The Role of Millionshschkov's Zero-Fourth Cumulant Hypothesis in Homogenous Isotropic Turbulence

Christos Mamaloukas, Amitabha Chanda, Himandri Pai Mazumdar

***Abstract*—**In this paper a new model of one-point, one-time fourth cumulant of homogeneous and isotropic turbulence in wavenumber space is suggested. The fourth cumulant in this model varies in wavenumber. In the final period decay it is identically zero, in compliance with the zero-fourth cumulant hypothesis, proposed by Millionshchikov. But as wavenumber goes to lower values the fourth cumulant becomes non-zero. The suggested model is based on a near-Gaussian distribution, built with the help of Gaussian distribution and truncated Hermite polynomial.

***Keywords*—**Millionschikov's hypothesis, zero-fourth cumulant hypothesis, homogeneous and isotropic turbulence, Hermite polynomial.

I. Introduction

In "zero fourth cumulant hypothesis" of Millionshchikov [1] the second and the fourth order moments are related as in normal distribution. The hypothesis goes also by the name of quasi-normal hypothesis. Millionshchikov has proposed that the two point distribution of simultaneous velocity amplitudes in a turbulent flow is quasi-normal. Lot of works have been done on this hypothesis. Works of legendary mathematicians like Heisenberg [2], Obukhov [3], Batchelor [4], Chandrasekhar [5] and Reid and Proudman [6] are worth mentioning. Experimental results have also been analyzed by Batchelor and Uberoi [7] to verify the authenticity of this hypothesis.

Heisenberg has extended the hypothesis to the distribution of Fourier coefficients in two times. On the other hand, Reid and Proudman have examined the hypothesis in the decay of homogenous turbulence using three point simultaneous velocity fluctuations. Chandrasekhar has studied the hypothesis in hydromagnetic turbulence. The study conducted by Ghosh [8] may also be mentioned.

In the recent times there is a spurt of work considering the viability or otherwise of this hypothesis. Bos & Rubenstein [9], Chen, H., Herring, J.R., Kerr, R.M., & Kraichnan R.H [10], Chen, S. & Kraichnan, R.H. [11], Mazumdar, H.P. & C. Mamaloukas [12] and Mamaloukas [13] have studied the hypothesis in detail.

Suggestions have also been forwarded from time to time to modify the hypothesis. Ogura [14] has suggested some modification. Mirabel [15] has observed some discrepancies in the suggestion, made by Ogura and has suggested some modification. The objection regarding the viability of Millionshchikov hypothesis forwarded by Kraichnan [16] is note-worthy. In the present paper we shall try to suggest a modification of the "zero-fourth cumulant hypothesis of Millionshchikov". We would confine our study to one-point one-time correlations in isotropic turbulence, that also in wavenumber space.

II. Equations of motion in isotropic turbulence

It is observed that in the final period of decay of homogenous and isotropic turbulence only the smallest eddies are active and viscous dissipation is the only mechanism through which turbulent energy decays in the form of thermal energy. The size or the length scale of these smallest eddies is given as $\lambda_0 = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}$. In wavenumber space these smallest eddies have the largest wavenumber, as $k_\infty = \left(\frac{\nu^3}{\epsilon}\right)^{-\frac{1}{4}}$. In the final period of decay neither inertial force nor pressure gradient takes part in the decay process. But these two terms, in fact, are the sources of third order moment. We now try to examine this point from the Navier-Stokes equation of homogenous and isotropic turbulence. The unit density, incompressible homogenous and isotropic fully developed turbulent flow in absence of external force is governed by the following equations.

$$\frac{\partial u(x,t)}{\partial t} + \left[u(x,t)\cdot\nabla u(x,t) - \nabla p(x,t)\right] = \nu\Delta u(x,t) \qquad (1)$$
$$\nabla u(x,t) = 0$$

C. Mamaloukas, Department of Statistics, Athens University of Economics and Business, Athens, 10434, Greece, email: mamkris@aueb.gr
C. Amitabha, Visiting Faculty, Calcutta University, P36/1, Manasbag, Kolkata 700056, India, email: amitabha39@yahoo.co.in and amitabha39@gmail.com.
H.P. Mazumdar, Indian Statistical Institute, Calcutta -- 700035, India, email: hpmi2003@yahoo.com

Kraichnan & Panda [17] observe that in turbulent flow the variance of the nonlinear term in the Navier-Stokes equations i.e. $\left[u(x,t)\cdot\nabla u(x,t)-\nabla p(x,t)\right]$ is on average smaller than what would be expected from a Gaussian estimate. More precisely, if one constructs a flow field consisting of random statistically independent Fourier modes exhibiting the same energy spectrum as the turbulent flow considered, the variance of the nonlinear term will be larger than that of the original field. This depletion of nonlinearity is the result of a self-organization process of the turbulent flow, a process which is, itself, due to the nonlinear term in the Navier-Stokes equations. Kraichnan and Panda give importance on the velocity-vorticity alignment in turbulent flows. They show it to be one expression of a more general, underlying property of nonlinear systems. We consider that this depletion of nonlinearity is an important feature of turbulent flows, since the nonlinearity of the Navier-Stokes equations is the heart of the turbulence problem.

The nonlinear term of the Navier-Stokes equations is a vector and its mean value is zero in isotropic turbulence. It is important to study the strength of the fluctuations of the nonlinear term and its depletion.

In order to examine the strength of the fluctuations of the nonlinear term and its depletion, we focus on the nonlinearity spectrum, which we shall define below. This spectrum measures the strength of the fluctuations of the nonlinear term as a function of scale, just like the energy spectrum does for the strength of velocity fluctuations. Whereas the characterization of the energy spectrum has received an enormous amount of attention in the field of turbulence research, only very few investigations consider the nonlinearity spectrum. To our knowledge, only the works by Chen, Herring & Kraichnan, Nelkin & Tabor [18] and Ishihara, T., Kaneda, Y., Yokokawa, M., Itakura, K. , & Uno, A [19] have considered this quantity.

Chen et al. performed low resolution Direct Numerical Simulations and compared their results to the Direct Interaction Approximation (DIA). No information was obtained on the inertial range behavior of this quantity, since the Reynolds number was too low in their simulations. Higher Reynolds numbers could in principle be obtained by using the DIA, but physically incorrect behavior is observed by Kraichnan in the inertial range dynamics of the original Eulerian DIA. Nelkin & Tabor considered only the scaling of the potential part of the advection term, assuming that the full nonlinear term scales as its potential part. Only the high resolution simulations by Ishihara et al. give an idea on the inertial range scaling of several fourth order spectra.

It is shown by Bos & Rubenstein that, in the inertial and dissipation range, the nonlinearity spectrum is given by

$$w(k)=\tilde{u}^{\frac{2}{3}}\epsilon^{\frac{3}{4}}k^{\frac{1}{3}}f(k\eta).$$

for very high Reynolds numbers. $\tilde{u}$ is the root-mean-square (rms) velocity fluctuation. The function $f(k\eta)$ tends to a constant value in the inertial range and its value is approximately 0.8 times the value of its Gaussian estimate.

The total depletion of nonlinearity is found to exhibit some sub-Gaussian behavior. Gryanik & Hartmann [20] have observed from CBL data that the Millionshchikov hypothesis of quasi-normal (Gaussian) distribution of the one-point fourth-order moments fails for convective boundary layer conditions. This is because the effect of the semiorganized coherent structures (plumes) leads to skewed distributions; the third-order moments are non-zero. It must be connected with a certain order in the flows, but how this manifests itself in an instantaneous flow field cannot be guessed from the statistical considerations presented here. The nonlinear term consists of two parts: the advection term and the pressure gradient term. The pressure spectrum, (Gotoh & Fukayama [21]) $\pi(k)$ scales approximately as $\pi(k)\approx\epsilon^{\frac{4}{3}}k^{-\frac{1}{3}}f(k\eta)$ and the pressure gradient spectrum scales as $\nabla\pi(k)\approx\epsilon^{\frac{3}{4}}k^{-\frac{7}{3}}f(k\eta)$

It may be noted, however, that this scaling appears only at relatively high Reynolds number, compared to the appearance of K41 [22] scaling for the energy spectrum. It is shown by Bos & Rubenstein that at large Reynolds numbers the mean square nonlinearity is proportional to the Gaussian value, the ratio being 0.65. The variance of the nonlinearity is therefore dominantly determined by the advection term. The depletion of nonlinearity implies hereby directly a depletion of the sweeping compared to the kinematic sweeping induced by a field consisting of independent Fourier modes. Recent works of Hans C. Eggers and Martin Greinerb [23] and Servidio, S., Matthaeus, W. H., & Dmitruk, P. [24] and Malecot, Y., Auriault, C., Kahaleras, H., Gagne, Y., Chanal, O., Chabaud, B. & Casting, B. [25], are worth mentioning.

In this context we refer to the work by Chen, H., Herring, J.R., Kerr, R.M., & Kraichnan, R.H., which discusses the possibility of a reduction of sweeping in turbulence. They argue that third order cumulants are small but nonzero quantities. But in the pre-viscous dissipation ranges, dominated by large eddies of smaller wave number k triple order interaction are so great in number that their total contribution cannot be ignored to get to the realistic analysis of turbulence. Chen, H., Herring, J.R., Kerr, R.M., & Kraichnan, R.H., argue that a complete reduction of sweeping is improbable for stochastically forced turbulence. Their arguments are not in disagreement with the experimental results, obtained thereafter.

The dependence of the large and small scales is influenced, and the sweeping, as estimated by purely kinematic arguments, is partially but definitely not completely suppressed. In this light, the depletion of nonlinearity can also be interpreted as a reduction of Eulerian acceleration, suggesting a larger Eulerian coherence for turbulence than for advection by random Fourier modes. The possible link of this enhanced coherence with inertial range and dissipation range intermittency is not clear at present. The super-Gaussian values of the large-scales of the nonlinearity spectrum were shown to be related to the non-Gaussianity of the Reynolds-stress-fluctuation spectrum. The physical importance of this relation for the dynamics of turbulent flow seems to deserve further research.

We mention here that a similar picture (large-scale super-Gaussian behavior and sub-Gaussian inertial range and dissipation range behavior), was observed in the depletion of advection (Bos et al.), where the inertial range scaling of the advection spectrum also displayed a constant reduction with respect to its Gaussian value.

III. THE PRESENT MODEL

Given above the background of study of nonlinearity vis-à-vis third order moment of turbulent flow we would now try to present a new model of one point one time moment in homogenous and isotropic turbulence of incompressible fluid with unit density. The main considerations used for constructing the model may be enumerated as below.

1. At the viscous dissipation range i.e where $k \to k_\nu$, turbulence energy dissipates only in the form of thermal energy through the set of smallest eddies of size $\left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}$. Here $k_\nu = \left(\frac{\nu^3}{\epsilon}\right)^{-\frac{1}{4}}$ stands for the wavenumber of the smallest dissipating eddies. In this range interaction among eddies of different size ceases to exist. Distribution of Fourier components of velocity fluctuation is assumed Gaussian. Zero-fourth cumulant hypothesis is strictly followed in this range.
2. As wave number becomes smaller and smaller a part of turbulence energy, not the whole, dissipates as thermal energy and the residual part is involved in interacting with Fourier modes of velocity fluctuations and cascading from smaller wavenumber to larger wavenumber. As k progresses to the smallest number, it crosses universal equilibrium range and large eddy range one after the other. In this passage the ratio between dissipated thermal energy and internal energy involved in interaction and cascading becomes less and less. In these ranges third order interaction is non-zero which indicates the strict presence of third order cumulant, however small it may be.

In consideration of the above points, we now present the following model, It may be noted that we are restricting our study to wavenumber space only of isotropic turbulence.

1. $\tilde{K}_4 = \overline{\tilde{u}^4} - 3\overline{\tilde{u}^2}^2 = 0$ for $k = k_\nu$, and
2. $\tilde{K}_4 = \overline{\tilde{u}^4} - 3\overline{\tilde{u}^2}^2 > 0$ for $k << k_\nu$

With targets given above we present a near-Gaussian model of distribution of *u*. The probability density of the proposed model is given as below.

$$\omega = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}} \left\{1 + U(k)\sum_{i=3}^{\infty} \frac{c_i}{i!} H_i\left(\frac{u}{\sigma}\right)\right\} \tag{2}$$

Here the function $U(k)$ in the above equation is such that $U(k) = 0$ for $k \approx k_\nu$, and U=1 for $k << k_\nu$. In addition c_i's are given as below. In the present paper we model $U(k)$ in the following manner.

$$U(k) = 1 - \frac{k}{k_\nu} \tag{3}$$

$$c_3 = \frac{\overline{x^3}}{\sigma^3},\ c_4 = \frac{\overline{x^4}}{\sigma^4} - 3,\ c_5 = \frac{\overline{x^5}}{\sigma^5} - 10\frac{\overline{x^3}}{\sigma^3} \tag{4}$$

H is the Hermite polynomial [27, 28]. The first three Hermite polynomials in the above expression are given as below,

$$\begin{aligned} H_3(x) &= 8x^3 - 12x \\ H_4(x) &= 16x^4 - 48x^2 + 12 \\ H_5(x) &= 32x^5 - 160x^3 + 120x \end{aligned} \tag{5}$$

ω being a probability density, it becomes equal to unity, if summed up over all Fourier modes. So, integrating over the whole range of *u* we get the following condition.

$$\int \omega du = 1 \tag{6}$$

ω becomes a Gaussian probability density ω' in viscous dissipation range and fourth cumulant becomes equal to zero satisfying zero-fourth cumulant hypothesis of Millionshchikov. But in the other ranges it is not so. We state these conditions in the following set of equations.

For viscous dissipation range, when $k \to \infty$ we have the following expression for ω,

$$\omega' = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}} \tag{7}$$

ω' in (7) is the probability density of Gaussian distribution, that supports zero-fourth cumulant hypothesis. For other ranges $(k << k_\infty)$ we shall use the expression ω'' for ω' to show the difference of these two probability densities. ω'' is given as below,

$$\omega'' = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}} \left\{1 + \sum_{i=3}^{\infty} \frac{c_i}{i!} H_i\left(\frac{u}{\sigma}\right)\right\} \tag{8}$$

It may be noted that the near-Gaussian probability, expressed by ω'' is far from Gaussian expressed by ω'

From (8) integrating over the whole range of values of u we can find the third order cumulant as below,

$$\int u^3\omega'' du = \int u^3 \left\{ \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}} \left\{ 1 + \sum_{i=3}^{\infty} \frac{c_i}{i!} H_i\left(\frac{u}{\sigma}\right) \right\} \right\} du \quad (9)$$

The third order cumulant is zero in Gaussian distribution. But for the present near-Gaussian distribution (8) we have the following expression for third order cumulant, after using (6)

$$\int u^3\omega'' du = \int u^3 \left\{ \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}} \sum_{i=3}^{\infty} \frac{c_i}{i!} H_i\left(\frac{u}{\sigma}\right) \right\} du \quad (10)$$

IV. Conclusion

It has been opined by Kraichnan that even if zero-fourth cumulant theory of Millionshchikov does not hold good in fully developed turbulence, third order interaction is weak. But because of large number of these interactions their total impact cannot be ignored. As a result, there is a nonzero third cumulant, indicating non-Gaussianity in distribution of velocity components. From this understanding it may be conjectured that velocity components are distributed in a near Gaussian pattern. In this paper we have considered one-time one-point moments in isotropic turbulence. In the background of the sets of weak third order interaction we have used Hermite polynomial to build a non-Gaussian probability density for ranges beyond viscous dissipation range.

We have ignored any change in the distribution of third order interaction in different ranges of turbulence. It may be observed that different non Gaussian probability densities may be considered by suitably truncating Hermite polynomials in their expressions. But the authenticity of such exercise would depend on experimental verification. This may be an open problem.

References

[1] M. D. Millionshchikov, Dokl. AN SSSR, Vol. 32, No.3, 1941.

[2] Heisenberg, W., Z. Physik, Vol 124, p 628, 1948.

[3] Obukhov, A.M., Doklady Akad. Nauk. S.S.S.R. Vol 66, p 17, 1949.

[4] Batchelor, G.K., Proc. Cambridge Phi. Soc., Vol 47, p 359, 1951.

[5] Chandrasekhar, S., Proc. Roy. Soc. (London), A210, p 18, 1951.

[6] Reid, W.H. & I. Proudman, Trans. Roy. Soc. (London), A247, p 163, 1954.

[7] Uberoi, J. Aeronautical Sc. Vol 20, p 197, 1953.

[8] Ghosh K.M., Indian Journal of Pure and Appl.Math.,Vol. 3, No.1, p. 157, 1972.

[9] Bos & Rubenstein. arXiv:1308.0213v1. Aug, 2013.

[10] Chen, H. , Herring, J.R. , Kerr, R.M. , & Kraichnan, R.H., Phys. Fluids A, Vol 1, p 1844, 1989.

[11] Chen, S & Kraichnan, R.H., Phys. Fluids A, Vol 1, p 2019, 1989.

[12] Mazumdar H.P., & Mamaloukas C., “On Millionschikov's Zero-Fourth Cumulant Hypothesis Applied to Turbulence”, ATINER Conference Paper Series, no. MAT 2012-0103, Athens 2012.

[13] Mamaloukas C., “A New Look at the Final-Period Decay of Homogeneous Isotropic Turbulence”, International Mathematical Forum, Vol. 8, No. 10, pp 495–499, 2013.

[14] Ogura Y., J Fluid Mech., Vol 16, No.1, 1963.

[15] Mirabel A.P., “Application of Millionshchikov’s Hypothesis to the Problem of Isotropic Turbulence Degeneration”, Izv. AN SSSR. Mekhanika Zhidkosti i Gaza, VoI. 4, No. 5, pp. 171-175, 1969.

[16] Kraichnan, R.H. “Relation of Fourth-Order to Second-Order Moments in Stationary Isotropic Turbulence”, Physical Review, Vol. 107, No. 6, pp 1485-1490, 1957.

[17] Kraichnan R.H. & R. Panda, Phys. Fluids, Vol 31, p 2395, 1988.

[18] Nelkin, M. & Tabor, M., Phys. Fluids A, Vol 2, p 81, 1990.

[19] Ishihara, T. , Kaneda, Y. , Yokokawa, M. , Itakura, K. , & Uno, A. "Spectra of Energy Dissipation, Enstrophy and Pressure by High-Resolution Direct Numerical Simulations of Turbulence in a Periodic Box”, J. Phys. Soc. Japan, Vol. 72, No. 5, pp 983-986, 2003.

[20] Gryanik, V.M & Hartmann, 2005. J. Workshop on Interdisciplinary Aspects of Turbulence. Schlo_ Ringberg at Lake Tegernsee.

[21] Gotoh, T. & Fukayama, D. “Pressure Spectrum in Homogeneous Turbulence”, Phys. Rev. Lett., Vol 86, No 17, pp 3775-3778, 2001.

[22] Kolmogorov, A.N., C.R. Akad. Sci. USSR, 30, 301, 1941.

[23] Hans C. Eggers and Martin Greiner, “Cumulant Ratios in Fully Developed Turbulence”, arXiv:hep-ph/0011291v1

[24] Servidio, S., Matthaeus, W.H. & Dmitruk, P., “Depression of Nonlinearity in Decaying Isotropic MHD Turbulence”, Phys. Rev. Lett., PRL100, 095005, 2008.

[25] Kolmogorov, A.N., “Kolmogorov’s $\frac{4}{5}$th Law and Intermittency in Turbulence”, Physical Review Letters, Vol. 73, No. 23, pp 3085-3088, 1994.

[26] Malecot Y., Auriault C., Kahalerras H., Gagne Y., Chanal O., Chabaud B. & Casting B., Physical J. B, Eur. Phys. J. B, Vol. 16, pp 549-561, 2000.

[27] Clemente Cesarano, Hermite Polynomials and some Generalizations on the Heat Equations, International journal of Systems Applications, Engineering and Development, Vol 8, pp 1993-197, ISSN: 2074-1308, 2014

[28] Clemente Cesarano, A Note on Generalized Hermite Polynomials, International journal of applied mathematics and informatics, Vol 8, pp 1-6, ISSN: 2074-1278, 2014

[29] Eusébio Z. E. Conceição, João M. M. Gomes, Daniel R. B. Geraldo, Mean and Turbulent Experimental Airflow

Inside a Vane Separator, International journal of Systems Applications, Engineering and Development, Vol 6, pp 288-295, ISSN: 2074-1308, 2012

[30] Valentina Koliskina, Inta Volodko, Analytical Solution of Eddy Current Problems for Multilayer Medium with Varying Electrical Conductivity and Magnetic Permeability, International journal of Mathematical Models and Methods in Applied Sciences, Vol. 7, pp 174-181, ISSN: 1998-0140, 2013