

Optimal stopping and restarting times for multi item production inventory systems with resource constraints

Zaid T. Balkhi

Abstract- Most of production inventory systems is interested in determining the optimal stopping and restarting times of producing certain commodity. In this paper, a multi-item production inventory model under resource constraints is considered. For any product, each of the production, the demand, and the deterioration rates in any cycle as well as all cost parameters are treated as known and arbitrary functions of time. Shortage for each product is allowed but it is partially backlogged. All cost components are affected by both inflation and time value of money. The existence of resource constraints implies the use of Linear Programming in order to determine the optimal production rates for each item. The objective is to find the optimal production and restarting times for each product in any cycle so that the overall total inventory cost for all products is minimized. A formulation of the problem is developed and rigorous optimization techniques are used to show the uniqueness and global optimality of the solution. An illustrative example which show the applicability of the theoretical results is provided.

Keywords- Linear programming, Inventory control, Multi-item production, Varying Parameters, Optimality.

I. INTRODUCTION

Inventory is known as materials, commodities, products ,...etc, which are usually carried out in stocks in order to be consumed or benefited from when needed. In fact, most of economic, trading, manufacturing, administrative,...etc, systems regardless of its size , needs to deal with its own Inventory Control System . Keeping inventory in stores has its own various costs which may, sometimes, be more than the value of the commodity being carried out in stores. As examples, nuclear and biological weapons, blood in blood banks, and some kinds of sensitive medications. However, any inventory system must answer the following two main questions. (i) How much to order or to produce for each inventory cycle?. (ii) When to order or to produce a new quantity?. Answering these two questions for certain inventory system leads to the so called “ Optimal Inventory Policies” for which “ the Total Inventory Costs” for this system is minimized.

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It is expected that all systems, in which controlling and managing inventory is an important factor, can greatly benefit from research results so as to minimize their relevant inventory cost operations. For example, according to Nahmias Book (Production and Operations Analysis (1997)), the investment in inventories in the United States held in the manufacturing, wholesale and retail sectors during the first quarter of 1995 was estimated to be \$1.25 trillion. Therefore, there is a great need to perform special research on inventory management for those giant systems.

In fact, many classical inventory models concern with single item. Among these are Resh et al [24] who considered a classical lot size inventory model with linearly increasing demand. Hong et al [20] considered an inventory model in which the production rate is uniform and finite ,where he introduced three production policies for linearly increasing demand .A new inventory model in which products deteriorate at a constant rate and in which demand , production rates are allowed to vary with time has been introduced by Balkhi and Benkhrouf [8] . In this model , an optimal production policy that minimizes the total relevant cost is established . Subsequently , Balkhi [7] ,[9], [10], [11], [12], and [14] and Balkhi et al [13] have introduced several inventory models in each of which , the demand , production , and deterioration rates are arbitrary functions of times ,and in some of which , shortages are allowed but are completely backlogged. In each of the last mentioned seven papers , closed forms of the total inventory cost was established , a solution procedure was introduced and the conditions that guarantee the optimality of the solution for the considered inventory system were introduced. Recently , Balkhi [2],[3] ,[6], and Alamri and Balkhi [1] have introduced more advanced inventory models with similar but with more relaxed and general assumptions.

Though so many papers have dealt with single item optimal inventory policy and though the literature concerned with multi- item are sparse, the analysis of multi-item optimal inventory policies, is, almost, parallel to that of single item. The multi-item inventory classical inventory models under resource constraints are available in the well known books of Hadley and Whitin [19] and in Nador [23] . Ben-Daya and Raouf [16] have developed an approach for more realistic and general single period for multi-item with budgetary and floor or shelf space constraints, where the demand of items follows a uniform probability distribution subject to the restrictions on available space and budget. Bhattacharya [17]

has studied two-item inventory model for deteriorating items. Lenard and Roy [22] have used different approaches for the determination of optimal inventory policies based on the notion of efficient policy and extended this notion to multi-item inventory control by defining the concept of family and aggregate items. Kar et al [21] obtained some interesting results about multi deteriorating items with constraint space and investment. Rosenblatt [25] has discussed multi-item inventory system with budgetary constraint comparison between the Lagrangean and the fixed cycle approach, whereas, Rosenblatt and Rothblum [26] have studied a single resource capacity where this capacity was treated as a decision variable. Balkhi and Foul [4] and [5] have treated the problem of multi-item inventory control but without resource constraints. For more details about multi-item inventory system, the readers are advised to consult the survey of Yasemin and Erenguc [28] and the references therein.

Our main concern, here, is to establish optimal inventory policies for different products that competes for several limited resources and which are manufactured by any multi-item production system. This means that we need to determine the optimal stopping and restarting production times for each produced item so that the total relevant inventory cost of all items, under some given resource constraints, is minimum. The paper is organized as follows. First, we explain the procedure of finding the optimal production rates for any system that produced several items which compete for resource constraints. Then, we introduce our assumptions and notations, before building the mathematical model of the underlying problem. The solution procedure of the developed model is established in section 5, and the optimality of the obtained solution is proved in section 6. An illustrative example by which we verify our theoretical results is provided in section 7. Finally we introduce a conclusion in which we summarize the main results of the paper as well as our proposals for further research.

II. OPTIMAL PRODUCTION RATES UNDER RESOURCE CONSTRAINTS

A consistent and practical case which we are going to introduce in this paper is about multi-item products systems. Such products are usually related to several competition factors such as limited resources like, budgets, raw materials, manpower, manufacturing capacity, technological and demand restrictions, Etc. . Also, each of these products has its own production conditions, profit, and cost. Within these known factors and conditions, and an objective of maximizing total net profits, we can easily find the optimal number of units to be produced from each of those products by a simple Linear Program. For more details, suppose that we have m items and K different resource and technological constraints as follows. B_j is the number of available units from resource j ($j=1,2,\dots, K$), a_{ij} is the number of units consumed from resource j by item i ($i=1,2,\dots,m$). Let P_i be

the number of units to be produced from item i (here, P_i are decision variables) and r_i is the unit net profit of item i . Then our problem in this stage is to find the optimal values of P_i ($i=1,2,\dots,m$) so that the total net profit of all produced items (say Z) is maximum. That is, we need to solve the following (LP) problem

$$\left. \begin{aligned} & \text{Maximize } Z = \sum_{i=1}^m r_i P_i \text{ subject to} \\ & \sum_{i=1}^m a_{ij} P_i \leq B_j \quad ; j=1,2,\dots,K \\ & P_i \geq 0 \quad ; i=1,2,\dots,m \end{aligned} \right\} (LP)$$

But in reality, each of the parameters r_i, a_{ij}, B_j varies (perhaps periodically) with time which in turn allows P_i to vary (periodically) with time. Therefore, the more general case is to allow the optimal production rates P_i which result from solving the above (LP) to vary arbitrarily with time.

III. ASSUMPTIONS AND NOTATIONS

Our assumptions and notations for the model to be treated here are as follows:

1. m different items are produced in an arbitrary cycle (Infinite time horizon case) and held in stock
2. All items are subject to deterioration while they are effectively in stock and there is no repair or replacement of deteriorated items.
3. The demand rate, production rate, deterioration rate, production cost, shortage cost for backordered items, shortage cost for lost items, set up cost, and holding cost are all known and arbitrary functions of time and all cost components are affected by the inflation rate and time value of money.
4. All costs are affected by inflation rate and time value of money. We shall denote by r_1 the inflation rate and by r_2 the discount rate representing the time value of money so that $r = r_2 - r_1$ is the discount rate net of inflation.

For $i=1,2,\dots,m$, the parameters of the model are known functions of time and are denoted as follows:

- $D_i(t)$: Demand rate for item i at time t .
- $P_i(t)$: Production rate for item i at time t .
- $\theta_i(t)$: Deterioration rate for item i at time t .
- $I_i(t)$: Inventory level for item i at time t .
- $c_i(t)$: Production cost for item i at time t .
- $h_i(t)$: Holding cost for item i per unit per unit of time at time t .
- $b_i(t)$: Shortage cost for item i per unit per unit of time at time t for backordered items.

$I_i(t)$: Shortage cost for item i per unit per unit of time at time t for lost items.

$k_i(t)$: Setup cost for item i at time t .

$\beta_i(\tau) = e^{-\tau}$, is the rate of backordered items for item i , where $\tau = T_{i3} - t$ is the waiting time for item i up to the new production when shortages start to be backlogged for this item. Note that, $\beta_i(\tau)$ is a decreasing function of τ , which reflects the fact that the less waiting time implies more backordered units for item i

For item i , the proposed inventory system operates as follows. The cycle starts at time $t = 0$ and the inventory accumulates at a rate $P_i(t) - D_i(t) - \theta_i(t) I_i(t)$ up to time $t = T_{i1}$ where the production stops. After that, the inventory level starts to decrease due to demand and deterioration at a rate $-D_i(t) - \theta_i(t) I_i(t)$ up to time $t = T_{i2}$, where shortages start to accumulate at a rate $\beta_i(\tau) D_i(t)$ up to time $t = T_{i3}$. Production restarts again at time $t = T_{i3}$ and ends at time $t = T_{i4}$ and the inventory accumulates again with a rate $P_i(t) - D_i(t)$ to recover both the previous shortages in the period $[T_{i2}, T_{i3}]$ and to satisfy the demand in the period $[T_{i3}, T_{i4}]$. The process is repeated. In this respect and in order to recover the backordered items within the period $[T_{i2}, T_{i3}]$ and to satisfies the demand in the period $[T_{i3}, T_{i4}]$ we require that

$$P_i(t) > D_i(t) + \beta_i(T_3 - t)D_i(t) =$$

$$[1 + \beta_i(T_3 - t)]D_i(t) = [1 + \beta_i(\tau)]D_i(t)$$

The variation of the underlying inventory system for item i for one cycle is shown in Fig.1 below.

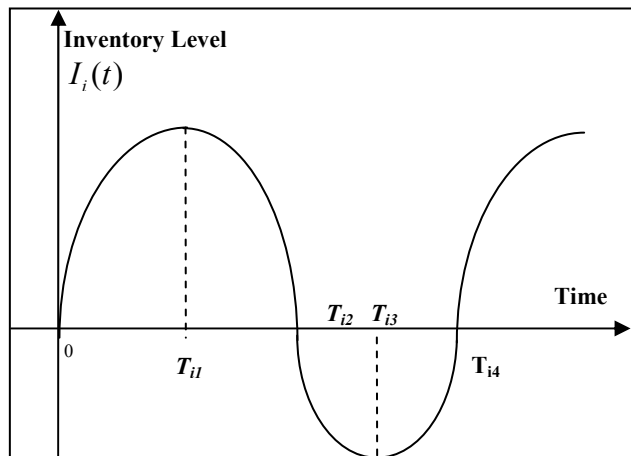


Fig 1. The Behavior of multi-item production
A. lot size inventory model

IV. MODEL BUILDING

The changes of the inventory level $I_i(t)$ for item i is governed by the following differential equation

$$\frac{dI_i(t)}{dt} = P_i(t) - D_i(t) - \theta_i(t)I_i(t); 0 \leq t \leq T_{i1} \quad (1)$$

$$\frac{dI_i(t)}{dt} = -D_i(t) - \theta_i(t)I_i(t); T_{i1} \leq t \leq T_{i2} \quad (2)$$

$$\frac{dI_i(t)}{dt} = -\beta_i(\tau)D_i(t); T_{i2} \leq t \leq T_{i3} \quad (3)$$

$$\frac{dI_i(t)}{dt} = P_i(t) - D_i(t); T_{i3} \leq t \leq T_{i4} \quad (4)$$

With the boundary conditions: $I_i(0) = 0$, $I_i(T_{i2}) = 0$, $I_i(T_{i3}) = 0$, $I_i(T_{i4}) = 0$ respectively.

The solutions of the above differential equations under their relative boundary conditions are

$$I_i(t) = e^{-g_i(t)} \int_0^t \{P_i(u) - D_i(u)\} e^{g_i(u)} du, \quad 0 \leq t \leq T_{i1} \quad (5)$$

$$I_i(t) = e^{-g_i(t)} \int_t^{T_{i2}} D_i(u) e^{g_i(u)} du; T_{i1} \leq t \leq T_{i2} \quad (6)$$

$$I_i(t) = - \int_{T_{i2}}^t \beta_i(\tau) D_i(u) du; T_{i2} \leq t \leq T_{i3} \quad (7)$$

$$I_i(t) = - \int_t^{T_{i4}} \{P_i(u) - D_i(u)\} du, T_{i3} \leq t \leq T_{i4} \quad (8)$$

respectively,

$$\text{where } g_i(t) = \int_0^t \theta_i(u) du \text{ with } g_i(0) = 0.$$

Next, we derive the present worth of each type of cost for item i .

Present worth of shortage cost of item i for backordered items (PWSCB):

Shortages occur over two periods, $[T_{i2}, T_{i3}]$ and $[T_{i3}, T_{i4}]$, which we denote by $PWSCB_{i1}$ and $PWSCB_{i2}$ respectively. Now,

$$PWSCB_{i1} = \int_{T_{i2}}^{T_{i3}} b_i(t) e^{-rt} I_i(t) dt$$

$$= \int_{T_{i2}}^{T_{i3}} b_i(t) e^{-rt} \left(\int_{T_{i2}}^t \beta(T_{i3} - u) D_i(u) du \right) dt.$$

Integrating by parts, we get:

$$PWSCB_{i1} = \int_{T_{i2}}^{T_{i3}} [B_i(T_{i3}) - B_i(t)] \beta_i(T_{i3} - t) D_i(t) dt \quad (9)$$

$$\text{Similarly } PWSCB_{i2} = \int_{T_{i3}}^{T_{i4}} b_i(t) e^{-rt} I_i(t) dt$$

$$= \int_{T_{i3}}^{T_{i4}} b_i(t) e^{-rt} \left(\int_t^{T_{i4}} \{P_i(u) - D_i(u)\} du \right) dt$$

Integrating by parts, we get:

$$PWSCB_{i2} = \int_{T_{i3}}^{T_{i4}} [B_i(t) - B_i(T_{i3})] \{P_i(t) - D_i(t)\} dt \quad (10)$$

Where

$$B_i(t) = \int_0^t b_i(u) e^{-ru} du, \text{ with } B_i(0) = 0 \quad (11)$$

Present worth of storage cost of item i for lost items ($PWSCL_i$).

In a small time period (dt) we lose a fraction $1 - \beta_i(T_{i3} - t) D_i(t) dt$, hence:

$$PWSCL_i = \int_{T_{i2}}^{T_{i3}} l_i(t) e^{-rt} [1 - \beta_i(T_{i3} - t)] D_i(t) dt$$

Using similar techniques we get the following:

Present worth of holding cost for item i ($PWHC_i$):

Items are held in stock in the two periods $[0, T_{i1}]$ and $[T_{i1}, T_{i2}]$. Using integration by parts as above we obtain

$$PWHC_i = \int_0^{T_{i1}} [H_i(T_{i1}) - H_i(t)] \{P_i(t) - D_i(t)\} e^{g_i(t)} dt + \int_{T_{i1}}^{T_{i2}} [H_i(t) - H_i(T_{i1})] D_i(t) e^{g_i(t)} dt \quad (13)$$

Where

$$H_i(t) = \int_0^t h_i(u) e^{-ru - g_i(u)} du, \text{ with } H_i(0) = 0 \quad (14)$$

Present worth of item production cost for item i ($PWPC_i$):

Since production occurs during the two periods $[0, T_{i1}]$ and $[T_{i3}, T_{i4}]$, so we have:

$$PWPC_i = \left\{ \int_0^{T_{i1}} c_i(t) P_i(t) e^{-rt} dt + \int_{T_{i3}}^{T_{i4}} c_i(t) P_i(t) e^{-rt} dt \right\} \quad (15)$$

Note that the last cost includes both consumed and deteriorated items.

Present worth of the set-up cost for item i ($PWSUC_i$):

The set-up of new production occurs twice during any cycle. The first is at $t = 0$, and the second is at $t = T_{i4}$. Therefore, the present worth of the set-up cost

$$PWSUC_i = k_i(0) e^{-r \cdot 0} + k_i(T_{i4}) e^{-rT_{i4}} = k_i(0) + k_i(T_{i4}) e^{-rT_{i4}} \quad (16)$$

Hence, the total relevant cost per unit time for item i as a function of $T_{i1}, T_{i2}, T_{i3}, T_{i4}$ which we shall denote by $w_i(T_{i1}, T_{i2}, T_{i3}, T_{i4})$ is given by

$$w_i = \frac{1}{T_{i4}} \{PWHC_{i1} + PWHC_{i2} + PWSCB_{i1} + PWSCB_{i2} + PWSCL_i + PWPC_i + PWSUC_i\} \text{ or,}$$

$$w_i = \frac{1}{T_{i4}} \left\{ \int_0^{T_{i1}} [H_i(T_{i1}) - H_i(t)] \{P_i(t) - D_i(t)\} e^{g_i(t)} dt + \int_{T_{i1}}^{T_{i2}} [H_i(t) - H_i(T_{i1})] D_i(t) e^{g_i(t)} dt + \int_{T_{i2}}^{T_{i3}} [B_i(T_{i3}) - B_i(t)] \beta_i(T_{i3} - t) D_i(t) dt + \int_{T_{i3}}^{T_{i4}} [B_i(t) - B_i(T_{i3})] \{P_i(t) - D_i(t)\} dt + \int_{T_{i2}}^{T_{i3}} l_i(t) e^{-rt} [1 - \beta_i(T_{i3} - t)] D_i(t) dt + \left[\int_0^{T_{i1}} c_i(t) P_i(t) e^{-rt} dt + \int_{T_{i3}}^{T_{i4}} c_i(t) P_i(t) e^{-rt} dt \right] + k_i(0) + k_i(T_{i4}) e^{-rT_{i4}} \right\} \quad (17)$$

Let

$$T_1 = (T_{11}, T_{12}, T_{13}, T_{14})^t, T_2 = (T_{21}, T_{22}, T_{23}, T_{24})^t, \dots, T_m = (T_{m1}, T_{m2}, T_{m3}, T_{m4})^t$$

Then, the total relevant cost for all items

say $W(T_1, T_2, T_3, T_4)$ is given by

$$W = \sum_{i=1}^m w_i \quad (18)$$

Our problem is to find the optimal values of $T_{i1}, T_{i2}, T_{i3}, T_{i4}$ for $i=1, 2, \dots, m$ which

minimize $W(T_1, T_2, T_3, T_4)$ given by (18)

subject to the following constraint:

$$T_{i1} < T_{i2} < T_{i3} < T_{i4} \quad , \quad i=1, 2, \dots, m \quad (19)$$

$$\int_{T_{i2}}^{T_{i3}} \beta_i(T_{i3} - t) D_i(t) dt = \int_{T_{i3}}^{T_{i4}} [P_i(t) - D_i(t)] dt, \text{ and}$$

$$e^{-g_i(T_{i1})} \int_0^{T_{i1}} \{P_i(t) - D_i(t)\} e^{g_i(t)} dt = e^{-g_i(T_{i1})} \int_{T_{i1}}^{T_{i2}} D_i(t) e^{g_i(t)} dt$$

The last two constraints are, respectively, equivalent to

$$C_{i1} : \int_{T_{i2}}^{T_{i3}} \beta_i(T_{i3} - t) D_i(t) dt - \int_{T_{i3}}^{T_{i4}} [P_i(t) - D_i(t)] dt = 0 \tag{20}$$

$$C_{i2} : \int_0^{T_{i1}} P_i(t) e^{g_i(t)} dt - \int_0^{T_{i2}} D_i(t) e^{g_i(t)} dt = 0 \tag{21}$$

$i=1,2,\dots,m$.

Note that constraints (19) are natural constraints since otherwise our problem would have no meaning. Constraint (20) comes from the fact that, for each i , the inventory levels given by (7) & (8) must be equal at $t = T_{i3}$ whereas constraint (21) comes from the fact that the inventory levels given by (5) & (6) must be equal at $t = T_{i1}$

Thus, our problem (call it (P)) is :

find the optimal values of $T_{i1}, T_{i2}, T_{i3}, T_{i4}$, $i=1,2,\dots,m$ which Minimize $W(T_1, T_2, T_3, T_4)$ subject to (19), (20) & (21).

V. SOLUTION PROCEDURES.

To solve problem (P) , we first ignore (19). This can be justified by the reasons that, if (19) do not hold, then the whole problem would have no meaning. However, we shall not consider any solutions that do not satisfy (19). Thus, our new problem is:

Minimize $W(T_1, T_2, T_3, T_4)$ subject to (20) & (21) (P_1)

Note that (P_1) is an optimization problem with two equality constraints, so it can be solved by the Lagrange Techniques. Now, let

$$\lambda_1 = (\lambda_{11}, \lambda_{12}, \dots, \lambda_{1m})^t, \lambda_2 = (\lambda_{21}, \lambda_{22}, \dots, \lambda_{2m})^t,$$

$$C_1 = (c_{11}, c_{12}, \dots, c_{1m})^t, C_2 = (c_{21}, c_{22}, \dots, c_{2m})^t$$

$L(T_1, T_2, T_3, T_4, \lambda_1, \lambda_2)$ be our Lagrangian then,

$$L(T_1, T_2, T_3, T_4, \lambda_1, \lambda_2) = W(T_1, T_2, T_3, T_4) + \lambda_1 C_1 + \lambda_2 C_2 \tag{22}$$

The necessary conditions for having optima are:

$$\frac{dL}{dT_{i1}} = 0, \frac{dL}{dT_{i2}} = 0, \frac{dL}{dT_{i3}} = 0, \frac{dL}{dT_{i4}} = 0, \frac{dL}{d\lambda_{i1}} = 0, \frac{dL}{d\lambda_{i2}} = 0 ; i=1,2,\dots,m \tag{23}$$

Now, from (17), (20) & (21) and for $i=1,2,\dots,m$, we have:

$$\frac{dL}{dT_{i1}} = \frac{1}{T_{i4}} \int_0^{T_{i1}} [H_i'(T_{i1})][P_i(t) - D_i(t)] e^{g_i(t)} dt - \frac{1}{T_{i4}} \int_{T_{i1}}^{T_{i2}} H_i'(T_{i1}) D_i(t) e^{g_i(t)} dt + \frac{1}{T_{i4}} e^{-rT_{i1}} c_i(T_{i1}) P_i(T_{i1}) + \lambda_{i1} \cdot 0 + \lambda_{i2} P_i(T_{i1}) e^{g_i(T_{i1})} = 0 \tag{24}$$

Recalling (21), we have:

$$\frac{1}{T_{i4}} e^{-rT_{i1}} c_i(T_{i1}) P_i(T_{i1}) + \lambda_{i2} P_i(T_{i1}) e^{g_i(T_{i1})} = 0 \Leftrightarrow \lambda_{i2} = \frac{-c(T_{i1}) e^{-rT_{i1} - g(T_{i1})}}{T_{i4}} \tag{25}$$

$$\frac{dL}{dT_{i2}} = \frac{1}{T_{i4}} [H_i(T_{i2}) - H_{ii}(T_{i1})] D_i(T_{i2}) e^{g_i(T_{i2})} - \frac{1}{T_{i4}} [B_i(T_{i3}) - B_i(T_{i2})] \beta_i(T_{i3} - T_{i2}) D_i(T_{i2}) - \frac{1}{T_{i4}} e^{-rT_{i2}} l_i(T_{i2}) [1 - \beta_i(T_{i3} - T_{i2})] D_i(T_{i2}) - \lambda_{i1} \beta_i(T_{i3} - T_{i2}) D_i(T_{i2}) - \lambda_{i2} D_i(T_{i2}) e^{g_i(T_{i2})} = 0 \tag{26}$$

Since $D_i(T_{i2}) > 0$, the above equation can be simplified to:

$$\frac{1}{T_{i4}} [H_i(T_{i2}) - H_i(T_{i1})] e^{g_i(T_{i2})} - \frac{1}{T_{i4}} [B_i(T_{i3}) - B_i(T_{i2})] \beta_i(T_{i3} - T_{i2}) - \frac{1}{T_{i4}} e^{-rT_{i2}} l_i(T_{i2}) [1 - \beta_i(T_{i3})] - \lambda_{i1} \beta_i(T_{i3} - T_{i2}) - \lambda_{i2} e^{g_i(T_{i2})} = 0 \tag{27}$$

$$\frac{dL}{dT_{i3}} = \frac{1}{T_{i4}} \frac{dW}{dT_{i3}} + \lambda_{i1} \frac{dC_{i1}}{dT_{i3}} + \lambda_{i2} \frac{dC_{i2}}{dT_{i3}} = 0$$

$$\frac{dL}{dT_{i3}} = \frac{1}{T_{i4}} \frac{dW}{dT_{i3}} + \lambda_{i1} \frac{dC_{i1}}{dT_{i3}} + \lambda_{i2} \frac{dC_{i2}}{dT_{i3}} = 0$$

Recalling (20), and noting that

$$\beta(T_3 - t) = e^{-(T_3 - t)}; \beta'(T_3 - t) = -e^{-(T_3 - t)}$$

; $\beta''(T_3 - t) = e^{-(T_3 - t)} = \beta(T_3 - t)$, the last equation is equivalent to

$$\begin{aligned}
 & -\frac{1}{T_{i4}} \int_{T_{i2}}^{T_{i3}} \beta_i(T_{i3}-t)[B_i(T_{i3})-B_i(t)]D_i(t)dt + \\
 & \frac{1}{T_{i4}} \int_{T_{i2}}^{T_{i3}} e^{-rt} l_i(t) \beta_i(T_{i3}-t) D_i(t) dt \\
 & -\frac{1}{T_{i4}} e^{-rT_{i3}} c_i(T_{i3}) P_i(T_{i3}) + \frac{1}{T_{i4}} k_i'(T_{i3}) e^{-rT_{i3}} \\
 & -\frac{1}{T_{i4}} r k_i(T_{i3}) e^{-rT_{i3}} + \lambda_{i1} \left[\int_{T_{i2}}^{T_{i3}} -\beta_i(T_{i3}-t) D_i(t) dt - \right.
 \end{aligned}$$

$$2D_i(T_{i3}) + P(T_{i3})] = 0 \tag{28}$$

To facilitate finding $\frac{dL}{dT_{i4}}$, let $W = \frac{V}{T_{i4}}$, then

from(17),(20)&(21)we have

$$\frac{dL}{dT_{i4}} = \frac{\frac{dV}{dT_{i4}} T_{i4} - V}{T_{i4}^2} + \lambda_{i1} \frac{dC_{i1}}{dT_{i4}} + \lambda_{i2} \frac{dC_{i2}}{dT_{i4}} = 0$$

This gives:

$$\begin{aligned}
 & \frac{1}{T_{i4}^2} (T_{i4} \{ [B_i(T_{i4}) - B_i(T_{i3})][P_i(T_{i4}) - D_i(T_{i4})] + \\
 & e^{-rT_{i4}} c_i(T_{i4}) P_i(T_{i4}) \} - V) - \lambda_{i1} [P_i(T_{i4}) - D_i(T_{i4})] = 0
 \end{aligned}$$

This leads to

$$\begin{aligned}
 W = \frac{V}{T_{i4}} &= [B_i(T_{i4}) - B_i(T_{i3})][P_i(T_{i4}) - D_i(T_{i4})] + \\
 & e^{-rT_{i4}} c_i(T_{i4}) P_i(T_{i4}) - \lambda_{i1} T_{i4} [P_i(T_{i4}) - D_i(T_{i4})] \tag{29}
 \end{aligned}$$

Note that, (29) gives the minimum total cost in terms of λ_{i1}, T_{i3} & T_{i4}

Equations (20),(21),(25),(27),(28),(29), are $6m$ equations with $6m$ variables Namely $T_{i1}, T_{i2}, T_{i3}, T_{i4}, \lambda_{i1}, \lambda_{i2}, i=1, 2, \dots, m$, so that the solution of these equations (if it exists) gives the critical points of $L(T_1, T_2, T_3, T_4, \lambda_1, \lambda_2)$ from which (T_1, T_2, T_3, T_4) is the corresponding critical point of $W(T_1, T_2, T_3, T_4)$.

VI. OPTIMALITY AND UNIQUENESS OF THE SOLUTION

In this section, we derive conditions that guarantee, uniqueness, and global optimality of any existing solution to problem (P). For that purpose, we first establish sufficient conditions under which the Hessian matrix $L(T_1^*, T_2^*, T_3^*, T_4^*)$ of the Lagrangean function

$L(T_1, T_2, T_3, T_4, \lambda_1, \lambda_2)$, calculated at any critical point $(T_1^*, T_2^*, T_3^*, T_4^*)$ of L , is positive semi- definite.

To compute the Hessian matrix of L we consider the following notations

$$\frac{\partial^2 L}{\partial T_{il}^2} = L_{T_{il}^2}, \quad \frac{\partial^2 L}{\partial T_{il} \partial T_{jl}} = L_{T_{il} T_{jl}},$$

$i, j=1, 2, \dots, m$ and $l=1, 2, 3, 4$

Then the related computations showed that for $i=1, 2, \dots, m$, $L(T_{i1}, T_{i2}, T_{i3}, T_{i4})$ has he following form

$$L(T_{i1}^*, T_{i2}^*, T_{i3}^*, T_{i4}^*) = \begin{bmatrix} L_{T_{i1}^2} & L_{T_{i1}T_{i2}} & 0 & L_{T_{i1}T_{i4}} \\ L_{T_{i1}T_{i2}} & L_{T_{i2}^2} & L_{T_{i2}T_{i3}} & 0 \\ 0 & L_{T_{i2}T_{i3}} & L_{T_{i3}^2} & L_{T_{i3}T_{i4}} \\ L_{T_{i1}T_{i4}} & 0 & L_{T_{i3}T_{i4}} & L_{T_{i4}^2} \end{bmatrix}_{(T_{i1}^*, T_{i2}^*, T_{i3}^*, T_{i4}^*)}$$

By Balkhi and Bebkherouf [8], Stewart [27] and Emet [18], this symmetric matrix is positive semi-definite if

$$L_{T_{i1}^2} \geq |L_{T_{i1}T_{i2}}| + |L_{T_{i1}T_{i4}}| \tag{30}$$

$$L_{T_{i2}^2} \geq |L_{T_{i1}T_{i2}}| + |L_{T_{i2}T_{i3}}| \tag{31}$$

$$L_{T_{i3}^2} \geq |L_{T_{i2}T_{i3}}| + |L_{T_{i3}T_{i4}}| \tag{32}$$

$$L_{T_{i4}^2} \geq |L_{T_{i1}T_{i4}}| + |L_{T_{i3}T_{i4}}| \tag{33}$$

Thus, the above arguments lead to the following theorem.

Theorem 1. Any existing solution of (P₁) is a minimizing solution to (P₁) if this solution satisfies (30) through (33).

Next we shall show that any minimizing solution of (P₁) is unique. To see this, we note, from (23) that, for a given i , each of $T_{i1}, T_{i2}, T_{i3}, T_{i4}$ can implicitly be determined as a function of (T_{i1}) . That is

$$T_{i1} = f_{i1}(T_{i1}) = T_{i1}, T_{i2} = f_{i2}(T_{i1}), T_{i3} = f_{i3}(T_{i1}), T_{i4} = f_{i4}(T_{i1})$$

Our argument in showing the uniqueness of any existing solution of (P₁) is based on the idea that the general value of W given by (17) must coincide with the minimum value of W given by (29). That is we must have

$$V(T_{i1}, f_{i2}(T_{i1}), f_{i3}(T_{i1}), f_{i4}(T_{i1})) / f_{i4}(T_{i1}) - W(T_{i1}, f_{i2}(T_{i1}), f_{i3}(T_{i1}), f_{i4}(T_{i1})) = 0 \quad (34)$$

Here, $V(T_{i1}, f_{i2}(T_{i1}), f_{i3}(T_{i1}), f_{i4}(T_{i1})) / f_{i4}(T_{i1})$ is taken from (29) and

$W(T_{i1}, f_{i2}(T_{i1}), f_{i3}(T_{i1}), f_{i4}(T_{i1}))$ is taken from (17).

Note that any minimizing solution of (P_i) (if it exists) is unique (hence global minimum) if equation (34), as an equation of T_{i1} , has a unique solution. This fact has been shown by Balkhi ([6], [7], [9] and [10]). Hence, the above arguments lead to the following theorem

Theorem 2. Any existing solution of (P_i) for which (30) through (33) hold is the unique and global optimal solution to (P_i) .

Next we shall verify our model by the following illustrative example

VII. ILLUSTRATIVE EXAMPLE AND ITS VERIFICATION

We have verified our model by the following illustrative example

$$D_i(t) = a_i t + a_{i0}, \quad \theta_i(t) = \theta_i, \quad P_i(t) = p_{i0} e^{p_i t}, \\ k_i(0) = k_{i1}, k_i(T_{i3}) = k_{i2}, c_i(t) = c_{i0} e^{c_i t}, h_i(t) = h_{i0} e^{h_i t}, \\ b_i(t) = b_{i0} e^{b_i t}, l_i(t) = l_{i0} e^{l_i t}, \beta_i(T_{i3} - t) = e^{-(T_{i3} - t)}$$

In order to verify the theoretical results of the introduced model, five different items with different parameter values have been chosen to verify this example.

The numerical results are shown by the Table-1 below. From the above numerical results, one can easily deduce that, production rate and the cost parameters have the major influence on the total cost W whereas, the influence of inflation and deterioration rate on W is minor.

VII. CONCLUSION

In this paper, we have considered a general multi-item production lot size inventory model under limited resources. The existence of resource constraints and some other technological conditions imply the use of linear programming techniques in order to determine the optimal production rates for all items being produced so that the total net profit is maximum. But, in reality, the parameters vary from time to time. This implies that the resulting production rates do vary with time. Therefore the more general case where the production rates are known and general functions of time is treated. Also, we considered the case where each of the demand, and deterioration as well as all cost parameters are known and general functions of time. Shortages are allowed but are partially backordered. Both inflation and time value of money are incorporated in all cost components. The objective is to minimize the overall total relevant inventory cost. We have built an exact mathematical model and introduced solution procedures by which we could determine the optimal stopping and restarting production times for any item in any cycle. Then, quite simple and feasible sufficient

conditions that guarantee the uniqueness and global optimality of the obtained solution are established. An illustrative example which explains the applicability of the theoretical results are also introduced and numerically verified. This seems to be the first time where such a general multi item (EPQ) is mathematically treated and numerically verified.

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Table-1 . Optimal stopping and restarting production times for five items with their corresponding minimum costs and their minimum total cost

<i>Parameters of item i</i>	a_i	a_{i0}	p_i	p_{i0}	c_i	c_{i0}	h_i	h_{i0}	k_{i1}	k_{i2}	l_i	l_{i0}	b_i	b_{i0}	r	θ_i
1	2	10	0.3	18	0.5	5	0.35	0.3	150	70	0.4	1	0.2	0.8	0.15	0.15
2	1	15	0.2	30	1	1	0.2	0.1	100	100	0.1	0.1	0.1	0.1	0.01	0.01
3	1.2	15	0.2	30	1	1	0.25	0.15	100	200	0.1	0.2	0.1	0.1	0.01	0.01
4	2	10	0.3	20	0.5	5	0.25	0.15	200	150	0.2	0.2	0.2	0.3	0.05	0.08
5	2	10	0.3	20	0.5	5	0.25	0.15	200	200	0.2	1	0.2	0.8	0.05	0.15
Optimal results for Item			λ_{i1}	λ_{i2}	T_{i1}		T_{i2}		T_{i3}		T_{i4}		W_i			
1			4.0028	-2.8773	0.72063		1.71409		1.76621		2.00714		184.288			
2			1.0054	-0.8441	0.4161		0.8832		1.0598		1.78133		205.36			
3			0.974	-0.8295	0.4815		1.0172		1.1564		1.9324		250.991			
4			4.0964	-2.838	0.55635		1.3144		1.62465		2.16453		299.762			
5			3.6712	-2.6755	0.50375		1.34669		1.64704		2.17368		332.141			
Total Cost W									=1272.542							