The Derivatives of Trigonometric Functions

Chii-Huei Yu

Abstract**—**This article uses the mathematical software Maple for the auxiliary tool to study the differential problem of some type of trigonometric functions. We can obtain the closed forms of any order derivatives of this type of functions by using binomial theorem. On the other hand, we propose two examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying our answers by using Maple.

Keywords**—**derivatives, trigonometric functions, binomial theorem, closed forms, Maple.

I. Introduction

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Beethoven, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research.

In calculus and engineering mathematics courses, determining the n -th order derivative value $f^{(n)}(c)$ of a function $f(x)$ at $x=c$, in general, needs to go through two procedures: firstly finding the n-th order derivative $f^{(n)}(x)$ of $f(x)$, and then taking $x=c$ into $f^{(n)}(x)$. These two procedures will make us face with increasingly complex calculations when calculating higher order derivative values of this function (i.e. n is large), and hence to obtain the answers by manual calculations is not easy. In this paper, we mainly study the differential problem of the following trigonometric function

$$f(x)=\sin^m(ax+b)\cos^n(ax+b) \tag{1}$$

where a,b are real numbers, and m,n are non-negative integers. We can obtain the closed forms of any order derivatives of this type of trigonometric functions using binomial theorem; this is the major result of this study (i.e., Theorem K), and hence greatly reduce the difficulty of calculating their higher order derivative values. As for the study of related differential problems can refer to [1-21]. In addition, we provide two examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods.

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II. Main Results

Firstly, we introduce a notation and two formulas used in this paper.

2.1. Notation

2.1.1. Suppose n,k are positive integers, $k\le n$. Define $\binom{n}{k}=\dfrac{n!}{k!(n-k)!}$, and $\binom{n}{0}=1$.

2.1.2. Let s be a real number, the notation $\lfloor s\rfloor$ represents the largest integer less than or equal to s.

2.2. Formulas

2.2.1. Euler's formula

$e^{iy}=\cos y+i\sin y$, where $i=\sqrt{-1}$, and y is any real number.

2.2.2. Suppose α,β are real numbers, then

$$\cos\alpha\cdot\cos\beta=\frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)].$$

Next, we introduce an important theorem used in this study.

2.3. Binomial theorem

$(u+v)^n=\sum_{k=0}^{n}\binom{n}{k}u^{n-k}v^k$, where u,v are real numbers, and n is a positive integer.

Before deriving our major result, we need a lemma.

2.4. Lemma Suppose a, b, x are real numbers, and n is a positive integer. Then

$$\cos^n(ax+b)$$

$$= \frac{1}{2^{n-1}} \cdot \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{k} \cos[(n-2k)(ax+b)] +$$

$$\frac{1+(-1)^n}{2} \cdot \frac{1}{2^n} \binom{n}{\lfloor n/2 \rfloor} \quad (2)$$

And

$$\sin^n(ax+b)$$

$$= \frac{1}{2^{n-1}} \cdot \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{k} \cos\left[(n-2k)\left(ax+b-\frac{\pi}{2}\right)\right] +$$

$$\frac{1+(-1)^n}{2} \cdot \frac{1}{2^n} \binom{n}{\lfloor n/2 \rfloor} \quad (3)$$

Proof $\cos^n(ax+b)$

$$= \left[\frac{1}{2}(e^{i(ax+b)} + e^{-i(ax+b)}\right]^n$$

(By Euler's formula)

$$= \frac{1}{2^n} \cdot \sum_{k=0}^{n} \binom{n}{k} [e^{i(ax+b)}]^{n-k} [e^{-i(ax+b)}]^k$$

(By binomial theorem)

$$= \frac{1}{2^n} \cdot \sum_{k=0}^{n} \binom{n}{k} e^{i(n-2k)(ax+b)}$$

$$= \frac{1}{2^n} \cdot \sum_{k=0}^{n} \binom{n}{k} \cos[(n-2k)(ax+b)]$$

$$= \frac{1}{2^{n-1}} \cdot \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{k} \cos[(n-2k)(ax+b)] +$$

$$\frac{1+(-1)^n}{2} \cdot \frac{1}{2^n} \binom{n}{\lfloor n/2 \rfloor}$$

And hence

$$\sin^n(ax+b)$$

$$= \cos^n\left(ax+b-\frac{\pi}{2}\right)$$

$$= \frac{1}{2^{n-1}} \cdot \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{k} \cos\left[(n-2k)\left(ax+b-\frac{\pi}{2}\right)\right] +$$

$$\frac{1+(-1)^n}{2} \cdot \frac{1}{2^n} \binom{n}{\lfloor n/2 \rfloor} \quad \blacksquare$$

Next, we determine the closed forms of any order derivatives of the trigonometric function (1).

2.5. Theorem K If a, b are real numbers, m, n are non-negative integers, and p is a positive integer. Suppose the domain of the function

$$f(x) = \sin^m(ax+b)\cos^n(ax+b)$$

is $(-\infty, \infty)$. Then the p -th order derivative of $f(x)$,

$$f^{(p)}(x)$$

$$= \frac{a^p}{2^{m+n-1}} \cdot \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{m}{k}\binom{n}{j} (m+n-2k-2j)^p$$

$$\times \cos\left[(m+n-2k-2j)(ax+b) - \frac{(m-2k-p)\pi}{2}\right]$$

$$+ \frac{a^p}{2^{m+n-1}} \cdot \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{m}{k}\binom{n}{j} (m-n-2k+2j)^p$$

$$\times \cos\left[(m-n-2k+2j)(ax+b) - \frac{(m-2k-p)\pi}{2}\right]$$

$$+ \frac{a^p[1+(-1)^m]}{2^{m+n}} \binom{m}{\lfloor m/2 \rfloor} \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{j} (n-2j)^p \cos\left[(n-2j)(ax+b) + \frac{p\pi}{2}\right]$$

$$+ \frac{a^p[1+(-1)^n]}{2^{m+n}} \binom{n}{\lfloor n/2 \rfloor} \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \binom{m}{k} (m-2k)^p \cos\left[(m-2k)\left(ax+b-\frac{\pi}{2}\right) + \frac{p\pi}{2}\right]$$

(4)

for all $x \in \boldsymbol{R}$.

Proof Because

$$f(x)$$

$$= \sin^m(ax+b)\cos^n(ax+b)$$

$$= \frac{1}{2^{m+n-1}} \cdot \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{m}{k}\binom{n}{j} \cos\left[(m+n-2k-2j)(ax+b) - \frac{(m-2k)\pi}{2}\right]$$

$$+ \frac{1}{2^{m+n-1}} \cdot \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{m}{k}\binom{n}{j} \cos\left[(m-n-2k+2j)(ax+b) - \frac{(m-2k)\pi}{2}\right]$$

$$+ \frac{1+(-1)^m}{2^{m+n}} \binom{m}{\lfloor m/2 \rfloor} \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{j} \cos[(n-2j)(ax+b)]$$

$$+ \frac{1+(-1)^n}{2^{m+n}} \binom{n}{\lfloor n/2 \rfloor} \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \binom{m}{k} \cos\left[(m-2k)\left(ax+b-\frac{\pi}{2}\right)\right]$$

$$+\frac{[1+(-1)^m][1+(-1)^n]}{2^{m+n+2}}\binom{m}{\lfloor m/2\rfloor}\binom{n}{\lfloor n/2\rfloor} \tag{5}$$

(By Formula 2.2.2 and Lemma)

Thus,

$$f^{(p)}(x)$$

$$=\frac{a^p}{2^{m+n-1}}\cdot\sum_{k=0}^{\lfloor(m-1)/2\rfloor}\sum_{j=0}^{\lfloor(n-1)/2\rfloor}\binom{m}{k}\binom{n}{j}(m+n-2k-2j)^p$$

$$\times\cos\left[(m+n-2k-2j)(ax+b)-\frac{(m-2k-p)\pi}{2}\right]$$

$$+\frac{a^p}{2^{m+n-1}}\cdot\sum_{k=0}^{\lfloor(m-1)/2\rfloor}\sum_{j=0}^{\lfloor(n-1)/2\rfloor}\binom{m}{k}\binom{n}{j}(m-n-2k+2j)^p$$

$$\times\cos\left[(m-n-2k+2j)(ax+b)-\frac{(m-2k-p)\pi}{2}\right]$$

$$+\frac{a^p[1+(-1)^m]}{2^{m+n}}\binom{m}{\lfloor m/2\rfloor}\sum_{j=0}^{\lfloor(n-1)/2\rfloor}\binom{n}{j}(n-2j)^p\cos\left[(n-2j)(ax+b)+\frac{p\pi}{2}\right]$$

$$+\frac{a^p[1+(-1)^n]}{2^{m+n}}\binom{n}{\lfloor n/2\rfloor}\sum_{k=0}^{\lfloor(m-1)/2\rfloor}\binom{m}{k}(m-2k)^p\cos\left[(m-2k)\left(ax+b-\frac{\pi}{2}\right)+\frac{p\pi}{2}\right]$$

for all $x\in\boldsymbol{R}$ ■

III. Examples

For the differential problem of trigonometric functions in this study, we propose two examples and use Theorem K to determine the closed forms of any order derivatives of these functions and evaluate some of their higher order derivative values. On the other hand, we employ Maple to calculate the approximations of these higher order derivative values and their solutions for verifying our answers.

3.1. Example 1 Suppose the domain of the trigonometric function

$$f(x)=\sin^8\left(3x+\frac{\pi}{4}\right)\cos^6\left(3x+\frac{\pi}{4}\right) \tag{6}$$

is $(-\infty,\infty)$. Then by (4), we obtain any p -th order derivative of $f(x)$,

$$f^{(p)}(x)$$

$$=\frac{3^p}{2^{13}}\cdot\sum_{k=0}^{3}\sum_{j=0}^{2}\binom{8}{k}\binom{6}{j}(14-2k-2j)^p$$

$$\times\cos\left[(14-2k-2j)\left(3x+\frac{\pi}{4}\right)-\frac{(8-2k-p)\pi}{2}\right]$$

$$+\frac{3^p}{2^{13}}\cdot\sum_{k=0}^{3}\sum_{j=0}^{2}\binom{8}{k}\binom{6}{j}(2-2k+2j)^p$$

$$\times\cos\left[(2-2k+2j)\left(3x+\frac{\pi}{4}\right)-\frac{(8-2k-p)\pi}{2}\right]$$

$$+\frac{3^p}{2^{13}}\binom{8}{4}\sum_{j=0}^{2}\binom{6}{j}(6-2j)^p\cos\left[(6-2j)\left(3x+\frac{\pi}{4}\right)+\frac{p\pi}{2}\right]$$

$$+\frac{3^p}{2^{13}}\binom{6}{3}\sum_{k=0}^{3}\binom{8}{k}(8-2k)^p\cos\left[(8-2k)\left(3x-\frac{\pi}{4}\right)+\frac{p\pi}{2}\right] \tag{7}$$

for all $x\in\boldsymbol{R}$.

Thus, the 9-th order derivative value of $f(x)$ at $x=\frac{\pi}{2}$,

$$f^{(9)}\left(\frac{\pi}{2}\right)$$

$$=\frac{3^9}{2^{13}}\cdot\sum_{k=0}^{3}\sum_{j=0}^{2}\binom{8}{k}\binom{6}{j}(14-2k-2j)^9\cos\frac{(2+3k+j)\pi}{2}$$

$$+\frac{3^9}{2^{13}}\cdot\sum_{k=0}^{3}\sum_{j=0}^{2}\binom{8}{k}\binom{6}{j}(2-2k+2j)^9\cos\frac{(3k-j)\pi}{2}$$

$$+\frac{3^9}{2^{13}}\binom{8}{4}\sum_{j=0}^{2}\binom{6}{j}(6-2j)^9\cos\frac{(2+j)\pi}{2}$$

$$+\frac{3^9}{2^{13}}\binom{6}{3}\sum_{k=0}^{3}\binom{8}{k}(8-2k)^9\cos\frac{(1+3k)\pi}{2} \tag{8}$$

In the following, we use Maple to verify the correctness of (8).

```
>f:=x->(sin(3*x+Pi/4))^8*(cos(3*x+Pi/4))^6;
>evalf((D@@9)(f)(Pi/2),18);
```

$$-6.1873982892\cdot 10^{10}$$

```
>evalf(3^9/2^13*sum(sum(8!/(k!*(8-k)!)*6!/(j!*(6-j)!)*(14-2*k-2*j)^9*cos((2+3*k+j)*Pi/2),j=0..2),k=0..3) +3^9/2^13*sum(sum(8!/(k!*(8-k)!)*6!/(j!*(6-j)!)*(2-2*k+2*j)^9*cos((3*k-j)*Pi/2),j=0..2),k=0..3)+ 3^9/2^13*8!/(4!*4!)*sum(6!/(j!*(6-j)!)*(6-2*j)^9*cos((2+j)*Pi/2),j=0..2)+ 3^9/2^13*6!/(3!*3!)*sum(8!/(k!*(8-k)!)*(8-2*k)^9*cos((1+3*k)*Pi/2),k=0..3),18);
```

$$-6.1873982892\cdot 10^{10}$$

3.2. Example 2 Suppose the domain of the trigonometric function

$$g(x)=\sin^7\left(5x-\frac{\pi}{6}\right)\cos^{10}\left(5x-\frac{\pi}{6}\right) \tag{9}$$

is $(-\infty,\infty)$. Also, using (4), we obtain

$$g^{(p)}(x)$$

$$=\frac{5^p}{2^{16}}\cdot\sum_{k=0}^{3}\sum_{j=0}^{4}\binom{7}{k}\binom{10}{j}(17-2k-2j)^p$$

$$\times\cos\left[(17-2k-2j)\left(5x-\frac{\pi}{6}\right)-\frac{(7-2k-p)\pi}{2}\right]$$

$$+\frac{5^p}{2^{16}}\cdot\sum_{k=0}^{3}\sum_{j=0}^{4}\binom{7}{k}\binom{10}{j}(-3-2k+2j)^p$$

$$\times\cos\left[(-3-2k+2j)\left(5x-\frac{\pi}{6}\right)-\frac{(7-2k-p)\pi}{2}\right]$$

$$+\frac{5^p}{2^{16}}\binom{10}{5}\sum_{k=0}^{3}\binom{7}{k}(7-2k)^p\cos\left[(7-2k)\left(5x-\frac{2\pi}{3}\right)+\frac{p\pi}{2}\right] \tag{10}$$

for all $x\in R$.

Hence,

$$g^{(8)}\left(\frac{2\pi}{3}\right)$$

$$=\frac{5^8}{2^{16}}\cdot\sum_{k=0}^{3}\sum_{j=0}^{4}\binom{7}{k}\binom{10}{j}(17-2k-2j)^8\cos\frac{(1+2k-j)\pi}{3}$$

$$+\frac{5^8}{2^{16}}\cdot\sum_{k=0}^{3}\sum_{j=0}^{4}\binom{7}{k}\binom{10}{j}(-3-2k+2j)^8\cos\frac{(3+2k+j)\pi}{3}$$

$$+\frac{5^8}{2^{16}}\binom{10}{5}\sum_{k=0}^{3}\binom{7}{k}(7-2k)^8\cos\frac{(2+2k)\pi}{3} \tag{11}$$

```
>g:=x->(sin(5*x-Pi/6))^7*(cos(5*x-Pi/6))^10;
>evalf((D@@8)(g)(2*Pi/3),18);
```

$$6.87371559530496597\cdot 10^{10}$$

```
>evalf(5^8/2^16*sum(sum(7!/(k!*(7-k)!)*10!/(j!*(10-j)!)*(17-2*k-2*j)^8*cos((1+2*k-j)*Pi/3),j=0..4),k=0..3) +5^8/2^16*sum(sum(7!/(k!*(7-k)!)*10!/(j!*(10-j)!)*(-3-2*k+2*j)^8*cos((3+2*k+j)*Pi/3),j=0..4),k=0..3)+ 5^8/2^16*10!/(5!*5!)*sum(7!/(k!*(7-k)!)*(7-2*k)^8*cos((2+2*k)*Pi/3),k=0..3),18);
```

$$6.87371559530496597\cdot 10^{10}$$

IV. Conclusion

In this paper, we provide a new method to evaluate any order derivatives of some type of trigonometric functions. We hope this method can be applied to solve another differential problems. In addition, the binomial theorem plays a significant role in the theoretical inferences of this study. In fact, the applications of this theorem are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to another calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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