Deformations of monic polynomial matrices. Analysis of perturbation of eigenvalues

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***Abstract*—** Let $P(\lambda)=\sum_{i=0}^{k}\lambda^i A_i(p)$ be a family of monic polynomial matrices smoothly dependent on a vector of real parameters $p=(p_1,\dots,p_n)$. In this work we study behavior of a multiple eigenvalue of the monic polynomial family $P(\lambda)$ as well as we study of behavior of a simple eigenvalue of a family of 1-degree singular polynomial matrices representing families of singular linear systems.

***Keywords*—** olynomial matrix, Eigenvalues, Perturbation.olynomial matrix, Eigenvalues, Perturbation.P

I. Introduction

Given a polynomial matrix $P(\lambda)=\sum_{i=0}^{k}\lambda^i A_i$ where A_i are square matrices over real or complex field, it is an important question from both the theoretical and the practical points of view to know how the eigenvalues and eigenvectors change when the elements of $P(\lambda)$ are subjected to small perturbations.

Eigenvalue problem for polynomial matrices $P(\lambda)v=0$, appears (among many other applications) modeling physical and engineering problems by means systems of k-order linear ordinary differential equations. The values of eigenvalues can correspond among others, to frequencies of vibration, critical values of stability parameters, or energy levels of atoms.

The eigenvalues of some matrices are sensitive to perturbations, it is well know that the eigenvalues of monic polynomial matrices are continuous functions of the entries of the matrix coefficients of the polynomial, but Small changes in the matrix elements can lead to large changes in the multiplicity of eigenvalues. For example a little perturbation of the matrix $\left(\begin{smallmatrix}\lambda & 1\\ 0 & \lambda\end{smallmatrix}\right)$ as $\left(\begin{smallmatrix}\lambda & 1\\ \varepsilon & \lambda\end{smallmatrix}\right)$ the double eigenvalue $\lambda=0$ is perturbed to two different eigenvalues $\lambda=\pm\sqrt{\varepsilon}$ changing completely the structure of the polynomial matrix. Obviously if we consider the perturbation $\left(\begin{smallmatrix}\lambda & 1+\varepsilon\\ 0 & \lambda\end{smallmatrix}\right)$ there are not changes in the structure.

Given a square complex matrix A, it is an important question from both the theoretical and the practical points of view to know how the eigenvalues and eigenvectors change when the elements of A are subjected to small perturbations. The usual formulation of the problem introduces a perturbation parameter ε belonging to some neighborhood of zero, and writes the perturbed matrix as $A+\varepsilon B$ for an arbitrary matrix B. In this situation, it is well known [15] section II.1.2, that each eigenvalue or eigenvector of $A+\varepsilon B$ admits an expansion in fractional powers of ε, whose zero-th order term is an eigenvalue or eigenvector of the unperturbed matrix A.

In this paper, in section 1 we present an overview over polynomial matrices $P(\lambda)$ and the analysis of perturbation of simple eigenvalue λ_0 of $P(\lambda)$ such that 0 is a simple eigenvalue of the linear map $P(\lambda_0)$. In section 3, inspired by the work of Arnold [2] on versal deformations of square matrices, we derive versal deformations providing a decomposition of arbitrary perturbation into tangent an orthogonal spaces of the set of equivalent polynomial matrices. In section 4, we study the perturbation of a multiple eigenvalue with a simple eigenvector of a monic polynomial matrix smoothly depending on parameters. Finally, in sections 5 and 6 we generalize the results to analysis of perturbation of simple eigenvalues of standard systems and singular systems.

The study of behavior of simple and multiple eigenvalues of a matrix depending smoothly of parameters has a great interest for its many applications. Perturbation theory for eigenvalues and eigenvectors of regular pencils is well established see [1],[17] for example and for vibrational systems in [16]. In this paper we extend some of these results to polynomial matrices.

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II. Preliminaries

A square polynomial matrix of size n and degree k is a polynomial of the form

$$P(\lambda)=\sum_{i=0}^{k}\lambda^i A_i,\quad A_0,\dots,A_k\in M_n(\mathbb{F}),\tag{1}$$

where $\mathbb{F}$ is the field of real or complex numbers. Our focus is on monic polynomial matrices. A square polynomial matrix $P(\lambda)$ is said to be monic if $A_k=I_n$ is identically. The polynomial matrix (1) naturally arises associated with linear systems of differential equations

$$A_k x^{(k)}(t)+A_{k-1}x^{(k-1)}(t)+\ldots+A_1x^1(t)+A_0x(t)=f(t)\tag{2}$$

where $x(t)$ is a vector-valued function (unknown) with n coordinates, $x^{(j)}(t)$ denotes the j-th derivative of $x(t)$ and $f(t)$ is another vector-valued function with n coordinates. Of particular relevance is the case of linear systems of second order, appearing in many engineering applications.

The eigenvalues of a polynomial matrix $P(\lambda)$ are the zeros of the nk-degree scalar polynomial $\det P(\lambda)$.

Let λ_0 be an eigenvalue of polynomial matrix $P(\lambda)$, then there exists a vector $v_0\neq 0$ such that $P(\lambda_0)(v_0)=0$, this vector is called an eigenvector.

We will call a Jordan chain of length $k+1$ for $P(\lambda)$ corresponding to complex number λ_0 to the sequence of n-dimensional vectors $v_0,\dots,v_k$ such that

$$\sum_{\ell=0}^{i}\frac{1}{\ell!}P^{(\ell)}(\lambda_0)v_{i-\ell}=0,\quad i=0,\dots,k\tag{3}$$

where $P^{(\ell)}$ denotes the ℓ-derivative of $P(\lambda)$ with respect the variable λ. If λ_0 is an eigenvalue there exists a Jordan chain of length at least 1 formed by the eigenvector.

Let λ_0 be an eigenvalue of $P(\lambda)$, then $\det P^t(\lambda_0)=\det P(\lambda_0)=0$, so λ_0 is an eigenvalue of $P^t(\lambda)$. For this eigenvalue there exists an eigenvector u_0, that is $P^t(\lambda_0)(u_0)=0$, equivalently $u_0^tP(\lambda_0)=0$. The vector u_0 is called left eigenvector corresponding to the eignevalue λ_0 of $P(\lambda)$.

For more information see [5], or [14] for example.

Let $P(\lambda)=\sum_{i=0}^{k}\lambda^i A_i$ be now, a polynomial matrix and we assume that the matrices A_i smoothly depend on the vector of real parameters $p=(p_1,\dots,p_r)$. The function $P(\lambda;p)=\sum_{i=0}^{k}\lambda^i A_i(p)$ is called a multi-parameter family of polynomial matrices. Eigenvalues of the polynomial matrix function are continuous functions of the vector of parameters. We are going to review the behavior of a simple eigenvalue of the family of polynomial matrices $P(\lambda;p)$.

Let $\lambda(p)$ be a simple eigenvalue of the polynomial matrix $P(\lambda;p)$. Since $\lambda(p)$ is a simple root of the scalar polynomial $\det P(\lambda)$, we have

$$\frac{\partial}{\partial\lambda}\det P(\lambda;p)\neq 0. \tag{4}$$

The expression (4) permit us to make use the implicit function theorem to the equation $\det P(\lambda;p)=0$, and we observe that the eigenvalue $\lambda(p)$ of the family of polynomial matrices smoothly depends on the vector of parameters, and its derivatives with respect to parameters are

$$\frac{\partial\lambda(p)}{\partial p_i}=-\frac{\frac{\partial}{\partial p_i}\det P(\lambda;p)}{\frac{\partial}{\partial\lambda}\det P(\lambda;p)},\quad i=1,\dots,r. \tag{5}$$

Taking into account that $\lambda(p)$ is a simple eigenvalue and that the sum of the lengths of Jordan chains in a canonical set is the multiplicity of the eigenvalue as zero of $\det P(\lambda;p)$, we have that the Jordan chains consist only of the eigenvectors.

The eigenvector $v_0(p)$ corresponding to the simple eigenvalue $\lambda(p)$ is determined up to a nonzero scaling factor α. This eigenvector determines a one-dimensional null-subspace of the matrix operator $P(\lambda(p);p)$ smoothly dependent on p. Hence, the eigenvector $v_0(p)$ can be chosen as a smooth function of the parameters.

An approximation of the eigenvalues as well of the corresponding eigenvectors by means their derivatives is given by the following result.

Theorem 1:

$$\frac{\partial\lambda}{\partial p_i}_{|(\lambda_0;p_0)}=-\frac{u_0^t\frac{\partial P(\lambda;p)}{\partial p_i}_{|(\lambda_0,p_0)}v_0(p_0)}{u_0^tP'(\lambda_0;p_0)v_0(p_0)} \tag{6}$$

and

$$\begin{gathered}\frac{\partial v_0(p)}{\partial p_i}_{|(\lambda_0,p_0)}=\\ -T_0^{-1}\left(\frac{\partial\lambda}{\partial p_i}\left(P'(\lambda;p)\right)+\frac{\partial P(\lambda;p)}{\partial p_i}\right)_{|(\lambda_0,p_0)}v_0(p_0).\end{gathered} \tag{7}$$

where $T_0=P(\lambda_0);p_0)+u_0u_0^tP'(\lambda_0;p_0)$, and

$$\frac{\partial^2\lambda}{\partial p_i\partial p_j}_{|(\lambda_0,p_0)}=-\frac{a}{b},$$

with

$$\begin{aligned}a=&\\ &\left(u_0^t\left(\frac{\partial\lambda}{\partial p_i}\frac{\partial\lambda}{\partial p_j}P'(\lambda;p)+\frac{\partial\lambda}{\partial p_i}\frac{\partial P'(\lambda;p)}{\partial p_j}\right.\right.\\ &\qquad\left.+\frac{\partial P'(\lambda;p)}{\partial p_i}\frac{\partial\lambda}{\partial p_j}+\frac{\partial^2P(\lambda;p)}{\partial p_i\partial p_j}\right)v_0(p)\\ &+u_0^t\left(P'(\lambda;p)\frac{\partial\lambda}{\partial p_j}+\frac{\partial P(\lambda;p)}{\partial p_j}\right)\frac{\partial v_0}{\partial p_i}\\ &\left.+u_0^t\left(P'(\lambda;p)\frac{\partial\lambda}{\partial p_i}+\frac{\partial P(\lambda;p)}{\partial p_i}\right)\frac{\partial v_0}{\partial p_j}\right)_{|(\lambda_0,p_0)},\end{aligned}$$

and

$$b=u_0^tP'(\lambda_0;p_0)v_0(p_0).$$

$$\begin{aligned}&\frac{\partial^2v_0(p)}{\partial p_i\partial p_j}_{|(\lambda_0,p_0)}=\\ &T_0^{-1}\left(\frac{\partial^2\lambda}{\partial p_i\partial p_j}P'(\lambda;p)v_0(p)+\right.\\ &\quad\left(\frac{\partial\lambda}{\partial p_i}\frac{\partial\lambda}{\partial p_j}P'(\lambda;p)+\frac{\partial\lambda}{\partial p_i}\frac{\partial P'(\lambda;p)}{\partial p_j}\right.\\ &\qquad\left.+\frac{\partial P'(\lambda;p)}{\partial p_i}\frac{\partial\lambda}{\partial p_j}+\frac{\partial^2P(\lambda;p)}{\partial p_i\partial p_j}\right)v_0(p)\\ &\quad+\left(P'(\lambda;p)\frac{\partial\lambda}{\partial p_j}+\frac{\partial P(\lambda;p)}{\partial p_j}\right)\frac{\partial v_0}{\partial p_i}\\ &\qquad\left.+\left(P'(\lambda;p)\frac{\partial\lambda}{\partial p_i}+\frac{\partial P(\lambda;p)}{\partial p_i}\right)\frac{\partial v_0}{\partial p_j}\right)_{|(\lambda_0,p_0)}.\end{aligned}$$

The proof is analogous to that given in [16] for matrix pencils and for vibrational systems.

III. VERSAL DEFORMATIONS

The Arnold technique of constructing a local canonical form, called versal deformation, of a differentiable family of square matrices under conjugation [2] provide a special parametrization of matrix spaces, which can be effectively applied to perturbation analysis and investigation of complicated objects like singularities and bifurcations in multi-parameter dynamical systems (see [2], [6] among others).

The general notion of versality is the following. Let $\mathcal{M}$ be a differential manifold with the equivalence relation defined by the action $\alpha(g,x)$ (where $x,\ g\circ x\in\mathcal{M}$ and $g\in\mathcal{G}$) of a Lie group $\mathcal{G}$. Let us consider a smooth mapping $x:\mathcal{U}_0\longrightarrow\mathcal{M}$, where $\mathcal{U}_0$ is a neighborhood of the origin of the space $\mathbb{F}^\ell$. The mapping $\varphi(\gamma)$ is called deformation of $x_0=x(0)$ with the parameter vector $\gamma\in\mathbb{F}^\ell$. Introducing a change of parameters $\phi:\mathcal{U}_0'\longrightarrow\mathcal{U}_0$, where $\mathcal{U}_0'$ is a neighborhood of the origin in $\mathbb{F}^k$, such that $\phi(0)=0$, we obtain the deformation $\varphi(\phi(\xi))$ of x_0 with the parameter vector $\xi\in\mathcal{U}_0'\subset\mathbb{F}^k$. Applying the equivalence transformation $g(\xi)$, where $g:\mathcal{U}_0'\longrightarrow\mathcal{G}$ is a smooth mapping such that $g(0)=e$ is the unit element of $\mathcal{G}$, we get the deformation

$$z(\xi)=\alpha(g(\xi),\varphi(\phi(\xi))) \tag{8}$$

of $z(0)=\alpha(e,x_0)=x_0$. Then, $\varphi(\gamma)$ is called versal deformation of x_0, if any deformation $z(\xi)$ of x_0 can be represented in the form (8) in some neighborhood of the origin $\mathcal{U}_0''\subset\mathbb{F}^k$. This definition implies that a versal deformation generates all deformations of x_0 and, hence, possesses properties (invariant under the equivalence transformation) of all deformations of the given element $x_0\in\mathcal{M}$.

Theorem 2: [2] The deformation $\varphi(\gamma)$ of x_0 is versal if and only if it is transversal to the orbit of x_0 under the action of $\mathcal{G}$.
This theorem reduces the problem of finding a versal deformation to solving a specific linear equation determined by x_0.

Proposition 1: A versal deformation of x_0 is given by

$$x_0+(T_{x_0}\mathcal{O}(x_0))^\perp$$

In fact we can take any complementary subspace F of $T_{x_0}\mathcal{O}(x_0)$.

In our particular setup we consider the Lie group $\mathcal{G}=Gl(n;\mathbb{R})$ acting over the space of monic polynomial matrices $\mathcal{P}_k(\lambda)=\{\sum_{i=0}^k\lambda^iA_i\mid A_k=I_n,A_i\in M_n(\mathbb{F}\}\approx\{(A_{k-1},\dots,A_1,A_0)\}=(M_n(\mathbb{R})^k$ in the following manner

$$\begin{array}{ll}\alpha:\mathcal{G}\times\mathcal{P}_k(\lambda)&\longrightarrow\mathcal{P}_k(\lambda)\\ (S,(A_{k-1},\dots,A_0))&\longrightarrow(S^{-1}A_{k-1}S,\dots,S^{-1}A_0S)\end{array} \tag{9}$$

Let us use the notation $T_I\mathcal{G}$ for the tangent space to the manifold $\mathcal{G}$ at the unit element I. Since $\mathcal{G}$ is an open subset of $M_n(\mathbb{F})$, we have $T_I\mathcal{G}=M_n(\mathbb{F})$,
and, since $\mathcal{P}_k(\lambda)$ (identified with $M_n(\mathbb{F})^n$) is a linear space, at the point $x_0=(A_{k-1_0},\dots,A_{1_0},A_{0_0})$, we have $T_{x_0}\mathcal{P}_k(\lambda)=\mathcal{P}_k(\lambda)$.

Proposition 2: Let $d\alpha_{x_0} : T_I\mathcal{G} \longrightarrow \mathcal{P}_k(\lambda)$ be the differential of α_{x_0} at the unit element I. Then

$$\begin{array}{l} d\alpha_{x_0}(S) = (A_{k-1}S - SA_{k-1}, \dots, A_0S - SA_0) \in \mathcal{P}_k(\lambda), \\ S \in T_I\mathcal{G}. \end{array} \tag{10}$$

Remark 1: It is well-known that the map $d\alpha_{x_0}$ provide a simple description of the tangent space $T_{x_0}\mathcal{O}(x_0)$.

Proposition 3:

$$T_{x_0}\mathcal{O}(x_0) = \operatorname{Im} d\alpha_{x_0} \subset \mathcal{P}_k(\lambda).$$

Hermitian product in $\mathcal{P}_k(\lambda)$ and $T_I\mathcal{G}$ we will deal with in this paper are the following ones:

$$\begin{array}{l} \langle x_1, x_2 \rangle = \operatorname{tr}(A_{k-1_1}A^*_{k-1_2} + \dots \operatorname{tr}(A_{0_1}A^*_{0_2}, \\ x_i = (A_{k-1_i} \dots, A_{0_i}) \in \mathcal{P}_k(\lambda), \end{array} \tag{11}$$

where A^* denotes the conjugate transpose of the matrix A.

Using the hermitian product (11) is easily to deduce a description of $T_{x_0}\mathcal{O}(x_0)^\perp$ for $x_0 \in \mathcal{P}_k(\lambda)$.

Proposition 4: Let $x_0 = (A_{k-1_0}, \dots, A_{0_0})$ be a polynomial matrix in $\mathcal{P}_k(\lambda)$. Then $(X_{k-1}, \dots, X_0) \in T_{x_0}\mathcal{O}(x_0)^\perp$ if and only if

$$X^*_{k-1}A_{k-1_0} - A_{k-1_0}X^*_{k-1_0} + \dots + X^*_0A_0 - A_0X^*_0 = 0$$

Let x_0 a polynomial matrix, the values of eigenvalues of all polynomial matrices "near" of x_0 are eigenvalues for some polynomial matrix in the versal deformation of x_0.

IV. Perturbation of eigenvalue of arbitrary multiplicity with single eigenvector

Let $P(\lambda;p) = \lambda^2 I_2 + A(p)$ with $A(p) = \begin{pmatrix} -1 & p \\ p & 0 \end{pmatrix}$ be a one parameter family of polynomial matrices. The eigenvalues are

$$\lambda_i = \pm\sqrt{\frac{1 \pm \sqrt{1+4p^2}}{2}}, \tag{12}$$

that they are branches of one quadruple-valued analytic function $\lambda(p) = \sqrt{\dfrac{1+\sqrt{1+4p^2}}{2}}$

the exceptional points are:

- $p = \frac{1}{2}i$ and the eigenvalues are $\pm\frac{\sqrt{2}}{2}$ both being double.
- $p = -\frac{1}{2}i$ and the eigenvalues are $\pm\frac{\sqrt{2}}{2}$ both being double.
-$p = 0$ and the eigenvalues are $+1$, -1 both being simple and 0 being double.

We observe that for $p = \frac{1}{2}i$, the polynomial matrix $P(\lambda;p)$ has a single eigenvector up to a non-zero scaling factor for the double eigenvalue $\lambda = \frac{\sqrt{2}}{2}$.

We next consider the behavior of the eigenvalues in the neighborhood of one of the exceptional points. Concretely we take $p = \frac{1}{2}i$. In this case the eigenvalues are not differentiable functions of the parameter at $p = \frac{1}{2}i$, just where the double eigenvalue appears, derivative of the eigenvalues tend to infinity as p approaches to $\frac{1}{2}i$. Therefore the analysis of perturbations of multiple eigenvalues with single eigenvector, must be treated in a different manner.

Let $P(\lambda;p)$ be a monic polynomial matrix family and λ_0 an eigenvalue of arbitrary multiplicity ℓ with single eigenvector up to a non-zero scaling factor at the point $p = p_0$, then, there exists a Jordan chain $v_0, \dots, v_{\ell-1}$ such that

$$\begin{array}{c} P(\lambda_0, p_0)v_0 = 0, \\ P'(\lambda_0, p_0)v_0 + P(\lambda_0, p_0)v_1 = 0, \\ \dfrac{1}{(\ell-1)!}P^{\ell-1}(\lambda_0, p_0)v_0 + \dots + P(\lambda_0, p_0)v_{\ell-1} = 0, \end{array} \tag{13}$$

and, there exists a left Jordan chain $u_0, \dots, u_{\ell-1}$ such that

$$\begin{array}{c} u_0^tP(\lambda_0, p_0) = 0, \\ u_0^tP'(\lambda_0, p_0) + u_1^tP(\lambda_0, p_0) = 0, \\ \dfrac{1}{(\ell-1)!}u_0^tP(\lambda_0, p_0) + \dots + u_{\ell-1}^tP(\lambda_0, p_0) = 0. \end{array} \tag{14}$$

Remark 2: a) $u_0^tP'(\lambda_0, p_0)v_0 = 0$,
b) $u_1^tP'(\lambda_0, p_0)v_0 = 0 \Leftrightarrow u_1^tP(\lambda_0, p_0)v_1 = 0 \Leftrightarrow u_0^tP'(\lambda_0, p_0)v_1 = 0$,
c) $u_0^tP'(\lambda_0; p_0)v_1 = u_1^tP'(\lambda_0; p_0)v_0$.

In order to analyze the behavior of two eigenvalues $\lambda(p)$ that merge to λ_0 at p_0, we consider a perturbation of the parameter along a smooth curve $p = p(\varepsilon)$, where $\varepsilon \geq 0$ is a small real perturbation parameter and $p(0) = p_0$.

Along the curve $p(\varepsilon) = (p_1(\varepsilon), \dots, p_r(\varepsilon))$ we have a one parameter matrix family $P(\lambda, p(\varepsilon))$, which can be represented in the form of Taylor expansion

$$P(\lambda, p(\varepsilon)) = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \dots,$$

with $P_0 = P(\lambda, p_0)$, $P_1 = \sum_{i=1}^r \dfrac{\partial P(\lambda, p(\varepsilon))}{\partial p_i}\dfrac{dp_i}{d\varepsilon}$,

$$P_2 = \frac{1}{2}\left(\sum_{i=1}^r \frac{\partial P(\lambda, p(\varepsilon))}{\partial p_i}\frac{d^2p_i}{d\varepsilon^2} + \sum_{i,j=1}^r \frac{\partial^2 P(\lambda, p(\varepsilon))}{\partial p_i \partial p_j}\frac{dp_i}{d\varepsilon}\frac{dp_j}{d\varepsilon}\right),$$

where the derivatives are evaluated at p_0.

Taking into account that $P(\lambda, p(\varepsilon)) = \sum_{i=0}^k \lambda i A_i(p(\varepsilon)$ ($A_k(p(\varepsilon) = I_n$), we have that

$$P(\lambda, p(\varepsilon)) = \sum_{i=0}^k \lambda^i(A_{i_0} + \varepsilon A_{i_1} + \varepsilon^2 A_{i_2} + \dots) \tag{15}$$

where $A_{k_0} + \varepsilon A_{k_1} + \varepsilon^2 A_{k_2} + \dots = I_n$, $A_{\ell_0} = A_\ell(p_0)$, $A_{\ell_1} = \sum_{i=1}^r \dfrac{\partial A_\ell(p(\varepsilon))}{\partial p_i}\dfrac{dp_i}{d\varepsilon}$, $A_{\ell_2} = \dfrac{1}{2}\left(\sum_{i=1}^r \dfrac{\partial A_\ell(p(\varepsilon))}{\partial p_i}\dfrac{d^2p_i}{d\varepsilon^2} + \sum_{i,j=1}^r \dfrac{\partial^2 A_\ell(p(\varepsilon))}{\partial p_i \partial p_j}\dfrac{dp_i}{d\varepsilon}\dfrac{dp_j}{d\varepsilon}\right)$. and the derivatives are evaluated at p_0.

If λ_0 is a ℓ-multiplicity eigenvalue of $P(\lambda; p_0)$ having a unique eigenvector v_0 up to a non-zero scaling factor the perturbation theory (see [15], for example) tell us that the ℓ-fold eigenvalue λ_0 generally splits into ℓ of simple eigenvalues λ under perturbation of the polynomial matrix $P(\lambda; p_0)$. These eigenvalues λ and the corresponding eigenvectors v can be represented in the form of the Puiseux series:

$$\begin{array}{l} \lambda = \lambda_0 + \varepsilon^{1/\ell}\lambda_1 + \varepsilon^{2/\ell}\lambda_2 + \varepsilon^{3/\ell}\lambda_3 + \varepsilon^{4/\ell}\lambda_4 + \dots \\ v = v_0 + \varepsilon^{1/\ell}w_1 + \varepsilon^{2/\ell}w_2 + \varepsilon^{3/\ell}w_3 + \varepsilon^{4/\ell}w_4 + \dots \end{array} \tag{16}$$

Lemma 1: Let p_0 be a point such that $\lambda(p_0) = \lambda_0$ is a ℓ-multiplicity eigenvalue with single eigenvector $v_0(p_0)$ and u_0 a corresponding left eigenvector. Then, $[u_0]^\perp = \operatorname{Im} P(\lambda_0, p_0)$.

Proof: Let $z \in \operatorname{Im} P(\lambda_0, p_0)$, then there exists a vector x such that $P(\lambda_0, p_0)x = z$. So

$$u_0^tz = u_0^tP(\lambda_0, p_0)x = 0^tx = 0,$$

consequently $\operatorname{Im} P(\lambda_0; p_0) \subset [u_0]^\perp$. And taking into account that

$$\operatorname{rank} P(\lambda_0, p_0) = \dim \operatorname{Im} P(\lambda_0, p_0) = n - 1 = \dim[u_0]^\perp,$$

we conclude the result. ■

Corollary 1: With the same conditions as the previous lemma, we have. $\dfrac{1}{\ell!}u_0^tP^\ell(\lambda_0; p_0)v_0 + \dfrac{1}{(\ell-1)!}u_0^tP^{\ell-1}(\lambda_0; p_0)v_1 + \dots + u_0^tP'(\lambda_0; p_0)v_{\ell-1} \neq 0$.

Proof: Suppose $\dfrac{1}{\ell!}u_0^tP^\ell(\lambda_0; p_0)v_0 + \dots + u_0^tP'(\lambda_0; p_0)v_{\ell-1} \neq 0$. Then $\dfrac{1}{\ell!}P^\ell(\lambda_0; p_0)v_0 + \dfrac{1}{(\ell-1)!}P^{\ell-1}(\lambda_0; p_0)v_1 +$

$\ldots + P'(\lambda_0;p_0)v_{\ell-1} \in \operatorname{Im} P(\lambda_0,p_0)$, and $\frac{1}{\ell!}P^{\ell}(\lambda_0;p_0)v_0 + \frac{1}{(\ell-1)!}P^{\ell-1}(\lambda_0;p_0)v_1 + \ldots + P'(\lambda_0;p_0)v_{\ell-1} = P(\lambda_0;p_0)x$. Equivalently:

$$\begin{aligned}&\frac{1}{\ell!}P^{\ell}(\lambda_0;p_0)v_0 + \frac{1}{(\ell-1)!}P^{\ell-1}(\lambda_0;p_0)v_1 + \ldots\\ &+P'(\lambda_0;p_0)v_{\ell-1} + P(\lambda_0;p_0)(-x) = 0,\end{aligned} \tag{17}$$

but the Jordan chains of the $P(\lambda;p_0)$ for $\lambda = \lambda_0$ are length ℓ, so there is no vector x verifying (17).

■

A. Perturbation of double eigenvalue with single eigenvector

Firstly and for a more understanding, we analyze the case where $\ell = 2$

Substituting (16) into (15) we obtain

$$\begin{aligned}&P(\lambda;p(\varepsilon)) =\\ &(\lambda_0^k I_n + \lambda_0^{k-1}A_{k-1_0} + \ldots + \lambda_0 A_{1_0} + A_{0_0}) +\\ &\varepsilon^{1/2}(k\lambda_0^{k-1}\lambda_1 I_n + (k-1)\lambda_0^{k-2}\lambda_1 A_{k-1_0} + \ldots + \lambda_1 A_{1_0}) +\\ &\varepsilon((k\lambda_0^{k-1}\lambda_2 + \frac{1}{2}k(k-1)\lambda_0\lambda_1^2)I_n + ((k-1)\lambda_0^{k-2}\lambda_2 +\\ &\quad \frac{1}{2}(k-1)(k-2)\lambda_0\lambda_1^2)A_{k-1_0} + \lambda_0^{k-1}A_{k-1_1} +\\ &\quad\quad \lambda_2 A_{1_0} + \lambda_0 A_{1_1} + \ldots + A_{0_1}) + \ldots\end{aligned}$$

If v is an eigenvector for the eigenvalue λ, we have that

$$P(\lambda;p(\varepsilon))v = P(\lambda;p(\varepsilon))(v_0 + \varepsilon^{1/2}w_1 + \varepsilon w_2 + \ldots) = 0.$$

Then, we find the chain of equations for the unknowns $\lambda_1, \lambda_2, \ldots$ and $w_1, w_2, \ldots$.

$$P(\lambda_0,p_0)v_0 = 0, \tag{18}$$

$$\lambda_1 P'(\lambda_0;p_0)v_0 + P(\lambda_0;p_0)w_1 = 0, \tag{19}$$

$$\begin{aligned}&P(\lambda_0;p_0)w_2 + \lambda_1 P'(\lambda_0;p_0)w_1 + \frac{1}{2}\lambda_1^2 P''(\lambda_0;p_0)v_0 +\\ &\quad \lambda_2 P'(\lambda_0;p_0)v_0 + P_1(\lambda_0;p_0)v_0 = 0,\end{aligned} \tag{20}$$

$$\begin{aligned}&P(\lambda_0;p_0)w_3 + \lambda_1 P'(\lambda_0;p_0)w_2 + \frac{1}{2}\lambda_1^2 P''(\lambda_0;p_0)w_1 +\\ &\lambda_2 P'(\lambda_0;p_0)w_1 + P_1(\lambda_0;p_0)w_1 + \lambda_1\lambda_2 P''(\lambda_0;p_0)v_0 +\\ &\lambda_1^3\frac{1}{3!}P'''(\lambda_0;p_0)v_0 + \lambda_3 P'(\lambda_0;p_0)v_0 + \lambda_1 P_1'(\lambda_0;p_0)v_0 = 0,\end{aligned} \tag{21}$$

where $P_1(\lambda_0;p_0) = \lambda_0^{k-1}A_{k-1_1} + \lambda_0 A_{k-2_1} + \ldots + \lambda_0 A_{11} + A_{01}$.

Equation (18) is satisfied because v_0 is an eigenvector corresponding to the eigenvalue λ_0. Comparing equation (32) with (3) for $i = 1$ we observe that $w_1 = \lambda_1 v_1 + \beta v_0$ for all β is a solution, we take $w_1 = \lambda_1 v_1$.

To find the value of λ_1 we premultiply equation (20) by u_0^t, using the given value for w_1 and taking into account $u_0^t P(\lambda_0;p_0) = 0$ and $u_0^t P'(\lambda_0;p_0)v_0 = 0$ we obtain

$$\lambda_1^2(u_0^t P'(\lambda_0;p_0)v_1 + \frac{1}{2}u_0^t P''(\lambda_0;p_0)v_0) + u_0^t P_1(\lambda_0;p_0)v_0 = 0.$$

Taking into account corollary 1 we can find

$$\lambda_1 = \pm\sqrt{\frac{-u_0^t P_1(\lambda_0;p_0)v_0}{u_0^t P'(\lambda_0;p_0)v_1 + \frac{1}{2}u_0^t P''(\lambda_0;p_0)v_0}}. \tag{22}$$

If $u_0^t P_1(\lambda_0;p_0)v_0 \neq 0$ we have two values of λ_1 that determine leading terms in expansions for two different eigenvalues λ that bifurcate from the double eigenvalue λ_0.

Suppose then, that $u_0^t P_1(\lambda_0;p_0)v_0 \neq 0$. Premultiplying (21) by u_0^t,

$$\begin{aligned}&\lambda_1 u_0^t P'(\lambda_0,p_0)w_2 + \frac{1}{2}\lambda_1^3 u_0^t P''(\lambda_0;p_0)v_1 +\\ &\lambda_1\lambda_2 u_0^t P'(\lambda_0;p_0)v_1 + \lambda_1 u_0^t P_1(\lambda_0;p_0)v_1 +\\ &\lambda_1\lambda_2 u_0^t P''(\lambda_0;p_0)v_0 + \lambda_1^3\frac{1}{3!}u_0^t P'''(\lambda_0;p_0)v_0 +\\ &\lambda_1 u_0^t P_1'(\lambda_0;p_0)v_0 = 0.\end{aligned}$$

Premultiplying (20) by u_1^t and according to 2, we have:

$$\begin{aligned}&u_0^t P'(\lambda_0;p_0)w_2 =\\ &\lambda_1 u_1^t P'(\lambda_0;p_0)w_1 + \frac{1}{2}\lambda_1^2 u_1^t P''(\lambda_0;p_0)v_0 +\\ &\lambda_2 u_1^t P'(\lambda_0;p_0)v_0 + u_1^t P_1(\lambda_0;p_0)v_0.\end{aligned}$$

So, taking into account (22)

$$\begin{aligned}&\lambda_1\lambda_2(2u_0^t P'(\lambda_0;p_0)v_1 + u_0^t P''(\lambda;p_0)v_0) =\\ &-(\lambda_1^3(u_1^t P'(\lambda_0;p_0)v_1 + \frac{1}{2}u_1^t P''(\lambda_0;p_0)v_0 + \frac{1}{2}u_0^t P''(\lambda_0;p_0)v_1\\ &+\frac{1}{3!}u_0^t P'''(\lambda_0;p_0)v_0) + \lambda_1(u_1^t P_1(\lambda_0;p_0)v_0 +\\ &u_0^t P_1(\lambda_0;p_0)v_1 + u_0^t P_1'(\lambda_0;p_0)v_0))\end{aligned}$$

Since $\lambda_1(u_0^t P'(\lambda_0;p_0)v_1 + \frac{1}{2}u_0^t P''(\lambda_0;p_0)v_0) \neq 0$ we obtain

$$\begin{aligned}\lambda_2 = &-\frac{\lambda_1^2(\frac{1}{2}u_0^t P''(\lambda_0;p_0)v_1 + \frac{1}{3!}u_0^t P'''(\lambda_0;p_0)v_0}{2(u_0^t P'(\lambda_0;p_0)v_1 + \frac{1}{2}u_0^t P''(\lambda_0;p_0)v_0)}\\ &-\frac{\lambda_1^2(u_1^t P'(\lambda_0;p_0)v_1 + \frac{1}{2}u_1^t P''(\lambda_0;p_0)v_0)}{2(u_0^t P'(\lambda_0;p_0)v_1 + \frac{1}{2}u_0^t P''(\lambda_0;p_0)v_0)}\\ &+\frac{u_0^t P_1(\lambda_0;p_0)v_1 + u_0^t P_1'(\lambda_0;p_0)v_0 + u_1^t P_1(\lambda_0;p_0)v_0}{2(u_0^t P'(\lambda_0;p_0)v_1 + \frac{1}{2}u_0^t P''(\lambda_0;p_0)v_0)}.\end{aligned} \tag{23}$$

Now, we can compute w_2. We have

$$\begin{aligned}&P(\lambda_0;p_0)w_2 = -\lambda_1 P'(\lambda_0;p_0)w_1 - \frac{1}{2}\lambda_1^2 P''(\lambda_0;p_0)v_0 -\\ &\lambda_2 P'(\lambda_0;p_0)v_0 + P_1(\lambda_0;p_0)v_0\end{aligned} \tag{24}$$

Lemma 2: Following condition $u_0^t P_1(\lambda_0;p_0)v_0 \neq 0$ we have that $P(\lambda_0;p_0) + u_0 u_0^t P_1(\lambda_0;p_0)v_0 v_0^t$ is an invertible matrix.

Proof: Let $x = \alpha v_0 + w$ with $w \in [v_0]^{\perp}$, be a vector in the null space, then $(P(\lambda_0;p_0) + u_0 u_0^t P_1(\lambda_0;p_0)v_0 v_0^t)x = 0$.

Premultiplying by u_0^t we have

$$u_0^t(P(\lambda_0;p_0) + u_0 u_0^t P_1(\lambda_0;p_0)v_0 v_0^t)x = 0,$$

$$\begin{aligned}&0 = u_0 u_0^t P_1(\lambda_0;p_0)v_0 v_0^t(\alpha v_0 + w) =\\ &|\alpha|\|u_0\|^2\|v_0\|^2 u_0^t P_1(\lambda_0;p_0)v_0.\end{aligned}$$

Then $\alpha = 0$.

Consequently, $x = w \in [v_0]^{\perp}$ and $x \in \operatorname{Ker} u_0 u_0^t P_1(\lambda_0;p_0)v_0 v_0^t$, so $x \in \operatorname{Ker} P(\lambda_0;p_0)$ and $x = \beta v_0$, but $x \in [v_0]^{\perp}$, then $\beta = 0$. ■

Now we consider the normalization condition $v_0^t w_2 = 0$, and adding $u_0 u_0^t P_1(\lambda_0;p_0)v_0 v_0^t$ from the left to equation (24) and using lemma 2, we find vector w_2.

Using these calculations we have the following theorem.

Theorem 3: Let λ_0 be a double eigenvalue of the polynomial matrix $P(\lambda;p_0)$, with a single eigenvector up to a non-zero scaling factor, and let v_0, v_1 be a Jordan chain and u_0, u_1 a left Jordan chain. We consider a perturbation of the parameter vector along the curve $p(\varepsilon)$ starting at p_0 satisfying the condition $\lambda_1 \neq 0$.

Then, the double eigenvalue λ_0 bifurcates into two simple eigenvalues given by the relation

$$\lambda = \lambda_0 + \varepsilon^{1/2}\lambda_1 + \varepsilon\lambda_2 + o(\varepsilon),$$

with λ_1 and λ_2 as (22) and (23) respectively.

B. Perturbation of a ℓ-multiplicity eigenvalue with single eigenvector

Now, we analyze the general case.
Analogously, substituting (16) into (15) we obtain

$$\begin{aligned}
&P(\lambda;p(\varepsilon)) = (\lambda_0+\varepsilon^{1/\ell}\lambda_1+\varepsilon^{2/\ell}\lambda_2+\ldots+\varepsilon\lambda_\ell+\ldots)^k I_n+\\
&(\lambda_0+\varepsilon^{1/\ell}\lambda_1+\ldots)^{k-1}(A_{k-1_0}+\ldots+\varepsilon^\ell A_{k-1_\ell}+\ldots)+\\
&\ldots+\\
&(\lambda_0+\varepsilon^{1/\ell}\lambda_1+\varepsilon^{2/\ell}\lambda_2+\ldots)(A_{1_0}+\varepsilon A_{1_1}+\varepsilon^2 A_{1_2}+\ldots)+\\
&A_{0_0}+\varepsilon A_{0_1}+\varepsilon^2 A_{0_2}+\ldots=\\
&(\lambda_0^k I_n+\lambda_0^{k-1}A_{k-1_0}+\ldots+\lambda_0 A_{1_0}+A_{0_0})+\\
&\varepsilon^{1/\ell}(k\lambda_0^{k-1}\lambda_1 I_n+(k-1)\lambda_0^{k-2}\lambda_1 A_{k-1_0}+\ldots+\lambda_1 A_{1_0})+\\
&\varepsilon^{2/\ell}((k\lambda_0^{k-1}\lambda_2+\frac{1}{2}k(k-1)\lambda_0\lambda_1^2)I_n+((k-1)\lambda_0^{k-2}\lambda_2+\\
&\quad\frac{1}{2}(k-1)(k-2)\lambda_0\lambda_1^2)A_{k-1_0}+\ldots+\lambda_2 A_{1_0})+\ldots
\end{aligned}$$

If v is an eigenvector for the eigenvalue λ we have that

$$P(\lambda;p(\varepsilon))v = P(\lambda;p(\varepsilon))(v_0+\varepsilon^{1/\ell}w_1+\varepsilon^{2/\ell}w_2+\ldots)=0$$

Then, we find the chain of equations for the unknowns $\lambda_1,\lambda_2,\ldots$ and $w_1,w_2,\ldots$.

$$P(\lambda_0,p_0)v_0=0, \tag{25}$$

$$\lambda_1 P'(\lambda_0;p_0)v_0+P(\lambda_0;p_0)w_1=0, \tag{26}$$

$$\begin{aligned}&P(\lambda_0;p_0)w_2+\lambda_1P'(\lambda_0;p_0)w_1+\\ &\frac{1}{2}\lambda_1^2P''(\lambda_0;p_0)v_0+\lambda_2P'(\lambda_0;p_0)v_0=0,\end{aligned} \tag{27}$$

$$\begin{aligned}&\lambda_3P'(\lambda_0,p_0)v_0+\frac{1}{3!}\lambda_1^3P'''(\lambda_0;p_0)v_0+\\ &\frac{1}{2}\lambda_1\lambda_2P''(\lambda_0;p_0)v_0+\lambda_2P'(\lambda_0;p_0)w_1+\\ &\frac{1}{2}\lambda_1^2P''(\lambda_0;p_0)w_1+\lambda_1P'(\lambda_0;p_0)w_2+P(\lambda_0;p_0)w_3=0,\end{aligned} \tag{28}$$

$\ldots$

$$\begin{aligned}&P(\lambda_0;p_0)w_\ell+\lambda_1P'(\lambda_0;p_0)w_{\ell-1}+\\ &\frac{1}{2}\lambda_1^2P''(\lambda_0;p_0)w_{\ell-2}+\lambda_2P'(\lambda_0;p_0)w_1+\ldots+\\ &\lambda_{\ell-1}P'(\lambda_0;p_0)w_1+P_1(\lambda_0;p_0)v_0=0,\end{aligned} \tag{29}$$

where $P_1(\lambda_0;p_0)=\lambda_0^{k-1}A_{k-1_1}+\lambda_0A_{k-2_1}+\ldots+\lambda_0A_{11}+A_{01}$.

Equation (25) is satisfied because v_0 is an eigenvector corresponding to the eigenvalue λ_0. Comparing equation (26) with (3) for $i=1$ we observe that $w_1=\lambda_1v_1+\beta v_0$ is a solution, comparing equation (27) with (3) for $i=2$ $w_2=\lambda_1^2v_2+\lambda_2v_1$ is a solution, following in this sense $w_3=\lambda_1^3v_3+\lambda_1\lambda_2v_2+\lambda_3v_1$ etc.

Theorem 4: Let λ_0 be a ℓ-multiplicity eigenvalue of the polynomial matrix $P(\lambda;p_0)$, with a single eigenvector up to a non-zero scaling factor, and let $v_0,\ldots,v_{\ell-1}$ be a Jordan chain and $u_0,\ldots,u_{\ell-1}$ a left Jordan chain. We consider a perturbation of the parameter vector along the curve $p(\varepsilon)$ starting at p_0. Suppose $u_0^tP_1(\lambda_0;p_0)v_0\neq 0$, then, the eigenvalue λ_0 bifurcates into ℓ simple eigenvalues given by the relation

$$\lambda=\lambda_0+\varepsilon^{1/\ell}\lambda_1+o(\varepsilon),$$

with

$$\lambda_1=\sqrt[\ell]{\frac{-u_0^tP_1(\lambda_0;p_0)v_0}{\frac{1}{\ell!}u_0^tP^\ell(\lambda_0;p_0)v_0+\ldots+u_0^tP'(\lambda_0;p_0)v_{\ell-1}}}.$$

Remark 3: Condition $u_0^tP_1(\lambda_0;p_0)v_0\neq 0$ holds for almost all perturbations.

Proof: To find the value of λ_1 using $w_1=\lambda_1v_1+\beta v_0$ in equation (20) and premultiply it by u_0^t and taking into account remark 2 and normalization condition $u_0^tP'(\lambda_0;p_0)v_i=0$, we obtain

$$\begin{aligned}&\lambda_1^\ell(\frac{1}{\ell!}u_0^tP^\ell(\lambda_0;p_0)v_0+\frac{1}{(\ell-1)!}u_0^tP^{\ell-1}(\lambda_0;p_0)v_1+\\ &\ldots+u_0^tP'(\lambda_0;p_0)v_{\ell-1})+u_0^tP_1(\lambda_0;p_0)v_0=0.\end{aligned}$$

Now, corollary 1 ensures the result. ■

V. Perturbation analysis of simple eigenvalues of standard systems

We consider systems in the form $\dot{x}=Ax+Bu$ with $A\in M_n(\mathbb{C})$ and $B\in M_{n\times m}(\mathbb{C})$, we will write the systems as a pair of matrices (A,B).

Remember that $\lambda_0\in\mathbb{C}$ is an eigenvalue of the system if and only if there exists a non-zero vector v_0 such that

$$A^tv_0=\lambda_0v_0,\quad B^tv_0=0,$$

and v_0 is called eigenvector of the system for this eigenvalue.

The eigenvalues of the system (A,B) correspond to the eigenvalues of the associate 1-degree singular polynomial matrix $\lambda\begin{pmatrix}I & 0\end{pmatrix}+\begin{pmatrix}A & B\end{pmatrix}$ and the eigenvectors correspond to the left eigenvectors of the pencil.

Remark 4: The vector v_0 is an eigenvector of A^t corresponding to eigenvalue λ_0, So, λ_0 is an eigenvalue of the matrix A, and the corresponding eigenvector u_0 is a left eigenvector of the matrix A^t.

Definition 1: An eigenvalue λ_0 of a system (A,B) is called simple if it is simple as eigenvalue of A.

Observe that an eigenvalue of A is not necessarily an eigenvector of (A,B).

Proposition 5 ([13]): Let λ_0 be a simple eigenvalue of (A,B). Then we can choose u_0 such that $u_0^tv_0\neq 0$.

Sometimes an eigenvalue of (A,B) is not simple but there may be a feedback such that the resulting closed-loop system has a simple eigenvalue.

Let λ_0 be a multiple eigenvalue of A^t which is a simple eigenvalue of $A^t+K^tB^t$ for some feedback K.

Proposition 6: Let λ_0 be an eigenvalue and v_0 a corresponding eigenvector of (A,B). Then λ_0 is an eigenvalue and v_0 the corresponding eigenvector of $(A+BK,B)$ for all K.

Proof: If $A^tv_0=\lambda_0v_0$ and $B^tv_0=0$ then $K^tB^tv_0=0$ and $(A^t+K^tB^t)v_0=\lambda_0v_0$.

Reciprocally, if $(A^t+K^tB^t)v_0=\lambda_0v_0$ and $B^tv_0=0$, then $A^tv_0=A^tv_0+K^tB^tv_0-K^tB^tv_0=(A^t+K^tB^t)v_0-K^tB^tv_0=\lambda_0v_0$. ■

Corollary 2: Let K such that μ_0 is an eigenvalue of $A^t+K^tB^t$ and w_0 a corresponding eigenvector. If μ_0 is not an eigenvalue of (A,B), then $B^tw_0\neq 0$.

Let (A,B) be a linear system and we assume that the matrices A, B smoothly depend on the vector of a real parameters $p=(p_1,\ldots,p_r)$. The function $(A(p),B(p))$ is called a multi-parameter family of linear systems. Eigenvalues of linear system function are continuous functions $\lambda(p)$ of the vector of parameters. In this section we are going to study the behavior of a simple eigenvalue of the family of linear systems $(A(p),B(p))$.

Let us consider a point p_0 in the parameter space and assume that $\lambda(p_0)=\lambda_0$ is a simple eigenvalue of $(A(p_0),B(p_0))=(A_0,B_0)$, and $v(p_0)=v_0$ an eigenvector, i.e.

$$A_0^tv_0=\lambda_0v_0,\quad B_0^tv_0=0.$$

Equivalently

$$(A_0^t+K^tB^t)v_0=\lambda_0v_0,\quad B_0^tv_0=0,\quad\forall K.$$

Now, we are going to review the behavior of a simple eigenvalue $\lambda(p)$ of the family of standard linear systems.

The eigenvector $v(p)$ corresponding to the simple eigenvalue $\lambda(p)$ determines a one-dimensional null-subspace of the matrix operator $\begin{pmatrix}A^t\\B^t\end{pmatrix}$ smoothly dependent on p. Hence, the eigenvector $v(p)$ can be chosen as a smooth function of the parameters. We will try to obtain an approximation by means their derivatives.

We write the eigenvalue problem as

$$\left.\begin{aligned}(A^t(p)+K^t(p)B^t(p))v(p)=\lambda(p)v(p)\\ B^t(p)v(p)=0.\end{aligned}\right\} \tag{30}$$

equivalently

$$\left.\begin{array}{r} A^t(p)v(p) = \lambda(p)v(p) \\ B^t(p)v(p) = 0. \end{array}\right\} \tag{31}$$

Taking the derivatives with respect to p_i

$$\left.\begin{array}{r} \dfrac{\partial A^t(p)}{\partial p_i}v(p) + A^t(p)\dfrac{\partial v(p)}{\partial p_i} = \dfrac{\partial \lambda}{\partial p_i}v(p) + \lambda(p)\dfrac{\partial v(p)}{\partial p_i} \\ \dfrac{\partial B^t(p)}{\partial p_i}v(p) + B^t(p)\dfrac{\partial v(p)}{\partial p_i} = 0 \end{array}\right\}$$

At the point p_0 we have.

$$\left.\begin{array}{r} \left(\dfrac{\partial A^t(p)}{\partial p_i} - \dfrac{\partial \lambda}{\partial p_i}I_n\right)_{|p_0} v_0 = (\lambda_0 I_n - A^t(p_0))\dfrac{\partial v(p)}{\partial p_i}_{|p_0} \\ \dfrac{\partial B^t(p)}{\partial p_i}_{|p_0} v_0 + B^t(p_0)\dfrac{\partial v(p)}{\partial p_i}_{|p_0} = 0 \end{array}\right\} \tag{32}$$

This is a linear algebraic system of equations for the unknowns $\frac{\partial \lambda}{\partial p_i}$ and $\frac{\partial v(p)}{\partial p_i}$ where the matrix $\lambda_0 I_n - A^t(p_0)$ is singular with rank equal $n-1$ because of λ_0 is a simple eigenvalue.

Lemma 3 ([13]): The matrix $(\lambda_0 I_n - A^t(p_0) - u_0 u_0^t$ is invertible.

Proposition 7: With the same conditions. The system (32) has a solution if and only if

$$\left.\begin{array}{r} u_0^t\left(\dfrac{\partial \lambda}{\partial p_i}_{|p_0} I_n - \dfrac{\partial A^t(p)}{\partial p_i}_{|p_0}\right)v_0 = 0 \\ \dfrac{\partial B^t(p)}{\partial p_i}_{|p_0} v_0 + B^t(p_0)\dfrac{\partial v(p)}{\partial p_i}_{|p_0} = 0 \end{array}\right\} \tag{33}$$

where u_0 is a left eigenvector for the simple eigenvalue λ_0 of the matrix A^t.

Proof: From first equation of (33) we obtain a solution for $\frac{\partial \lambda}{\partial p_i}_{|(\lambda_0;p_0)}$:

$$\frac{\partial \lambda}{\partial p_i}(u_0^t v_0) = u_0^t \frac{\partial A^t(p)}{\partial p_i}v_0,$$

$$\frac{\partial \lambda}{\partial p_i} = \frac{u_0^t \dfrac{\partial A^t(p)}{\partial p}v_0}{u_0^t v_0}. \tag{34}$$

We can choice u_0 in such away that $u_0^t v_0 = 1$

Replacing this solution in first equation of (32) we obtain

$$\frac{\partial v(p)}{\partial p_i} = (\lambda_0 I_n - A^t(p_0) - u_0 u_0^t)^{-1}\left(\frac{\partial A^t(p)}{\partial p_i}_{|p_i} - \frac{\partial \lambda}{\partial p_i}I_n\right)v_0.$$

Now we need to see if this expression verifies the second equation of (32). ■

Taking the partial derivative $\partial^2/\partial p_i \partial p_j$ of both sides of eigenvalue problem (30) we have:

$$\left.\begin{array}{l} \dfrac{\partial^2 A^t(p)}{\partial p_i \partial p_j}v(p) + \dfrac{\partial A^t(p)}{\partial p_i}\dfrac{\partial v(p)}{\partial v(p)}\partial p_j + \\ \quad \dfrac{\partial A^t(p)}{\partial p_j}\dfrac{\partial v(p)}{\partial v(p)}\partial p_i + A^t(p)\dfrac{\partial^2 v(p)}{\partial p_i \partial p_j} = \\ \quad \dfrac{\partial^2 \lambda(p)}{\partial p_i \partial p_j}v(p) + \dfrac{\partial \lambda(p)}{\partial p_i}\dfrac{\partial v(p)}{\partial p_j} + \\ \quad \dfrac{\partial \lambda(p)}{\partial p_j}\dfrac{\partial v(p)}{\partial p_i} + \lambda(p)\dfrac{\partial^2 v(p)}{\partial p_i \partial p_j}, \\ \dfrac{\partial^2 B^t}{\partial p_i \partial P_j}v(p) + \dfrac{\partial B^t(p)}{\partial p_i}\dfrac{\partial v(p)}{\partial p_j} + \\ \quad \dfrac{\partial B^t(p)}{\partial p_j}\dfrac{\partial v(p)}{\partial p_i} + B^t\dfrac{\partial^2 v(p)}{\partial p_i \partial p_j} = 0 \end{array}\right\}$$

At p_0 and premultiplying the equation by u_0^t we can deduce an expression for derivatives $\frac{\partial^2 \lambda(p)}{\partial p_i \partial p_j}_{|p_0}$

$$\begin{array}{l} \dfrac{\partial^2 \lambda(p)}{\partial p_i \partial p_j}_{|p_0} u_0^t v_0 = u_0^t \dfrac{\partial^2 A^t(p)}{\partial p_i \partial p_j}_{|p_0} v_0 + \\ \quad u_0 \dfrac{\partial A^t(p)}{\partial p_j}_{|p_0}\dfrac{\partial v(p)}{\partial p_i}_{|p_0} + u_0 \dfrac{\partial A^t(p)}{\partial p_i}_{|p_0}\dfrac{\partial v(p)}{\partial p_j}_{|p_0} \\ -u_0^t \dfrac{\partial \lambda(p)}{\partial p_j}_{|p_0}\dfrac{\partial v(p)}{\partial p_i}_{|p_0} - u_0^t \dfrac{\partial \lambda(p)}{\partial p_i}_{|p_0}\dfrac{\partial v(p)}{\partial p_j}_{|p_0} \end{array}$$

Knowing $\frac{\partial^2 \lambda}{\partial p_i \partial p_j}_{|p_0}$ we can deduce the values of $\frac{\partial^2 v(p)}{\partial p_i \partial p_j}_{|p_0}$ calling $S = (A^t(p) - \lambda(p)I - u_0 u_0^t)^{-1}$

$$\begin{array}{l} \dfrac{\partial^2 v(p)}{\partial p_i \partial p_j}_{|p_0} = \\ S\left(\dfrac{\partial^2 \lambda(p)}{\partial p_i \partial p_j}v(p) + \dfrac{\partial \lambda(p)}{\partial p_i}\dfrac{\partial v(p)}{\partial p_j} + \dfrac{\partial \lambda(p)}{\partial p_j}\dfrac{\partial v(p)}{\partial p_i}\right. \\ \left. -\dfrac{\partial^2 A^t(p)}{\partial p_i \partial p_j}v(p) - \dfrac{\partial A^t(p)}{\partial p_i}\dfrac{\partial v(p)}{\partial p_j} - \dfrac{\partial A^t(p)}{\partial p_j}\dfrac{\partial v(p)}{\partial p_i}\right). \end{array}$$

VI. Perturbation analysis of simple eigenvalues of singular systems

Finally, we consider systems in the form $E\dot{x} = Ax + Bu$ with $E, A \in M_n(\mathbb{C})$ and $B \in M_{n\times m}(\mathbb{C})$, we will write the systems as a triple of matrices (E, A, B).

Let $M(\lambda) = (\lambda E + A, B)$ be a matrix pencil associated to the triple (E, A, B), λ_0 is an eigenvalue of (E, A, B), if rank $M(\lambda_0) <$ rank $M(\lambda)$. (In the case where the matrix pencil $\lambda E + A$ is regular this is equivalent to $\det(\lambda_0 E + A) = 0$.

$v_0 \in \mathbb{C}^n$ is an eigenvector corresponding to the eigenvalue λ_0, if $(\lambda_0 E^t + A^t)v_0 = 0$ and $B^t v_0 = 0$.

Proposition 8: Let λ_0 be an eigenvalue and v_0 a corresponding eigenvector of (E, A, B). Then λ_0 is an eigenvalue and v_0 the corresponding eigenvector of $(E + BK_1, A + BK_2, B)$ for all K.

Suppose that matrices E, A, B, defining the singular system, smoothly depend on the vector of a real parameters $p = (p_1, \ldots, p_r)$. The function $(E(p), A(p), B(p))$ is called a multi-parameter family of singular systems.

We write the eigenvalue problem as

$$\left.\begin{array}{rl} (\lambda E^t(p) + A^t(p))v(p) & = 0 \\ B^t(p)v(p) & = 0 \end{array}\right\}$$

equivalently

$$\left.\begin{array}{rl} (\lambda(E^t(p) + K_1(p)B^t(p)) + (A^t(p) + K_2^t(p)B^t(p)))v(p) & = 0 \\ B^t(p)v(p) & = 0 \end{array}\right\}$$

Taking derivatives

$$\left.\begin{array}{l} \left(\dfrac{\partial \lambda}{\partial p_i}E^t(p) + \lambda\dfrac{\partial E^t(p)}{\partial p_i} + \dfrac{\partial A^t(p)}{\partial p_i}\right)v(p) + \\ (\lambda E^t(p) + A^t(p))\dfrac{\partial v(p)}{\partial p_i} = 0 \\ \dfrac{\partial B^t(p)}{\partial p_i}v(p) + B^t(p)\dfrac{\partial v(p)}{\partial p_i} = 0 \end{array}\right\}$$

At the point (λ_0, p_0) is

$$\left.\begin{array}{rl} \left(\left(\dfrac{\partial \lambda}{\partial p_i}E^t(p) + \lambda\dfrac{\partial E^t(p)}{\partial p_i} + \dfrac{\partial A^t(p)}{\partial p_i}\right)v(p) + \right. & \\ \left.(\lambda E^t(p) + A^t(p))\dfrac{\partial v(p)}{\partial p_i}\right)_{|(\lambda_0,p_0)} & = 0 \\ \left(\dfrac{\partial B^t(p)}{\partial p_i}v(p) + B^t(p)\dfrac{\partial v(p)}{\partial p_i}\right)_{|(\lambda_0,p_0)} & = 0 \end{array}\right\}$$

Premultiplying by u_0^t the first equality we have

$$\left.\begin{array}{rl} u_0^t\left(\dfrac{\partial\lambda}{\partial p_i}E^t(p)+\lambda_0\dfrac{\partial E^t(p)}{\partial p_i}+\dfrac{\partial A^t(p)}{\partial p_i}\right)_{|(\lambda_0,p_0)}v_0 & =0 \\ \dfrac{\partial B^t(p)}{\partial p_i}_{|(\lambda_0,p_0)}v_0+B_0^t\dfrac{\partial v(p)}{\partial p_i}_{|(\lambda_0,p_0)} & =0 \end{array}\right\}$$

$$\left.\begin{array}{r} \dfrac{\partial\lambda}{\partial p_i}_{|(\lambda_0,p_0)}u_0^tE^t(p_0)v_0= \\ -\lambda_0u_0^t\dfrac{\partial E^t(p)}{\partial p_i}_{|(\lambda_0,p_0)}v_0-u_0^t\dfrac{\partial A^t(p)}{\partial p_i}_{|(\lambda_0,p_0)}v_0=0 \\ \dfrac{\partial B^t(p)}{\partial p_i}_{|(\lambda_0,p_0)}v_0+B_0^t\dfrac{\partial v(p)}{\partial p_i}_{|(\lambda_0,p_0)}=0 \end{array}\right\}$$

Suppose that rank $(\lambda_0E(p_0)+A(p_0))=n-1$, in this case we can chose u_0 in such away that $u_0^tv_0\neq 0$.

Using the normalization condition $u_0^tv(p)=1$ (it is possible because the function $u_0^tv(p)$ in $p=p_0$ is non zero) we have that $u_0^t\dfrac{\partial v(p)}{\partial p_i}_{|(\lambda_0,p_0)}=0$.

Lemma 4: There exists a left eigenvector such that $u_0^tE(p_0)v_0\neq 0$.

Proof: Taking into account that λ_0 is a simple eigenvalue $E^tv_0+(\lambda E^tv_1+A^tv_1)\neq 0$ for all vector v_1. Taking $v_1=0$ we have that $E^tv_0\neq 0$.

If $u_0^tE^tv_0=0$ we have that $Eu_0,Au_0\in[v_0]^\perp$, so u_0 is an eigenvector of the linear map $(\lambda_0E+A)_{|[v_0]^{perp}}$ for the zero eigenvalue of, but zero is a simple eigenvalue of λ_0E+A. ■

Lemma 5: The matrix $T_0=\lambda_0E^t(p_0)+A^t(p_0)+u_0u_0^t$ is invertible.

Proof: $u_0u_0^t$ is a symmetric map of rank 1, u_0 is an eigenvector of eigenvalue $\|u_0\|^2$ and $[u_0]^\perp$ is the null-space.

Let $w\in\operatorname{Ker}T_0$, we can write $w=\alpha u_0+w_1$ with $w_1\in[u_0]^\perp$. Then $0=T_0w$ and

$$\begin{array}{l} 0=u_0^tT_0w=u_0^t(\lambda_0E^t(p_0)+A^t(p_0)+u_0u_0^t)(\alpha u_0+w_1)= \\ u_0^t(u_0u_0^t)(\alpha u_0+w_1)=\alpha(u_0^tu_0)^2. \end{array}$$

Then $\alpha=0$ and $w=w_1\in\operatorname{Ker}u_0u_0^t$, consequently $(\lambda_0E^t(p_0)+A^t(p_0))w=0$, and taking into account that λ_0 is a simple eigenvalue we have $w=w_1=\beta v_0\in[u_0]^\perp$. Finally, condition $u_0^tv_0(p_0)\neq 0$ implies $\beta=0$ and T_0 is invertible. ■

VII. Conclusion

In this paper the perturbation of a multiple eigenvalue with a simple eigenvector of a monic polynomial matrix smoothly depending on parameters is analyzed, as well as the perturbation of a simple eigenvalue of a standard and a singular linear system smoothly depending on parameters.

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