Diana Savin

About the diophantine equations $\frac{x^5+y^5}{x+y}=5z^5$ and $\frac{x^5+y^5}{x+y}=z^5$, in special conditions

***Abstract*—** In this paper we solve the Diophantine equations $\frac{x^5+y^5}{x+y}=5z^5$ and $\frac{x^5+y^5}{x+y}=z^5$, in special conditions.

***Keywords*—**Cyclotomics fields, Diophantine equations.

I. INTRODUCTION

Other equations related of the equation from the title have just been studied. For example in 1998 B. Poonen studied the Diophantine equation $x^5 + y^5 = z^3$. For this purpose he used the properties of elliptic curves, concretely the properties of Frey curves.

In 1999, Y. Bugeaud and later Bugeaud and M. Mignotte studied the Diophantine equation $\frac{x^n-1}{x-1}=y^q$.

They proved that the only solution (x, y, n, q) of the equation $\frac{x^n-1}{x-1}=y^q$, with $n\equiv 1 \pmod q$ is (3, 11, 5, 2). Using the hypergeometric method they showed more results about this equation.

The main result of this paper is:

Main Theorem. *The Diophantine equation* $\frac{x^5+y^5}{x+y}=5z^5$ *does not have integer solutions,* $x,y,z\in\mathbf{Z}$, $(x, y) = 1$.

In our proof we used techniques of cyclotomic fields and Dirichlet's theorem.

First we recall some theoretical results, necessary in the proof of the Main Theorem.

Proposition 1.1. ([1], [5]**).** *Let r be a prime odd positive integer, ξ be a primitive root of order r of unity and the cyclotomic field* $\mathbf{Q}(\xi)$. *The following statements hold:*

i)$1 - \xi$ is *a irreducible element in* $\mathbf{Z}[\xi]$.

ii) $1 - \xi^k = u_k(1-\xi)$, *where* $k\notin r\mathbf{Z}$, $u_k\in U(\mathbf{Z}[\xi])$.

iii) $r = u(1-\xi)^{r-1}$, *where* $u\in U(\mathbf{Z}[\xi])$.

Proposition 1.2. ([1]**).** *Let r be a prime odd positive integer, ξ be a primitive root of order r of unity and the cyclotomic field* $\mathbf{Q}(\xi)$. *Let m,n be rational positive integers, x, y be rational integers and suppose that r does not divide m-n. Then, the ideals*

$$(x + \xi^m y) \text{ and } (x + \xi^n y)$$

are coprime *ideals of the ring* $\mathbf{Z}[\xi]$ *if and only if g.c.d.*$(x, y) =1$ and $x+y$ *is not divisible by r.*

Proposition 1.3 ([8]). *Let r be a prime odd positive integer, ξ be a primitive r-th root of unity and the cyclotomic field* $\mathbf{Q}(\xi)$. *Let m, n be rational positive integers, x, y be rational integers such that g.c.d.(x,y)=1, r divides x+y and r does not divide m-n. Then, the greastest commun divisor of the elements* $x + \xi^m y$ *and* $x + \xi^m y$ (*in the ring* $\mathbf{Z}[\xi]$) *is* $1-\xi$.

Proposition 1.4. *Let r be a prime odd positive integer, ξ be a primitive r-th root of unity and the cyclotomic field* $\mathbf{Q}(\xi)$. *Let* G *be the Galois group of the cyclotomic field* $\mathbf{Q}(\xi)$ *over* $\mathbf{Q}$. *Then, the following statements holds:*

i) for any $\sigma\in G$ *and for any* $P\in Spec(\mathbf{Z}[\xi])$, *we have* $\sigma(P)\in Spec(\mathbf{Z}[\xi])$.

ii) the Galois group G *is isomorph with the multiplicative group* $U(\mathbf{Z}_r)$.

Let K be a field of algebraic numbers, $[K: \mathbf{Q}] = n$ and $\alpha\in K$ be such that $K=\mathbf{Q}(\alpha)$. We denote with $\mathbf{Z}_K$ the ring of integers of the field K.

Let $\alpha_1=\alpha$, $\alpha_2, \ldots, \alpha_n$ be the conjugates of α over $\mathbf{Q}$. We consider $\alpha_1, \alpha_2,\ldots, \alpha_r\in\mathbf{R}$, $\alpha_{r+1}, \alpha_{r+2},\ldots,\alpha_{r+s}\in\mathbf{C}$, $\alpha_{j+s}=\overline{\alpha}_j$, $r<j\leq s$, with $r + 2s = n$.

Let us denote with:

$$\varphi_j : K\rightarrow \mathbf{R},\ \varphi_j(\alpha)=\alpha_j,\ j=\overline{1,r},$$

the embeddings of K in $\mathbf{R}$, and

$$\varphi_j : K \to \mathbf{C}\ \ \varphi_j(\alpha) = \alpha_j,\ j = \overline{r+1, n}$$

the embeddings of K in **C**.

We can define:

$$\psi : U(\mathbf{Z_K}) \to \mathbf{R^{r+s}},$$

$\psi(\varepsilon) = (\log \left| |\varphi_1(\varepsilon)| \right|, \log \left| |\varphi_2(\varepsilon)| \right|, \ldots, \log \left| \varphi_{r+s}(\varepsilon) \right|)$, $\forall\ \varepsilon \in U(\mathbf{Z_K})$.

ψ is called logarithmic representation of algebraic numbers.

Dirichlet's units theorem.

i) *Kerψ is a finite group and Kerψ =W_K,*

where

W_K ={ $\beta \in$ K / β *is a root of unity* }.

We denote with: $\omega = \left| W_K \right|$.

ii) Imψ is a discrete abelian group, free of rank r+s-1 and there exists the isomorphism

$$U(\mathbf{Z_K}) \cong \mathbf{Z} / \omega \mathbf{Z} \times \mathbf{Z^{r+s-1}}.$$

If u$\in$ $\mathbf{Z_K}$, it results from Dirichlet's units theorem that there exists $\xi \in$ W_K and t = r+s-1 units of finite order $u_1, u_2, \ldots, u_t$ such that

$$u = \xi^a \cdot u_1^{h_1} \cdot \ldots \cdot u_t^{h_t},$$

where a,h_1, h_2,…, $h_t \in$ **N,** a $\le \omega$.

The set {u_1, u_2, … , u_t} is **called fundamental system of units** of the field K.

Lemma 1.5. *An element* $a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 \in$ **Z[**ξ**]** *is divisible by* 5 *if and only if* $5 \mid a_i$ (*in* **Z**), ($\forall$) i$\in \{0,1,2,3\}$.

II. Results

At the beginning we made some calculations in Mathematica 6.0 on the expression $\frac{x^5 + y^5}{x + y}$.

We gave x=$\overline{1,200}$, y=$\overline{1,200}$ and we remarked that does not exist rational integers z such that

$$\frac{x^5 + y^5}{x + y} = 5z^5.$$

We show in the following the calculus in Mathematica 6.0 for x=$\overline{1,20}$, y = $\overline{1,20}$.

```
In[1]:=Do[Print[FactorInteger[(i^5+j^5)/(i+j)]],{i,20},{j,20}]

{{1,1}}
{{11,1}}
{{61,1}}
{{5,1},{41,1}}
{{521,1}}
{{11,1},{101,1}}
{{11,1},{191,1}}
{{11,1},{331,1}}
{{5,1},{1181,1}}
{{9091,1}}
{{13421,1}}
{{19141,1}}
{{11,1},{2411,1}}
{{5,1},{71,1},{101,1}}
{{31,1},{1531,1}}
{{61681,1}}
{{11,1},{71,1},{101,1}}
{{11,1},{9041,1}}
{{5,1},{11,1},{2251,1}}
{{152381,1}}
{{11,1}}
{{2,4}}
{{5,1},{11,1}}
{{2,4},{11,1}}
{{11,1},{41,1}}
{{2,4},{61,1}}
{{1871,1}}
{{2,4},{5,1},{41,1}}
{{41,1},{131,1}}
{{2,4},{521,1}}
{{12391,1}}
{{2,4},{11,1},{101,1}}
{{5,1},{4951,1}}
{{2,4},{11,1},{191,1}}
{{11,1},{31,1},{131,1}}
{{2,4},{11,1},{331,1}}
{{74731,1}}
{{2,4},{5,1},{1181,1}}
{{117911,1}}
{{2,4},{9091,1}}
{{61,1}}
{{5,1},{11,1}}
{{3,4}}
{{181,1}}
{{421,1}}
{{3,4},{11,1}}
{{5,1},{11,1},{31,1}}
{{3001,1}}
{{3,4},{61,1}}
{{11,1},{701,1}}
{{41,1},{281,1}}
{{3,4},{5,1},{41,1}}
{{11,1},{2111,1}}
{{31,1},{1021,1}}
{{3,4},{521,1}}
{{55201,1}}
{{5,1},{11,1},{1291,1}}
```

{{3,4},{11,1},{101,1}}
{{31,1},{3631,1}}
{{61,1},{2281,1}}
{{5,1},{41,1}}
{{2,4},{11,1}}
{{181,1}}
{{2,8}}
{{461,1}}
{{2,4},{5,1},{11,1}}
{{1621,1}}
{{2,8},{11,1}}
{{4621,1}}
{{2,4},{11,1},{41,1}}
{{5,1},{2161,1}}
{{2,8},{61,1}}
{{11,2},{181,1}}
{{2,4},{1871,1}}
{{31,1},{1291,1}}
{{2,8},{5,1},{41,1}}
{{11,1},{6151,1}}
{{2,4},{41,1},{131,1}}
{{11,1},{9791,1}}
{{2,8},{521,1}}
{{521,1}}
{{11,1},{41,1}}
{{421,1}}
{{461,1}}
{{5,4}}
{{991,1}}
{{11,1},{151,1}}
{{11,1},{251,1}}
{{4441,1}}
{{5,4},{11,1}}
{{31,1},{331,1}}
{{14821,1}}
{{11,1},{31,1},{61,1}}
{{71,1},{401,1}}
{{5,4},{61,1}}
{{61,1},{821,1}}
{{71,1},{911,1}}
{{11,1},{7481,1}}
{{11,1},{9391,1}}
{{5,5},{41,1}}
{{11,1},{101,1}}
{{2,4},{61,1}}
{{3,4},{11,1}}
{{2,4},{5,1},{11,1}}
{{991,1}}
{{2,4},{3,4}}
{{31,1},{61,1}}
{{2,4},{181,1}}
{{3,4},{5,1},{11,1}}
{{2,4},{421,1}}
{{9931,1}}
{{2,4},{3,4},{11,1}}
{{71,1},{281,1}}
{{2,4},{5,1},{11,1},{31,1}}
{{3,4},{11,1},{41,1}}
{{2,4},{3001,1}}
{{62071,1}}
{{2,4},{3,4},{61,1}}
{{5,1},{31,1},{641,1}}
{{2,4},{11,1},{701,1}}
{{11,1},{191,1}}
{{1871,1}}
{{5,1},{11,1},{31,1}}
{{1621,1}}
{{11,1},{151,1}}
{{31,1},{61,1}}
{{7,4}}
{{5,1},{661,1}}
{{11,1},{431,1}}
{{6871,1}}
{{41,1},{241,1}}
{{11,1},{31,1},{41,1}}
{{5,1},{3881,1}}
{{7,4},{11,1}}
{{35281,1}}
{{11,1},{4211,1}}
{{31,1},{1931,1}}
{{5,1},{101,1},{151,1}}
{{95881,1}}
{{11,1},{10831,1}}
{{11,1},{331,1}}
{{2,4},{5,1},{41,1}}
{{3001,1}}
{{2,8},{11,1}}
{{11,1},{251,1}}
{{2,4},{181,1}}
{{5,1},{661,1}}
{{2,12}}
{{11,1},{491,1}}
{{2,4},{461,1}}
{{101,2}}
{{2,8},{5,1},{11,1}}
{{71,1},{271,1}}
{{2,4},{1621,1}}
{{11,1},{31,1},{101,1}}
{{2,12},{11,1}}
{{5,1},{11621,1}}
{{2,4},{4621,1}}
{{92921,1}}
{{2,8},{11,1},{41,1}}
{{5,1},{1181,1}}
{{41,1},{131,1}}
{{3,4},{61,1}}
{{4621,1}}
{{4441,1}}

{{3,4},{5,1},{11,1}}
{{11,1},{431,1}}
{{11,1},{491,1}}
{{3,8}}
{{11,1},{761,1}}
{{5,1},{31,1},{71,1}}
{{3,4},{181,1}}
{{31,1},{631,1}}
{{25951,1}}
{{3,4},{421,1}}
{{5,1},{8861,1}}
{{11,1},{5171,1}}
{{3,8},{11,1}}
{{11,1},{8231,1}}
{{41,1},{2741,1}}
{{9091,1}}
{{2,4},{521,1}}
{{11,1},{701,1}}
{{2,4},{11,1},{41,1}}
{{5,4},{11,1}}
{{2,4},{421,1}}
{{6871,1}}
{{2,4},{461,1}}
{{11,1},{761,1}}
{{2,4},{5,4}}
{{31,1},{401,1}}
{{2,4},{991,1}}
{{31,1},{661,1}}
{{2,4},{11,1},{151,1}}
{{5,5},{11,1}}
{{2,4},{11,1},{251,1}}
{{181,1},{311,1}}
{{2,4},{4441,1}}
{{211,1},{421,1}}
{{2,4},{5,4},{11,1}}
{{13421,1}}
{{12391,1}}
{{41,1},{281,1}}
{{5,1},{2161,1}}
{{31,1},{331,1}}
{{9931,1}}
{{41,1},{241,1}}
{{101,2}}
{{5,1},{31,1},{71,1}}
{{31,1},{401,1}}
{{11,4}}
{{71,1},{251,1}}
{{41,1},{541,1}}
{{5,1},{5591,1}}
{{35401,1}}
{{71,1},{631,1}}
{{131,1},{431,1}}
{{31,1},{2281,1}}
{{5,1},{17581,1}}
{{108421,1}}
{{19141,1}}
{{2,4},{11,1},{101,1}}
{{3,4},{5,1},{41,1}}
{{2,8},{61,1}}
{{14821,1}}
{{2,4},{3,4},{11,1}}
{{11,1},{31,1},{41,1}}
{{2,8},{5,1},{11,1}}
{{3,4},{181,1}}
{{2,4},{991,1}}
{{71,1},{251,1}}
{{2,8},{3,4}}
{{5,1},{11,2},{41,1}}
{{2,4},{31,1},{61,1}}
{{3,4},{461,1}}
{{2,8},{181,1}}
{{11,1},{5231,1}}
{{2,4},{3,4},{5,1},{11,1}}
{{11,1},{61,1},{131,1}}
{{2,8},{421,1}}
{{11,1},{2411,1}}
{{5,1},{4951,1}}
{{11,1},{2111,1}}
{{11,2},{181,1}}
{{11,1},{31,1},{61,1}}
{{71,1},{281,1}}
{{5,1},{3881,1}}
{{71,1},{271,1}}
{{31,1},{631,1}}
{{31,1},{661,1}}
{{41,1},{541,1}}
{{5,1},{11,2},{41,1}}
{{13,4}}
{{11,1},{3061,1}}
{{11,1},{3671,1}}
{{11,1},{4451,1}}
{{5,1},{11941,1}}
{{72931,1}}
{{101,1},{881,1}}
{{31,1},{3491,1}}
{{5,1},{71,1},{101,1}}
{{2,4},{11,1},{191,1}}
{{31,1},{1021,1}}
{{2,4},{1871,1}}
{{71,1},{401,1}}
{{2,4},{5,1},{11,1},{31,1}}
{{7,4},{11,1}}
{{2,4},{1621,1}}
{{25951,1}}
{{2,4},{11,1},{151,1}}
{{5,1},{5591,1}}
{{2,4},{31,1},{61,1}}
{{11,1},{3061,1}}

{{2,4},{7,4}}
{{41,1},{1091,1}}
{{2,4},{5,1},{661,1}}
{{11,1},{5741,1}}
{{2,4},{11,1},{431,1}}
{{91331,1}}
{{2,4},{6871,1}}
{{31,1},{1531,1}}
{{11,1},{31,1},{131,1}}
{{3,4},{521,1}}
{{31,1},{1291,1}}
{{5,4},{61,1}}
{{3,4},{11,1},{41,1}}
{{35281,1}}
{{11,1},{31,1},{101,1}}
{{3,4},{421,1}}
{{5,5},{11,1}}
{{35401,1}}
{{3,4},{461,1}}
{{11,1},{3671,1}}
{{41,1},{1091,1}}
{{3,4},{5,4}}
{{58321,1}}
{{11,1},{41,1},{151,1}}
{{3,4},{991,1}}
{{11,1},{41,1},{211,1}}
{{5,4},{181,1}}
{{61681,1}}
{{2,4},{11,1},{331,1}}
{{55201,1}}
{{2,8},{5,1},{41,1}}
{{61,1},{821,1}}
{{2,4},{3001,1}}
{{11,1},{4211,1}}
{{2,12},{11,1}}
{{5,1},{8861,1}}
{{2,4},{11,1},{251,1}}
{{71,1},{631,1}}
{{2,8},{181,1}}
{{11,1},{4451,1}}
{{2,4},{5,1},{661,1}}
{{58321,1}}
{{2,16}}
{{131,1},{571,1}}
{{2,4},{11,1},{491,1}}
{{5,1},{11,1},{1831,1}}
{{2,8},{461,1}}
{{11,1},{71,1},{101,1}}
{{74731,1}}
{{5,1},{11,1},{1291,1}}
{{11,1},{6151,1}}
{{71,1},{911,1}}
{{62071,1}}
{{31,1},{1931,1}}
{{5,1},{11621,1}}
{{11,1},{5171,1}}
{{181,1},{311,1}}
{{131,1},{431,1}}
{{11,1},{5231,1}}
{{5,1},{11941,1}}
{{11,1},{5741,1}}
{{11,1},{41,1},{151,1}}
{{131,1},{571,1}}
{{17,4}}
{{5,1},{18911,1}}
{{31,1},{3491,1}}
{{11,1},{11351,1}}
{{11,1},{9041,1}}
{{2,4},{5,1},{1181,1}}
{{3,4},{11,1},{101,1}}
{{2,4},{41,1},{131,1}}
{{11,1},{7481,1}}
{{2,4},{3,4},{61,1}}
{{5,1},{101,1},{151,1}}
{{2,4},{4621,1}}
{{3,8},{11,1}}
{{2,4},{4441,1}}
{{31,1},{2281,1}}
{{2,4},{3,4},{5,1},{11,1}}
{{72931,1}}
{{2,4},{11,1},{431,1}}
{{3,4},{991,1}}
{{2,4},{11,1},{491,1}}
{{5,1},{18911,1}}
{{2,4},{3,8}}
{{117991,1}}
{{2,4},{11,1},{761,1}}
{{5,1},{11,1},{2251,1}}
{{117911,1}}
{{31,1},{3631,1}}
{{11,1},{9791,1}}
{{11,1},{9391,1}}
{{5,1},{31,1},{641,1}}
{{95881,1}}
{{92921,1}}
{{11,1},{8231,1}}
{{211,1},{421,1}}
{{5,1},{17581,1}}
{{11,1},{61,1},{131,1}}
{{101,1},{881,1}}
{{91331,1}}
{{11,1},{41,1},{211,1}}
{{5,1},{11,1},{1831,1}}
{{31,1},{3491,1}}
{{117991,1}}
{{19,4}}
{{11,1},{101,1},{131,1}}
{{152381,1}}

{{2,4},{9091,1}}
{{61,1},{2281,1}}
{{2,8},{521,1}}
{{5,5},{41,1}}
{{2,4},{11,1},{701,1}}
{{11,1},{10831,1}}
{{2,8},{11,1},{41,1}}
{{41,1},{2741,1}}
{{2,4},{5,4},{11,1}}
{{108421,1}}
{{2,8},{421,1}}
{{31,1},{3491,1}}
{{2,4},{6871,1}}
{{5,4},{181,1}}
{{2,8},{461,1}}
{{11,1},{11351,1}}
{{2,4},{11,1},{761,1}}
{{11,1},{101,1},{131,1}}
{{2,8},{5,4}}

We think that the equation $\frac{x^5+y^5}{x+y} = 5z^5$ does not have integer solutions , with g.c.d.(x,y)=1.

Proof of the Main Theorem.

Let $(x, y, z) \in \mathbf{Z}\times \mathbf{Z}\times \mathbf{Z}$ a solution of the Diophantine equation $\frac{x^5+y^5}{x+y} = 5z^5$, with (x, y) = 1.

Applying Small Fermat's Theorem we have

$$x^5 \equiv x \pmod 5,\ y^5 \equiv y \pmod 5,$$

so

$$x^5 + y^5 \equiv x+y \pmod 5.$$

If 5 does not divide x+y, it results that 5 does not divide $x^5 + y^5$. A contradiction was obtained, so 5 divides x+y.

We denote with l the biggest positive integer with the property:

$$5^l \mid (x+y).$$ the last

So $5^{l+1} \mid (x^5 + y^5)$.

Let ξ be a primitive 5-th root of unity and G be the Galois group of the cyclotomic field $\mathbf{Q}(\xi)$ over $\mathbf{Q}$.

It is known that the ring $\mathbf{Z}[\xi]$ is principal. Since $\overline{2}$ is a generator of the group $(\mathbf{Z}_5^*, \cdot)$ it results that $\sigma \in G$,

$$\sigma(\xi) = \xi^2$$

is a generator of the group G.

Our equation is equivalent to

$$(x+\xi y)(x+\xi^2 y)(x+\xi^3 y)(x+\xi^4 y)=u(1-\xi)^4 z^5,$$

with $u \in U(\mathbf{Z}[\xi])$.

Since

$$\sigma(x+\xi^2 y) = x+\xi^4 y,$$

$$\sigma(\sigma(x+\xi^2 y)) = x+\xi^3 y,$$

$$\sigma(\sigma(\sigma(x+\xi^2 y))) = x+\xi y,$$

applying Proposition 1.4 and Proposition 1.3 we obtain

$$x+\xi^j y = u_j(1-\xi)\alpha_j^5,$$

where $\alpha_j \in \mathbf{Z}[\xi]$, j = 1,…,4, and α_i and α_j are coprime for any i, $j=\overline{1,4}$, $i \neq j$, $u_j \in U(\mathbf{Z}[\xi])$, $j = \overline{1,4}$.

We have:

$$\alpha_1 = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3,$$

with $a_i \in \mathbf{Z}$, i = 1, 2, 3,

$$\alpha_2 = b_0 + b_1\xi + b_2\xi^2 + b_3\xi^3,$$

with $b_i \in \mathbf{Z}$, i = 1, 2, 3.

We obtain

$$\alpha_1^5 \equiv a_0^5 + a_1^5\xi^5 + a_2^5\xi^{10} + a_3^5\xi^{15} \equiv$$
$$\equiv a_0 + a_1 + a_2 + a_3 \pmod 5.$$

Analogously

$$\alpha_2^5 \equiv b_0 + b_1 + b_2 + b_3 \pmod 5.$$

Applying Dirichlet's units theorem we have

$u_1 = \xi^{s_1} v_1$, with v_1 a real unity from $\mathbf{Z}[\xi]$,

$u_2 = \xi^{s_2} v_2$, with v_2 a real unity from $\mathbf{Z}[\xi]$.

We obtain

$$x+\xi y \equiv (1-\xi)\xi^{s_1} v_1 (a_0 + a_1 + a_2 + a_3) \pmod 5,$$

$$x+\xi^2 y \equiv (1-\xi)\xi^{s_2} v_2 (b_0 + b_1 + b_2 + b_3) \pmod 5.$$

After calculus we get:

$$v_1, v_2 \in \left\{ \pm\left(\frac{1+\sqrt{5}}{2}\right)^k / k \in Z \right\}.$$

Denoting with:

$$\beta_1 = v_1 (a_0 + a_1 + a_2 + a_3),$$

$$\beta_2 = v_2 (b_0 + b_1 + b_2 + b_3),$$

the last congruences become

$$x+\xi y \equiv \beta_1 (1-\xi)\xi^{s_1} \pmod 5,$$

$$x+\xi^2 y \equiv \beta_2 (1-\xi)\xi^{s_2} \pmod 5.$$

We obtain:

$$y \equiv \xi^{s_1-1}(\beta_1 - \beta_2\xi^{s_2-s_1}) \pmod{(1-\xi)^3},$$

$$x \equiv \xi^{1+s_1}(\beta_2\xi^{s_2-s_1-1} - \beta_1) \pmod{(1-\xi)^3}.$$

It results

$$x+y \equiv \beta_1\xi^{s_1-1}(1-\xi^2) - \beta_2\xi^{s_2-1}(1-\xi) \pmod{(1-\xi)^3}.$$

The last congruence is equivalent with

$$x+y \equiv (1-\xi)[\beta_1\xi^{s_1-1}(1+\xi) - \beta_2\xi^{s_2-1}] \pmod{(1-\xi)^3}.$$

Since

$$x+y\equiv 0 \pmod{(1-\xi)^3}$$

we obtain

$$(1-\xi)\beta_1\xi^{s_1-1}(1+\xi)\equiv(1-\xi)\,\beta_2\xi^{s_2-1} \pmod{(1-\xi)^3}.$$

Multiplying the last congruence with ξ it results

$$(x+\xi y)(1+\xi)\equiv x+\xi^2 y \pmod{(1-\xi)^3} \quad (1).$$

But

$$x+\xi^2 y=\sigma(x+\xi y),$$

so

$$x+\xi^2 y\equiv \sigma((1-\xi)\beta_1\xi^{s_1}) \pmod 5.$$

The last congruence is equivalent to

$$x+\xi^2 y\equiv$$
$$\equiv(1-\xi^2)\left[\pm\left(\frac{1+\sqrt5}{2}\right)^k\right](a_0+a_1+a_2+a_3)\,\xi^{2s_1} \pmod 5,$$

therefore

$$x+\xi^2 y\equiv(x+\xi y)(1+\xi)\xi^{s_1}\left[\pm\left(\frac{1+\sqrt5}{2}\right)^{k'}\right] \pmod 5 \quad (2).$$

From the relations (1) and (2) we obtain

$$(x+\xi y)(1+\xi)\equiv$$
$$\equiv(x+\xi y)(1+\xi)\xi^{s_1}\cdot\left[\pm\left(\frac{1+\sqrt5}{2}\right)^{k'}\right] \pmod{(1-\xi)^3}.$$

The last congruence is equivalent to

$$(x+\xi y)(1+\xi)\left[1-\xi^{s_1}\left(\pm\left(\frac{1+\sqrt5}{2}\right)^{k'}\right)\right]\equiv 0 \pmod{(1-\xi)^3}.$$

This congruence is equivalent with

$$(x+\xi y)(1+\xi)\left[1-\xi^{s_1}+\xi^{s_1}-\xi^{s_1}\left(\pm\left(\frac{1+\sqrt5}{2}\right)^{k'}\right)\right]\equiv$$
$$\equiv 0 \pmod{(1-\xi)^3}.$$

Applying Proposition 1.1 we get

$$(x+\xi y)(1+\xi)\left[(1-\xi)u_{s_1}+\xi^{s_1}\left(1\mp\left(\frac{1+\sqrt5}{2}\right)^{k'}\right)\right]\equiv$$
$$\equiv 0 \pmod{(1-\xi)^3},$$

with $u_{s_1}\in U(\mathbf{Z}[\xi])$.

If $(1-\xi)$ divides $1\mp\left(\frac{1+\sqrt5}{2}\right)^{k'}$, since $1\mp\left(\frac{1+\sqrt5}{2}\right)^{k'}$ is a real number it results that 5 divides $1\mp\left(\frac{1+\sqrt5}{2}\right)^{k'}$. So the biggest positive integer l' with the property $(1-\xi)^{l'}$ divides

$$(x+\xi y)(1+\xi)\left[(1-\xi)u_{s_1}+\xi^{s_1}\left(1\mp\left(\frac{1+\sqrt5}{2}\right)^{k'}\right)\right]$$

is $l'=1$.

If $(1-\xi)$ does not divide $1\mp\left(\frac{1+\sqrt5}{2}\right)^{k'}$, it results that $(1-\xi)$ does not divide

$$(x+\xi y)(1+\xi)\left[(1-\xi)u_{s_1}+\xi^{s_1}\left(1\mp\left(\frac{1+\sqrt5}{2}\right)^{k'}\right)\right].$$

It results that $(1-\xi)^2$ divides $(x+\xi y)(1+\xi)$.

Knowing that $1-\xi$ is a prime element in the ring $\mathbf{Z}[\xi]$ and $1-\xi$ does not divide $1+\xi$, we obtain that $(1-\xi)^2$ divides $x+\xi y$.

But $(1-\xi)^2$ divides $x+y$, therefore $(1-\xi)^2$ divides

$$x+\xi y-x-y=-y(1-\xi).$$

It results that $1-\xi$ divides y and since $1-\xi$ divides x+y we get that $1-\xi$ divides x. We obtain a contradiction with the fact that x and y are coprime.

III. The Diophantine Equation $\frac{x^5+y^5}{x+y}=5z^5$ in Some General Conditions

In the following we study the Diophantine equation $\frac{x^5+y^5}{x+y}=5z^5$, in the case when g.c.d.$(x,y)\neq 1$.

Let q a prime positive integer such that $q\mid x$ and $q\mid y$. We denote with k the biggest positive integer with the property $q^k\mid x$ and $q^k\mid y$.

Case I: if $q\neq 5$, simplifying with q^{4k} we obtain an equation of the same type (that is $\frac{x^5+y^5}{x+y}=5z^5$, with g.c.d.(x,y)= 1).

Case II: if q=5, simplifying with 5^{4k} we obtain the

Diophantine equation $\frac{x_1^5+y_1^5}{x_1+y_1}=z_1^5$, where $x=5^k x_1$, $y=5^k y_1$, $z=5^{\frac{4k-1}{5}}\cdot z_1$.

We remark that g.c.d.$(x_1,y_1)=1$.

We change the notations: $x_1\to x$, $y_1\to y$, $z_1\to z$ and we study the Diophantine $\frac{x^5+y^5}{x+y}=z^5$ equation if we put the rectrictions g.c.d.(5,x)=1, g.c.d.(5,y)=1, x is not congruent with y modulo 5.

It is well known that if p is a prime positive integer and x and y are coprime integers, then every prime factor of $\frac{x^p+y^p}{x+y}$ is p and it appears with exponent 1 or it is congruent with 1 modulo p.

It is clear that 5 does not divide z, so 5 does not divide x^5+y^5.

Since $x^5+y^5\equiv x+y \pmod 5$, it results that 5 does not divide x+y. The equation is equivalent to

$$(x+\xi y)(x+\xi^2 y)(x+\xi^3 y)(x+\xi^4 y)=z^5.$$

Applying Proposition 1.2 it results that $x+\xi^i y$ and $x+\xi^j y$ are coprime, any $i,j\in\{1,2,3,4\}$, $i\neq j$.

Since **Z**[ξ] is a factorial ring, it results that $x+\xi^j y=u_j\alpha_j^5$, $\alpha_j\in$ **Z**[ξ], $(\forall)$ $j\in\{1,2,3,4\}$, α_i,α_j are coprime $(\forall)$i, $j\in\{1,2,3,4\}$, $i\neq j$, $u_j\in U(\mathbf{Z}[\xi])$, $(\forall)$ $j\in\{1,2,3,4\}$. We known that **Z[ξ]** is free module of the rank 4 over **Z** and $\{1,\xi,\xi^2,\xi^3\}$ is a basis for **Z[ξ].** Using this fact we obtain that

$$\alpha_1=a_0+a_1\xi+a_2\xi^2+a_3\xi^3,$$

with $a_i\in$ **Z,** $(\forall)$ $i\in\{0,1,2,3\}$. Analogously with the proof of Main Theorem we obtain that

$$\alpha_1^5\equiv a_0+a_1+a_2+a_3 \pmod 5.$$

Applying Dirichlet's units theorem we have $u_1=\xi^{s_1}v_1$, with v_1 a real unity from **Z[ξ].**

Denoting with:

$$\beta_1=v_1(a_0+a_1+a_2+a_3),$$

we obtain

$$x+\xi y\equiv\beta_1\xi^{s_1} \pmod 5.$$

The last congruence can be written in the form

$$\xi^{-s_1}(x+\xi y)\equiv\beta_1 \pmod 5.$$

Knowing that, if an element $z\in$ **Z[ξ],** it results $\overline{z}\in$ **Z[ξ],** from the last congruence is results

$$\xi^{s_1}(x+\xi^{-1}y)\equiv\beta_1 \pmod 5.$$

We obtain

$$x\xi^{s_1}+y\xi^{s_1-1}-x\xi^{-s_1}-y\xi^{1-s_1}\equiv 0 \pmod 5 \qquad (3)$$

We study the exponents of ξ from the congruence (3). If some exponent of ξ from the congruence (3), for example s_1 (analogous to proceed if instead of s_1 we consider any other exponent of ξ) is congruent with 4 modulo 5, replacing $\xi^{s_1}=\xi^4=-1-\xi-\xi^2-\xi^3$, congruence (3) becomes

$$x-2x\xi-(x+y)\xi^2+(y-x)\xi^3\equiv 0 \pmod 5.$$

Applying Lemma 1.5 we obtain that $x+y\equiv 0 \pmod 5$, contradiction.

We show that any two from the exponents of ξ from the congruence (3) are not congruent modulo 5. It is clear that s_1 and $1-s_1$ are not congruent modulo 5. If $s_1\equiv -s_1 \pmod 5$ it results that $5\mid s_1$ and congruence (3) becomes

$$x+y\xi^4-x-y\xi\equiv 0 \pmod 5,$$

which is equivalent with

$$y+2y\xi+y\xi^2+y\xi^3\equiv 0 \pmod 5.$$

Applying Lemma 1.5 we obtain that $y\equiv 0 \pmod 5$, contradiction.

If $s_1-1\equiv 1-s_1 \pmod 5$ it results that $s_1\equiv 1 \pmod 5$ and congruence (3) becomes

$$x\xi+y-x\xi^4-y\equiv 0 \pmod 5.$$

This congruence is equivalent with

$$x+2x\xi+x\xi^2+x\xi^3\equiv 0 \pmod 5.$$

Applying Lemma 1.5 we obtain that $x\equiv 0 \pmod 5$, contradiction.

If $s_1\equiv 1-s_1 \pmod 5$ it results that $s_1\equiv 3 \pmod 5$ and congruence (3) becomes

$$(x-y)\xi^3+(y-x)\xi^2\equiv 0 \pmod 5.$$

Applying Lemma 1.5 we obtain that $x\equiv y \pmod 5$, contradiction.

From the previously proved we obtain that the Diophantine equation $\frac{x^5+y^5}{x+y}=z^5$ does not have integer solutions, with g.c.d.(x, y) = 1, g.c.d.(5,x)=1, g.c.d.(5,y)=1, x is not congruent with y modulo 5.

IV. Conclusion

From the previously proved we obtain that the Diophantine equation $\frac{x^5+y^5}{x+y}=5z^5$ does not have solutions $(x, y, z)\in \mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}$, with g.c.d. (x, y) = 1.

In the case when g.c.d.(x, y) $\neq$ 1, our problem reduces to study the Diophantine equation $\frac{x^5+y^5}{x+y}=z^5$, which we have shown that does not have solutions $(x, y, z)\in\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}$, with g.c.d.(x, y) = 1, g.c.d.(5,x)=1, g.c.d.(5,y)=1, x is not congruent with y modulo 5.

REFERENCES

[1] T. Albu, I.D. Ion, *Chapters of the algebraic number theory* (*in Romanian*), Ed. Academiei, Bucureşti, 1984.

[2] Y. Bugeaud, *Linear forms in p-adic logarithms and the Diophantine equation* $\frac{x^n - 1}{x - 1} = y^q$, Math. Proc. Cambridge Phil. Soc. 127 (1999), pp. 373 – 381.

[3] Y. Bugeaud, *On the diophantine equation a* $\frac{x^n - 1}{x - 1} = y^q$, Proceedings of the Number Theory Conference held in Turku, Ed. M. Jutila and T. Metsankila, De Gruyter (2001), pp. 19-24.

[4] **Y.** Bugeaud and M. Mignotte, L'equation de Nagell – Ljunggren $\frac{x^n - 1}{x - 1} = y^q$, Enseign. Math. (2) 48 (2002), pp. 147-168.

[5] K. Ireland, M. Rosen, *A Classical Introduction to Modern Number Theory,* Second Edition, Springer-Verlag, 1992.

[6] F. Lemmermeyer, *Reciprocity Laws,* Springer-Verlag, 2000.

[7] B. Poonen, *Some diophantine equations of the form* $x^n + y^n = z^m$, Acta Arithmetica LXXXVI, 1998, pp. 193 – 205.

[8] D. Savin, *Some properties of cyclotomic fields and Kummer fields*, International Journal of Mathematics and Computation, vol. 6, No. M10, March 2010, pp. 22 - 26.

[9] D. Savin, *On the Diophantine Equation* $\frac{x5 + y5}{x + y} = 5z^5$, Proceedings of the 12 th WSEAS International Conference on Mathematical Methods, Computational Techniques and Intelligent Systems, (MAMECTIS'10), May 3-6, 2010, Sousse, Tunisia.