^{(0)2}$$
$$\leq \left(\omega_1 + \omega_0 \left(\frac{\pi}{L}\right)^4\right) \sum_{i=n+1}^{\infty} \frac{1}{i^p}$$
$$+ \omega_0 \left(\frac{\pi}{L}\right)^2 \left(\alpha + \beta \int_0^L \left(w^{0'}(x)\right)^2 dx\right) \sum_{i=n+1}^{\infty} \frac{1}{i^{p+2}} \qquad (46)$$
$$\leq \frac{1}{(p-1) n^{p-1}} \left[\omega_1 + \omega_0 \left(\frac{\pi}{L}\right)^4\right.$$
$$\left. + \omega_0 \frac{1}{n^2} \left(\frac{\pi}{L}\right)^2 (p-1)(p+1)^{-1} \left(\alpha + \beta \int_0^L \left(w^{0'}(x)\right)^2 dx\right)\right].$$

Applying to (40) inequalities (44)-(46) and (23) successively, we come to the conclusion that (39) is fulfilled and also that

$$c_4 = \frac{\beta L c_0}{2(p-1)} \left(\alpha + \left(\frac{\pi}{L}\right)^2\right)^{-1} \left[\omega_1 + \omega_0 \left(\frac{\pi}{L}\right)^4\right.$$
$$+ \omega_0 \frac{1}{n^2} \left(\frac{\pi}{L}\right)^2 (p-1)(p+1)^{-1}$$
$$\left. \times \left(\alpha + \beta \int_0^L \left(w^{0'}(x)\right)^2 dx\right)\right]$$
$$\times \exp\left[\frac{1}{2} c_0 \beta L \left(\alpha + \left(\frac{\pi}{L}\right)^2\right)^{-1} T\right]. \ \square$$

Let us formulate the main result.

Theorem. The inequality

$$\left(\left\|\left(\Delta \boldsymbol{w}_n(t)\right)'\right\|_n^2 + \left\|\Delta \boldsymbol{w}_n(t)\right\|_{Q_n^4}^2 + \alpha \left\|\Delta \boldsymbol{w}_n(t)\right\|_{Q_n^2}^2\right)^{\frac{1}{2}}$$
$$\leq \frac{c(t)}{n^{p-1}}, \qquad (47)$$

where $c(t)$ is defined below, is fulfilled for the error of the Galerkin method.

Proof. After the scalar multiplication of (21) by $2\left(\Delta \boldsymbol{w}_n(t)\right)'$ we obtain

$$F_n'(t)=\frac{1}{2}\beta L\Bigg[\|\Delta \boldsymbol{w}_n(t)\|^2_{Q_n^2}\left(\|\boldsymbol{w}_n(t)\|^2_{Q_n^2}\right)' + 2\left(\|p_n\boldsymbol{w}(t)\|^2_{Q_n^2}-\|\boldsymbol{w}_n(t)\|^2_{Q_n^2}\right) \times\left(Q_n^2 p_n \boldsymbol{w}(t),(\Delta\boldsymbol{w}_n(t))'\right)_n\Bigg] + 2\left(\boldsymbol{z}_n(t),(\Delta\boldsymbol{w}_n(t))'\right)_n, \quad (48)$$

where

$$F_n(t)=\left\|(\Delta\boldsymbol{w}_n(t))'\right\|^2_n+\|\Delta\boldsymbol{w}_n(t)\|^2_{Q_n^4} +\left(\alpha+\frac{1}{2}\beta L\|\boldsymbol{w}_n(t)\|^2_{Q_n^2}\right)\|\Delta\boldsymbol{w}_n(t)\|^2_{Q_n^2}. \quad (49)$$

Let us estimate some terms from the right-hand part of relation (48). For this we will have to make repeated use of (13).

By (32), (33) and (36) we get

$$\left(\|\boldsymbol{w}_n(t)\|^2_{Q_n^2}\right)'\le\|\boldsymbol{w}_n'(t)\|^2_n+\|\boldsymbol{w}_n(t)\|^2_{Q_n^4}\le\Phi_n(t)\le c_3. \quad (50)$$

From (16), (17), (23) and (31) follows

$$\left|\|p_n\boldsymbol{w}(t)\|^2_{Q_n^2}-\|\boldsymbol{w}_n(t)\|^2_{Q_n^2}\right| \le\left(\frac{\pi}{L}\right)^2\sum_{i=1}^n i^2\left|w_i^2(t)-w_{ni}^2(t)\right| \le\sqrt{2}\left(\|p_{\boldsymbol{n}}\boldsymbol{w}(t)\|_{Q_n^2}+\|\boldsymbol{w}(t)\|_{Q_n^2}\right)\|\Delta\boldsymbol{w}_n(t)\|_{Q_n^2} \le\sqrt{2}(c_1+c_2)\|\Delta\boldsymbol{w}_n(t)\|_{Q_n^2}. \quad (51)$$

Finally, again using (23) we find

$$\left|\left(Q_n^2p_n\boldsymbol{w}(t),(\Delta\boldsymbol{w}_n(t))'\right)_n\right| \le\|p_n\boldsymbol{w}(t)\|_{Q_n^4}\left\|(\Delta\boldsymbol{w}_n(t))'\right\|_n\le c_0\left\|(\Delta\boldsymbol{w}_n(t))'\right\|_n. \quad (52)$$

Relations (48)-(52) together with (13), (22) and (39) allow us to conclude that

$$F_n(t)=\int_0^t F_n'(\tau)d\tau\le\frac{c_4^2T}{n^{2(p-1)}}+\max(c_5,c_6)\int_0^t F_n(\tau)d\tau,$$

where

$$c_5=1+\nu,\quad c_6=\left(\alpha+\left(\frac{\pi}{L}\right)^2\right)^{-1}\left(\nu+\frac{1}{2}c_3\beta L\right), \quad \nu=\frac{L}{\sqrt{2}}c_0\beta(c_1+c_2). \quad (53)$$

Applying the Gronwall inequality and definition (49), we obtain the proven inequality (47) together with the formula for the coefficient $c(t)$

$$c(t)=c_4\sqrt{Te^{\max(c_5,c_6)t}}.\ \square$$

Note that if we weaken the accuracy requirement, relations (53) can be simplified. By virtue of (38) we can take c_1 instead of c_2 and replace the value $\Phi_n(0)$ contained in c_3 by one of its upper bounds from (37).

VI. Solution of the Galerkin System

Here we consider a method of solving the system (11), (12). Let us introduce, on the time segment $[0,T]$, a grid with step $\tau=T/M$ and nodes $t_m=m\tau$, $m=0,1,\ldots,M$. An approximate value of $w_{ni}(t_m)$ denoted by w_{ni}^m is determined by a difference scheme of the form

$$w_{ni\bar{t}t}^{m-1}+\left(\frac{\pi i}{L}\right)^4\frac{w_{ni}^m+w_{ni}^{m-2}}{2}+\left(\frac{\pi i}{L}\right)^2 \times\left(\alpha+\beta\frac{\pi^2}{2L}\sum_{j=1}^n j^2\frac{\left(w_{nj}^m\right)^2+\left(w_{nj}^{m-2}\right)^2}{2}\right)\frac{w_{ni}^m+w_{ni}^{m-2}}{2} =\frac{1}{2}\left(f_i^m+f_i^{m-2}\right), \quad (54)$$
$$i=1,2,\ldots n,\quad m=2,3,\ldots M,$$

with the conditions

$$w_{ni}^0=a_i^{(0)},$$
$$w_{ni}^1=a_i^{(0)}+\tau a_i^{(1)}-\frac{\tau^2}{2}\left[\left(\frac{\pi i}{L}\right)^4+\left(\frac{\pi i}{L}\right)^2 \times\left(\alpha+\beta\frac{\pi^2}{2L}\sum_{j=1}^n j^2a_j^{(0)2}\right)\right]a_i^{(0)}, \quad (55)$$

where $f_i^{m-l}=f_i(t_{m-l})$, $l=0,2$.

From (54) and (55) it follows that if the counting is performed from level to level, then, knowing the results for the preceding levels, at the mth time level, $m = 2,3,\ldots,M$, i.e. for $t = t_m$, we have to solve a system of nonlinear equations with respect to w_{ni}^m, $i = 1,2,\ldots,n$, which has the form

$$\left[1+\frac{r^2}{2}\left(\frac{\pi i}{L}\right)^2\left(\alpha+\left(\frac{\pi i}{L}\right)^2\right.\right.$$
$$\left.\left.+\beta\frac{\pi^2}{4L}\sum_{j=1}^{n} j^2\left(\left(w_{nj}^m\right)^2+\left(w_{nj}^{m-2}\right)^2\right)\right)\right]\left(w_{ni}^m+w_{ni}^{m-2}\right) \quad (56)$$
$$= 2w_{ni}^{m-1}+\frac{\tau^2}{2}\left(f_i^{\,m}+f_i^{\,m-2}\right),$$
$$i = 1,2,\ldots,n.$$

System (56) is solved by the iteration method consisting in calculating successive approximations by Jacobi's rule [19]

$$\left\{1+\frac{r^2}{2}\left(\frac{\pi i}{L}\right)^2\left[\alpha+\left(\frac{\pi i}{L}\right)^2\right.\right.$$
$$+\beta\frac{\pi^2}{4L}\left(i^2\left(\left(w_{ni,k+1}^m\right)^2+\left(w_{ni,F}^{m-2}\right)^2\right)\right.$$
$$\left.\left.\left.+\sum_{\substack{j=1\\ j\neq i}}^{n} j^2\left(\left(w_{nj,k}^m\right)^2+\left(w_{nj,F}^{m-2}\right)^2\right)\right)\right]\right\} \quad (57)$$
$$\times\left(w_{ni,k+1}^m+w_{ni,F}^{m-2}\right)$$
$$= 2w_{ni,F}^{m-1}+\frac{\tau^2}{2}\left(f_i^{\,m}+f_i^{\,m-2}\right),$$
$$i = 1,2,\ldots n, \quad k = 0,1,\ldots,$$

where $w_{ni,k}^m$ and $w_{ni,F}^{m-l}$ are the kth and the final iteration approximation of w_{ni}^m and w_{ni}^{m-l}, $l = 1,2$.

For fixed i, (57) is a cubic equation with respect to $w_{ni,k+1}^m$. The Cardano formula [15] allows us to determine $w_{ni,k+1}^m$ in an explicit form. We get

$$iw_{ni,k+1}^m = -\frac{iw_{ni,F}^{m-2}}{3}$$
$$-\sum_{l=0}^{1}\left[\frac{s_i}{2}+(-1)^l\left(\frac{s_i^2}{4}+\frac{r_i^3}{27}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}, \quad (58)$$
$$k = 0,1,\ldots, \quad i = 1,2,\ldots,n,$$

where

$$r_i = q_i+\frac{2}{3}\left(iw_{ni,F}^{m-2}\right)^2+\frac{1}{\tau^2 i^2\pi\beta}\left(\frac{2L}{\pi}\right)^3,$$
$$s_i = \frac{2}{3}iw_{ni,F}^{m-2}\left(q_i+\frac{10}{9}\left(iw_{ni,F}^{m-2}\right)^2\right)-\frac{1}{\tau^2 i^2\pi\beta}\left(\frac{2L}{\pi}\right)^3$$
$$\times\left(-\frac{2iw_{ni,F}^{m-2}}{3}+2iw_{ni,F}^{m-1}+\frac{\tau^2 i}{2}\left(f_i^{\,m}+f_i^{\,m-2}\right)\right),$$
$$q_i = \frac{4L}{\pi^2\beta}\left(\alpha+\left(\frac{\pi i}{L}\right)^2\right.$$
$$\left.+\sum_{\substack{j=1\\ j\neq i}}^{n} j^2\left(\left(w_{nj,k}^m\right)^2+\left(w_{nj,F}^{m-2}\right)^2\right)\right).$$

Thus the proposed algorithm is reduced to the calculation by formula (58). Having $w_{ni,k}^m$, we can construct the series $\sum_{i=1}^{n} w_{ni,k}^m \sin\frac{i\pi}{L}x$, which gives an approximate value of the exact solution $w(x,t)$ of problem (1), (2) for $t = t_m$.

The case where $f(x,t) = 0$ in (1) was considered in the author's paper [21].

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