Numerical Solution of Fredholm Integral Equations of the Second Kind by using 2-Point Explicit Group Successive Over-Relaxation Iterative Method

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***Abstract*—**In this paper, we introduce and analyse the performance of 2-Point Explicit Group Successive Over-Relaxation (2-EGSOR) iterative method for the solution of dense linear systems that arise from second kind Fredholm integral equations. The derivation and implementation of the proposed method are described. We present results of some test examples and computational complexity analysis to illustrate the efficiency of the proposed method.

***Keywords*—**Fredholm integral equations, Explicit Group method, Successive Over-Relaxation approach, Composite closed Newton-Cotes scheme, Dense linear system.

I. Introduction

INTEGRAL equations (IEs) have been one of the principal mathematical models in various areas of science and engineering. The IEs are encountered in numerous applications including continuum mechanics, potential theory, geophysics, electricity and magnetism, kinetic theory of gases, hereditary phenomena in physics and biology, renewal theory, quantum mechanics, radiation, optimization, optimal control systems, communication theory, mathematical economics, population genetics, queuing theory, medicine, mathematical problems of radiative equilibrium, particle transport problems of astrophysics and reactor theory, acoustics, fluid mechanics, steady state heat conduction, fracture mechanics and radiative heat transfer problems ([1], [2], [3], [4]). Consequently, in this paper, a type of IEs i.e. linear Fredholm integral equations of the second kind is considered.

The general form of linear Fredholm integral equations of the second kind can be defined as follows

$$\varphi(x)-\int_{\alpha}^{\beta} K(x,t)\varphi(t)dt = f(x),\ x\in[\alpha,\beta]. \quad (1)$$

The right-hand side function f and kernel K are given. Meanwhile, φ is the unknown function to be determined. The kernel function K is assumed to be absolutely integrable and satisfy the Fredholm alternative theorem [5]. The application of numerical methods for solving the problem (1) is the focus of this paper. There is a huge literature on numerical methods for solving problem (1), for instance refer [6]-[11]. The implementations of numerical methods on problem (1) mostly lead to dense linear systems. Thus, efficient iterative solvers are required to solve the resulting dense linear systems.

Recently, a family of block iterative methods known as Explicit Group (EG) iterative methods has been applied widely in solving various types of linear systems. Thus, in this paper, performance of an iterative method under EG methods i.e. 2-Point Explicit Group Successive Over-Relaxation (2-EGSOR) will be investigated in solving first order composite closed Newton-Cotes quadrature (1-CCNC) algebraic equations. The performance of the 2-EGSOR method on 1-CCNC algebraic equations is comparatively studied by their application in solving problem (1). The concept of the 2-EGSOR method is derived by combining the standard 2-Point Explicit Group (2-EG) method with Successive Over-Relaxation (SOR) approach. Numerical performance of the 2-EGSOR method will be compared with the standard 2-EG method. The standard 2-EG method is also known as 2-Point Explicit Group Gauss-Seidel (2-EGGS) method.

This paper is organised in six main sections. Section II explains the derivation of 1-CCNC algebraic equations for problem (1) followed by the formulations of the 2-EGGS and 2-EGSOR methods in Section III. The computational complexity analysis and numerical results from the simulations are given in Section IV and V respectively. Finally, concluding remarks are given in Section VI.

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II. 1-CCNC ALGEBRAIC EQUATIONS

In this section, discretisation of the problem (1) by using 1-CCNC scheme is discussed. An application of the 1-CCNC scheme for problem (1) leads to 1-CCNC algebraic equations which will be solved by using 2-EGGS and 2-EGSOR methods. Now, let the interval $[\alpha,\beta]$ divided uniformly into N subintervals and the discrete set of points of x and t given by $x_i = \alpha + ih$ $(i = 0,1,2,\cdots,N-2,N-1,N)$ and $t_j = \alpha + jh$ $(j = 0,1,2,\cdots,N-2,N-1,N)$ respectively, where the constant step size, h is defined as follows

$$h = \frac{\beta-\alpha}{N}. \tag{2}$$

Before further explanations, the following notations i.e., $K_{i,j} \equiv K(x_i,t_j)$, $\hat{\varphi}_i \equiv \hat{\varphi}(x_i)$, $\hat{\varphi}_j \equiv \hat{\varphi}(t_j)$ and $f_i \equiv f(x_i)$ will be applied subsequently for simplicity.

An application of the 1-CCNC scheme reduces problem (1) into algebraic equations as follows ([8], [12])

$$\hat{\varphi}_i - \sum_{j=0}^{N} w_j K_{i,j} \hat{\varphi}_j = f_i \tag{3}$$

for $i = 0,1,2,\cdots,N-2,N-1,N$. The solution $\hat{\varphi}$ is an approximation of the exact solution φ to (1) and w_j is the weights of 1-CCNC scheme that satisfies the following condition

$$w_j = \begin{cases} \frac{h}{2}, & j = 0,N \\ h, & \text{otherwise} \end{cases}. \tag{4}$$

Following the conventional process, the generated 1-CCNC algebraic equations (3) can be written as the following matrix form

$$A\hat{\varphi} = f, \tag{5}$$

where $A = [a_{i,j}] \in \Re^{(N+1)\times(N+1)}$ is a real and dense coefficient matrix with elements

$$a_{i,j} = \begin{cases} 1 - w_j K_{i,j}, & i = j \\ -w_j K_{i,j}, & i \neq j \end{cases}. \tag{6}$$

III. 2-EGGS AND 2-EGSOR ITERATIVE METHODS

As afore-mentioned, the formulation and implementation of the 2-EGGS and 2-EGSOR methods for solving the generated 1-CCNC algebraic equations will be discussed. Now, let consider any group of two points i.e., x_i and x_{i+1} that are used simultaneously to calculate the values of $\hat{\varphi}$ based on algebraic equations (3). Therefore, at point x_i, the solution is approximated by

$$\hat{\varphi}_i - \sum_{j=0}^{N} w_j K_{i,j} \hat{\varphi}_j = f_i \text{ (i.e. equation (3)).} \tag{7}$$

Whereas, at point x_{i+1} the solution is given by

$$\hat{\varphi}_{i+1} - \sum_{j=0}^{N} w_j K_{i+1,j} \hat{\varphi}_j = f_{i+1}. \tag{8}$$

Now, the equations (7) and (8) can be written simultaneously in the matrix form as follows

$$\begin{bmatrix} a_{i,i} & a_{i,i+1} \\ a_{i+1,i} & a_{i+1,i+1} \end{bmatrix} \begin{bmatrix} \hat{\varphi}_i \\ \hat{\varphi}_{i+1} \end{bmatrix} = \begin{bmatrix} f_i - \sum_{j=0}^{i-1} a_{i,j}\hat{\varphi}_j - \sum_{j=i+2}^{N} a_{i,j}\hat{\varphi}_j \\ f_{i+1} - \sum_{j=0}^{i-1} a_{i+1,j}\hat{\varphi}_j - \sum_{j=i+2}^{N} a_{i+1,j}\hat{\varphi}_j \end{bmatrix} \tag{9}$$

where coefficient matrix with size (2×2) can be easily inverted. Thus, the equation (9) can be written in explicit form as

$$\begin{bmatrix} \hat{\varphi}_i \\ \hat{\varphi}_{i+1} \end{bmatrix} = \frac{1}{|B|} \begin{bmatrix} a_{i+1,i+1} & -a_{i,i+1} \\ -a_{i+1,i} & a_{i,i} \end{bmatrix} \begin{bmatrix} f_i - \sum_{j=0}^{i-1} a_{i,j}\hat{\varphi}_j - \sum_{j=i+2}^{N} a_{i,j}\hat{\varphi}_j \\ f_{i+1} - \sum_{j=0}^{i-1} a_{i+1,j}\hat{\varphi}_j - \sum_{j=i+2}^{N} a_{i+1,j}\hat{\varphi}_j \end{bmatrix} \tag{10}$$

where $|B| = \det B = (a_{i,i})(a_{i+1,i+1}) - (a_{i+1,i})(a_{i,i+1})$. This simplifies to the formulae

$$\begin{bmatrix} \hat{\varphi}_i \\ \hat{\varphi}_{i+1} \end{bmatrix} = \frac{1}{|B|} \begin{bmatrix} a_{i+1,i+1}(C) - a_{i,i+1}(D) \\ -a_{i+1,i}(C) + a_{i,i}(D) \end{bmatrix} \tag{11}$$

with

$$C = f_i - \sum_{j=0}^{i-1} a_{i,j}\hat{\varphi}_j - \sum_{j=i+2}^{N} a_{i,j}\hat{\varphi}_j \tag{12}$$

and

$$D = f_{i+1} - \sum_{j=0}^{i-1} a_{i+1,j} \hat{\varphi}_j - \sum_{j=i+2}^{N} a_{i+1,j} \hat{\varphi}_j . \qquad (13)$$

Hence, the iterative scheme for 2-EGGS method is given by

$$\begin{bmatrix} \hat{\varphi}_i \\ \hat{\varphi}_{i+1} \end{bmatrix}^{(k+1)} = \frac{1}{|B|} \begin{bmatrix} a_{i+1,i+1}(C) - a_{i,i+1}(D) \\ -a_{i+1,i}(C) + a_{i,i}(D) \end{bmatrix} \qquad (14)$$

for $i = 0,2,4,\cdots,N-3,N-1$, where

$$C = f_i - \sum_{j=0}^{i-1} a_{i,j} \hat{\varphi}_j^{(k+1)} - \sum_{j=i+2}^{N} a_{i,j} \hat{\varphi}_j^{(k)} \qquad (15)$$

and

$$D = f_{i+1} - \sum_{j=0}^{i-1} a_{i+1,j} \hat{\varphi}_j^{(k+1)} - \sum_{j=i+2}^{N} a_{i+1,j} \hat{\varphi}_j^{(k)} . \qquad (16)$$

By adding an accelerated parameter, ω into formulae (11), the iterative scheme for 2-EGSOR method can be rewritten as

$$\begin{bmatrix} \hat{\varphi}_i \\ \hat{\varphi}_{i+1} \end{bmatrix}^{(k+1)} = (1-\omega) \begin{bmatrix} \hat{\varphi}_i \\ \hat{\varphi}_{i+1} \end{bmatrix}^{(k)} + \frac{\omega}{|B|} \begin{bmatrix} a_{i+1,i+1}(C) - a_{i,i+1}(D) \\ -a_{i+1,i}(C) + a_{i,i}(D) \end{bmatrix} \qquad (17)$$

for $i = 0,2,4,\cdots,N-3,N-1$, where C and D are as shown in equations (15) and (16) respectively. When $\omega = 1$, the 2-EGSOR method reduced trivially to the 2-EGGS method.

For an even subintervals, N, the number of discrete node points is odd i.e., $N+1$, which results in one ungrouped point. Therefore, the ungrouped point i.e., x_N, will be computed based on the following point iterations

$$\hat{\varphi}_N^{(k+1)} = \frac{1}{a_{N,N}} \left[f_i - \sum_{j=0}^{N-1} \left(a_{N,j} \hat{\varphi}_j^{(k+1)} \right) \right] \qquad (18)$$

and

$$\hat{\varphi}_N^{(k+1)} = (1-\omega) \hat{\varphi}_N^{(k)} + \frac{\omega}{a_{N,N}} \left[f_i - \sum_{j=0}^{N-1} \left(a_{N,j} \hat{\varphi}_j^{(k+1)} \right) \right] \qquad (19)$$

for 2-EGGS and 2-EGSOR methods respectively. By considering formulations of 2-EGGS and 2-EGSOR methods, algorithm for both cases i.e. complete grouped (Case 1) and incomplete grouped (with one single point ungrouped) (Case 2) are described in Algorithms 1 and 2 respectively.

Algorithm 1: 2-EGGS and 2-EGSOR methods for Case 1

i. Set all the parameters
ii. Iteration cycle

for $i = 0,2,4,\cdots,N-3,N-1$

$$\begin{bmatrix} \hat{\varphi}_i \\ \hat{\varphi}_{i+1} \end{bmatrix}^{(k+1)} \leftarrow (1-\omega) \begin{bmatrix} \hat{\varphi}_i \\ \hat{\varphi}_{i+1} \end{bmatrix}^{(k)} + \frac{\omega}{|B|} \begin{bmatrix} a_{i+1,i+1}(C) - a_{i,i+1}(D) \\ -a_{i+1,i}(C) + a_{i,i}(D) \end{bmatrix}$$

iii. Convergence test. If the converge criterion i.e., the maximum norm $\left\| \hat{\varphi}^{(k+1)} - \hat{\varphi}^{(k)} \right\| \le \varepsilon$ (where ε is the convergence criterion) is satisfied, go to Step iv. Otherwise, go to Step ii.
iv. Stop.

Algorithm 2: 2-EGGS and 2-EGSOR methods for Case 2

i. Set all the parameters
ii. Iteration cycle

for $i = 0,2,4,\cdots,N-3,N-2$

$$\begin{bmatrix} \hat{\varphi}_i \\ \hat{\varphi}_{i+1} \end{bmatrix}^{(k+1)} \leftarrow (1-\omega) \begin{bmatrix} \hat{\varphi}_i \\ \hat{\varphi}_{i+1} \end{bmatrix}^{(k)} + \frac{\omega}{|B|} \begin{bmatrix} a_{i+1,i+1}(C) - a_{i,i+1}(D) \\ -a_{i+1,i}(C) + a_{i,i}(D) \end{bmatrix}$$

for $i = N$

$$\hat{\varphi}_i^{(k+1)} \leftarrow (1-\omega) \hat{\varphi}_i^{(k)} + \frac{\omega}{a_{i,i}} \left[f_i - \sum_{j=0}^{N-1} \left(a_{i,j} \hat{\varphi}_j^{(k+1)} \right) \right]$$

iii. Convergence test. If the converge criterion i.e., the maximum norm $\left\| \hat{\varphi}^{(k+1)} - \hat{\varphi}^{(k)} \right\| \le \varepsilon$ (where ε is the convergence criterion) is satisfied, go to Step iv. Otherwise, go to Step ii.
iv. Stop.

IV. Computational Complexity Analysis

An estimation amount of the computational work has been conducted in order to evaluate the computational complexity of 2-EGSOR method. The computational work is estimated by considering the arithmetic operations performed per iteration. In estimating the computational work, the values of $a_{i,j}$ in A and $|B|$ are stored beforehand. Based on Algorithm 1 (for Case 1), the total arithmetic operations (excluding the convergence test) involved for 2-EGSOR methods is

$$\left(N^2 + 2N + 1\right) ADD/SUB + \left(N^2 + 6N + 5\right) MUL/DIV$$

per iteration. Meanwhile, for Case 2, the number of arithmetic operations required is

$$\left(N^2 + 2N + 1\right) ADD / SUB + \left(N^2 + 6N + 3\right) MUL / DIV$$

per iteration. The ADD/SUB and MUL/DIV represent additions/subtractions and multiplications/divisions operations respectively.

V. Simulation Results

The following two linear Fredholm integral equations of the second kind are used as the test problems in order to compare the performance of the methods.

Test Problem 1 [1]
Consider the Fredholm integral equation of the second kind

$$\varphi(x) - \int_0^1 \left(4xt - x^2\right)\varphi(t)dt = x, \; x \in [0,1], \qquad (20)$$

and the exact solution is given by

$$\varphi(x) = 24x - 9x^2.$$

Test Problem 2 [8]
Consider the Fredholm integral equation of the second kind

$$\varphi(x) - \int_0^1 \left(x^2 + t^2\right)\varphi(t)dt = x^6 - 5x^3 + x + 10, \; x \in [0,1], \quad (21)$$

with the exact solution

$$\varphi(x) = x^6 - 5x^3 + \frac{1045}{28}x^2 + x + \frac{2141}{84}.$$

For the numerical simulations, three criteria are used for a comparative analysis i.e.

k - Number of iterations,
CPU - CPU time (in seconds) when the converged solution is obtained,
$RMSE$ - Root mean square error [13].

The value of initial datum, $\hat{\varphi}^{(0)}$, is set to zero for both the test problems. The computations are performed on a personal computer with Intel(R) Core(TM) i3-2120 CPU and 4.00GB RAM, and the programming codes are compiled by using C language. Throughout the simulations, the convergence test considered is $\varepsilon = 10^{-10}$ and tested on eight different values of N i.e. 60, 120, 240, 480, 960, 1920, 3840 and 7680. Meanwhile, the experimental values of ω were obtained within ± 0.01 by running the programs for different values of ω and choosing the one that gave the minimum number of iterations. For the case of more than one ω (based on minimum number of iterations), the optimum value of ω is chosen by considering the minimum $RMSE$. The numerical results of the tested methods for test problems 1 and 2 are presented in Tables I and II respectively.

TABLE I. Numerical Results of Test Problem 1

N	*Methods*	*k*	*CPU*	*RMSE*
60	2-EGGS	183	0.13	2.29894×10^{-02}
	2-EGSOR	40	0.04	2.29894×10^{-02}
		($\omega = 1.53$)		
120	2-EGGS	189	0.31	5.71079×10^{-03}
	2-EGSOR	40	0.06	5.71079×10^{-03}
		($\omega = 1.54$)		
240	2-EGGS	192	1.10	1.42375×10^{-03}
	2-EGSOR	40	0.25	1.42375×10^{-03}
		($\omega = 1.54$)		
480	2-EGGS	193	4.28	3.55480×10^{-04}
	2-EGSOR	41	0.94	3.55481×10^{-04}
		($\omega = 1.54$)		
960	2-EGGS	194	17.02	8.88150×10^{-05}
	2-EGSOR	41	3.69	8.88154×10^{-05}
		($\omega = 1.54$)		
1920	2-EGGS	194	67.92	2.21967×10^{-05}
	2-EGSOR	41	14.64	2.21972×10^{-05}
		($\omega = 1.54$)		
3840	2-EGGS	195	274.62	5.54797×10^{-06}
	2-EGSOR	41	58.48	5.54847×10^{-06}
		($\omega = 1.55$)		
7680	2-EGGS	195	1091.82	1.38651×10^{-06}
	2-EGSOR	41	235.13	1.38702×10^{-06}
		($\omega = 1.55$)		

TABLE II. Numerical Results of Test Problem 2

N	*Methods*	*k*	*CPU*	*RMSE*
60	2-EGGS	54	0.05	2.15612×10^{-02}
	2-EGSOR	23	0.02	2.15612×10^{-02}
		($\omega = 1.27$)		
120	2-EGGS	55	0.09	5.35413×10^{-03}
	2-EGSOR	23	0.05	5.35413×10^{-03}
		($\omega = 1.28$)		
240	2-EGGS	55	0.32	1.33417×10^{-03}
	2-EGSOR	23	0.14	1.33417×10^{-03}
		($\omega = 1.28$)		
480	2-EGGS	56	1.24	3.33005×10^{-04}
	2-EGSOR	23	0.56	3.33005×10^{-04}
		($\omega = 1.28$)		
960	2-EGGS	56	4.92	8.31845×10^{-05}
	2-EGSOR	23	2.06	8.31846×10^{-05}
		($\omega = 1.29$)		
1920	2-EGGS	56	19.81	2.07878×10^{-05}
	2-EGSOR	23	8.06	2.07879×10^{-05}
		($\omega = 1.29$)		
3840	2-EGGS	56	79.49	5.19584×10^{-06}
	2-EGSOR	23	31.89	5.19593×10^{-06}
		($\omega = 1.29$)		
7680	2-EGGS	56	317.22	1.29876×10^{-06}
	2-EGSOR	23	127.17	1.29885×10^{-06}
		($\omega = 1.29$)		

The following tables show the approximation solutions of φ at some discrete points for test problems 1 and 2.

TABLE III. NUMERICAL SOLUTIONS FOR CASE $N = 60$ OF TEST PROBLEM 1

x	*Exact Solution*	$2-EGGS$	$2-EGSOR$
0.00	0.00000000	0.00000000 (0.000000E+00)	0.00000000 (0.000000E+00)
0.10	2.31000000	2.31522864 (5.228640E-03)	2.31522864 (5.228640E-03)
0.20	4.44000000	4.45005637 (1.005637E-02)	4.45005637 1.005637E-02
0.30	6.39000000	6.40448317 (1.448317E-02)	6.40448317 (1.448317E-02)
0.40	8.16000000	8.17850906 (1.850906E-02)	8.17850906 (1.850906E-02)
0.50	9.75000000	9.77213403 (2.213403E-02)	9.77213403 (2.213403E-02)
0.60	11.16000000	11.18535808 (2.535808E-02)	11.18535808 (2.535808E-02)
0.70	12.39000000	12.41818121 (2.818121E-02)	12.41818121 (2.818121E-02)
0.80	13.44000000	13.47060342 (3.060342E-02)	13.47060342 (3.060342E-02)
0.90	14.31000000	14.34262471 (3.262471E-02)	14.34262471 (3.262471E-02)
1.00	15.00000000	15.03424509 (3.424509E-02)	15.03424509 (3.424509E-02)

(Value in the bracket shows an error of numerical solution)

TABLE IV. NUMERICAL SOLUTIONS FOR CASE $N = 120$ OF TEST PROBLEM 1

x	*Exact Solution*	$2-EGGS$	$2-EGSOR$
0.00	0.00000000	0.00000000 (0.000000E+00)	0.00000000 (0.000000E+00)
0.10	2.31000000	2.31130491 (1.304910E-03)	2.31130491 (1.304910E-03)
0.20	4.44000000	4.44250977 (2.509770E-03)	4.44250977 (2.509770E-03)
0.30	6.39000000	6.39361457 (3.614570E-03)	6.39361457 (3.614570E-03)
0.40	8.16000000	8.16461931 (4.619310E-03)	8.16461931 (4.619310E-03)
0.50	9.75000000	9.75552400 (5.524000E-03)	9.75552400 (5.524000E-03)
0.60	11.16000000	11.16632862 (6.328620E-03)	11.16632862 (6.328620E-03)
0.70	12.39000000	12.39703319 (7.033190E-03)	12.39703319 (7.033190E-03)
0.80	13.44000000	13.44763771 (7.637710E-03)	13.44763771 (7.637710E-03)
0.90	14.31000000	14.31814216 (8.142160E-03)	14.31814216 (8.142160E-03)
1.00	15.00000000	15.00854656 (8.546560E-03)	15.00854656 (8.546560E-03)

(Value in the bracket shows an error of numerical solution)

TABLE V. NUMERICAL SOLUTIONS FOR CASE $N = 240$ OF TEST PROBLEM 1

x	*Exact Solution*	$2-EGGS$	$2-EGSOR$
0.00	0.00000000	0.00000000 (0.000000E+00)	0.00000000 (0.000000E+00)
0.10	2.31000000	2.31032609 (3.260900E-04)	2.31032609 (3.260900E-04)
0.20	4.44000000	4.44062717 (6.271700E-04)	4.44062717 (6.271700E-04)
0.30	6.39000000	6.39090325 (9.032500E-04)	6.39090325 (9.032500E-04)
0.40	8.16000000	8.16115433 (1.154330E-03)	8.16115433 (1.154330E-03)
0.50	9.75000000	9.75138041 (1.380410E-03)	9.75138041 (1.380410E-03)
0.60	11.16000000	11.16158148 (1.581480E-03)	11.16158148 (1.581480E-03)
0.70	12.39000000	12.39175754 (1.757540E-03)	12.39175754 (1.757540E-03)
0.80	13.44000000	13.44190861 (1.908610E-03)	13.44190861 (1.908610E-03)
0.90	14.31000000	14.31203467 (2.034670E-03)	14.31203467 (2.034670E-03)
1.00	15.00000000	15.00213572 (2.135720E-03)	15.00213572 (2.135720E-03)

(Value in the bracket shows an error of numerical solution)

TABLE VI. NUMERICAL SOLUTIONS FOR CASE $N = 480$ OF TEST PROBLEM 1

x	*Exact Solution*	$2-EGGS$	$2-EGSOR$
0.00	0.00000000	0.00000000 (0.000000E+00)	0.00000000 (0.000000E+00)
0.10	2.31000000	2.31008151 (8.151000E-05)	2.31008151 (8.151000E-05)
0.20	4.44000000	4.44015678 (1.567800E-04)	4.44015678 (1.567800E-04)
0.30	6.39000000	6.39022579 (2.257900E-04)	6.39022579 (2.257900E-04)
0.40	8.16000000	8.16028855 (2.885500E-04)	8.16028855 (2.885500E-04)
0.50	9.75000000	9.75034506 (3.450600E-04)	9.75034506 (3.450600E-04)
0.60	11.16000000	11.16039533 (3.953300E-04)	11.16039533 (3.953300E-04)
0.70	12.39000000	12.39043934 (4.393400E-04)	12.39043934 (4.393400E-04)
0.80	13.44000000	13.44047710 (4.771000E-04)	13.44047710 (4.771000E-04)
0.90	14.31000000	14.31050861 (5.086100E-04)	14.31050861 (5.086100E-04)
1.00	15.00000000	15.00053387 (5.338700E-04)	15.00053387 (5.338700E-04)

(Value in the bracket shows an error of numerical solution)

TABLE VII. NUMERICAL SOLUTIONS FOR CASE $N = 960$ OF TEST PROBLEM 1

x	*Exact Solution*	$2-EGGS$	$2-EGSOR$
0.00	0.00000000	0.00000000 (0.000000E+00)	0.00000000 (0.000000E+00)
0.10	2.31000000	2.31002038 (2.038000E-05)	2.31002038 (2.038000E-05)
0.20	4.44000000	4.44003919 (3.919000E-05)	4.44003919 (3.919000E-05)
0.30	6.39000000	6.39005645 (5.645000E-05)	6.39005645 (5.645000E-05)
0.40	8.16000000	8.16007214 (7.214000E-05)	8.16007214 (7.214000E-05)
0.50	9.75000000	9.75008626 (8.626000E-05)	9.75008626 (8.626000E-05)
0.60	11.16000000	11.16009883 (9.883000E-05)	11.16009883 (9.883000E-05)
0.70	12.39000000	12.39010983 (1.098300E-04)	12.39010983 (1.098300E-04)
0.80	13.44000000	13.44011927 (1.192700E-04)	13.44011927 (1.192700E-04)
0.90	14.31000000	14.31012715 (1.271500E-04)	14.31012715 (1.271500E-04)
1.00	15.00000000	15.00013346 (1.334600E-04)	15.00013346 (1.334600E-04)

(Value in the bracket shows an error of numerical solution)

TABLE VIII. NUMERICAL SOLUTIONS FOR CASE $N = 1920$ OF TEST PROBLEM 1

x	*Exact Solution*	$2-EGGS$	$2-EGSOR$
0.00	0.00000000	0.00000000 (0.000000E+00)	0.00000000 (0.000000E+00)
0.10	2.31000000	2.31000509 (5.090000E-06)	2.31000509 (5.090000E-06)
0.20	4.44000000	4.44000980 (9.800000E-06)	4.44000980 (9.800000E-06)
0.30	6.39000000	6.39001411 (1.411000E-05)	6.39001411 (1.411000E-05)
0.40	8.16000000	8.16001803 (1.803000E-05)	8.16001803 (1.803000E-05)
0.50	9.75000000	9.75002157 (2.157000E-05)	9.75002157 (2.157000E-05)
0.60	11.16000000	11.16002471 (2.471000E-05)	11.16002471 (2.471000E-05)
0.70	12.39000000	12.39002746 (2.746000E-05)	12.39002746 (2.746000E-05)
0.80	13.44000000	13.44002982 (2.982000E-05)	13.44002982 (2.982000E-05)
0.90	14.31000000	14.31003179 (3.179000E-05)	14.31003179 (3.179000E-05)
1.00	15.00000000	15.00003337 (3.337000E-05)	15.00003337 (3.337000E-05)

(Value in the bracket shows an error of numerical solution)

TABLE IX. NUMERICAL SOLUTIONS FOR CASE $N = 3840$ OF TEST PROBLEM 1

x	*Exact Solution*	$2-EGGS$	$2-EGSOR$
0.00	0.00000000	0.00000000 (0.000000E+00)	0.00000000 (0.000000E+00)
0.10	2.31000000	2.31000127 (1.270000E-06)	2.31000127 (1.270000E-06)
0.20	4.44000000	4.44000245 (2.450000E-06)	4.44000245 (2.450000E-06)
0.30	6.39000000	6.39000353 (3.530000E-06)	6.39000353 (3.530000E-06)
0.40	8.16000000	8.16000451 (4.510000E-06)	8.16000451 (4.510000E-06)
0.50	9.75000000	9.75000539 (5.390000E-06)	9.75000539 (5.390000E-06)
0.60	11.16000000	11.16000618 (6.180000E-06)	11.16000618 (6.180000E-06)
0.70	12.39000000	12.39000686 (6.860000E-06)	12.39000686 (6.860000E-06)
0.80	13.44000000	13.44000745 (7.450000E-06)	13.44000745 (7.450000E-06)
0.90	14.31000000	14.31000795 (7.950000E-06)	14.31000795 (7.950000E-06)
1.00	15.00000000	15.00000834 (8.340000E-06)	15.00000834 (8.340000E-06)

(Value in the bracket shows an error of numerical solution)

TABLE X. NUMERICAL SOLUTIONS FOR CASE $N = 7680$ OF TEST PROBLEM 1

x	*Exact Solution*	$2-EGGS$	$2-EGSOR$
0.00	0.00000000	0.00000000 (0.000000E+00)	0.00000000 (0.000000E+00)
0.10	2.31000000	2.31000032 (3.200000E-07)	2.31000032 (3.200000E-07)
0.20	4.44000000	4.44000061 (6.100000E-07)	4.44000061 (6.100000E-07)
0.30	6.39000000	6.39000088 (8.800000E-07)	6.39000088 (8.800000E-07)
0.40	8.16000000	8.16000113 (1.130000E-06)	8.16000113 (1.130000E-06)
0.50	9.75000000	9.75000135 (1.350000E-06)	9.75000135 (1.350000E-06)
0.60	11.16000000	11.16000154 (1.540000E-06)	11.16000154 (1.540000E-06)
0.70	12.39000000	12.39000172 (1.720000E-06)	12.39000172 (1.720000E-06)
0.80	13.44000000	13.44000186 (1.860000E-06)	13.44000186 (1.860000E-06)
0.90	14.31000000	14.31000199 (1.990000E-06)	14.31000199 (1.990000E-06)
1.00	15.00000000	15.00000208 (2.080000E-06)	15.00000209 (2.090000E-06)

(Value in the bracket shows an error of numerical solution)

TABLE XI. NUMERICAL SOLUTIONS FOR CASE $N = 60$ OF TEST PROBLEM 2

x	*Exact Solution*	$2-EGGS$	$2-EGSOR$
0.00	25.48809524	25.50110512 (1.300988E-02)	25.50110512 (1.300988E-02)
0.10	25.95631052	25.96953836 (1.322784E-02)	25.96953836 (1.322784E-02)
0.20	27.14101638	27.15489809 (1.388171E-02)	27.15489809 (1.388171E-02)
0.30	29.01275281	29.02772430 (1.497149E-02)	29.02772430 (1.497149E-02)
0.40	31.54361981	31.56011699 (1.649718E-02)	31.56011699 (1.649718E-02)
0.50	34.70907738	34.72753616 (1.845878E-02)	34.72753616 (1.845878E-02)
0.60	38.49046552	38.51132182 (2.085630E-02)	38.51132182 (2.085630E-02)
0.70	42.87824424	42.90193396 (2.368972E-02)	42.90193396 (2.368972E-02)
0.80	47.87595352	47.90291258 (2.695906E-02)	47.90291258 (2.695906E-02)
0.90	53.50489338	53.53555769 (3.066431E-02)	53.53555769 (3.066431E-02)
1.00	59.80952381	59.84432928 (3.480547E-02)	59.84432928 (3.480547E-02)

(Value in the bracket shows an error of numerical solution)

TABLE XII. NUMERICAL SOLUTIONS FOR CASE $N = 120$ OF TEST PROBLEM 2

x	*Exact Solution*	$2-EGGS$	$2-EGSOR$
0.00	25.48809524	25.49134627 (3.251030E-03)	25.49134627 (3.251030E-03)
0.10	25.95631052	25.95961602 (3.305500E-03)	25.95961602 (3.305500E-03)
0.20	27.14101638	27.14448527 (3.468890E-03)	27.14448527 (3.468890E-03)
0.30	29.01275281	29.01649402 (3.741210E-03)	29.01649402 (3.741210E-03)
0.40	31.54361981	31.54774227 (4.122460E-03)	31.54774227 (4.122460E-03)
0.50	34.70907738	34.71369002 (4.612640E-03)	34.71369002 (4.612640E-03)
0.60	38.49046552	38.49567728 (5.211760E-03)	38.49567728 (5.211760E-03)
0.70	42.87824424	42.88416403 (5.919790E-03)	42.88416403 (5.919790E-03)
0.80	47.87595352	47.88269028 (6.736760E-03)	47.88269028 (6.736760E-03)
0.90	53.50489338	53.51255604 (7.662660E-03)	53.51255604 (7.662660E-03)
1.00	59.80952381	59.81822129 (8.697480E-03)	59.81822129 (8.697480E-03)

(Value in the bracket shows an error of numerical solution)

TABLE XIII. NUMERICAL SOLUTIONS FOR CASE $N = 240$ OF TEST PROBLEM 2

x	*Exact Solution*	$2-EGGS$	$2-EGSOR$
0.00	25.48809524	25.48890791 (8.126700E-04)	25.48890791 (8.126700E-04)
0.10	25.95631052	25.95713681 (8.262900E-04)	25.95713681 (8.262900E-04)
0.20	27.14101638	27.14188351 (8.671300E-04)	27.14188351 (8.671300E-04)
0.30	29.01275281	29.01368801 (9.352000E-04)	29.01368801 (9.352000E-04)
0.40	31.54361981	31.54465031 (1.030500E-03)	31.54465031 (1.030500E-03)
0.50	34.70907738	34.71023041 (1.153030E-03)	34.71023041 (1.153030E-03)
0.60	38.49046552	38.49176832 (1.302800E-03)	38.49176832 (1.302800E-03)
0.70	42.87824424	42.87972402 (1.479780E-03)	42.87972402 (1.479780E-03)
0.80	47.87595352	47.87763753 (1.684010E-03)	47.87763753 (1.684010E-03)
0.90	53.50489338	53.50680883 (1.915450E-03)	53.50680883 (1.915450E-03)
1.00	59.80952381	59.81169794 (2.174130E-03)	59.81169794 (2.174130E-03)

(Value in the bracket shows an error of numerical solution)

TABLE XIV. NUMERICAL SOLUTIONS FOR CASE $N = 480$ OF TEST PROBLEM 2

x	*Exact Solution*	$2-EGGS$	$2-EGSOR$
0.00	25.48809524	25.48829840 (2.031600E-04)	25.48829840 (2.031600E-04)
0.10	25.95631052	25.95651709 (2.065700E-04)	25.95651709 (2.065700E-04)
0.20	27.14101638	27.14123316 (2.167800E-04)	27.14123316 (2.167800E-04)
0.30	29.01275281	29.01298660 (2.337900E-04)	29.01298660 (2.337900E-04)
0.40	31.54361981	31.54387743 (2.576200E-04)	31.54387743 (2.576200E-04)
0.50	34.70907738	34.70936563 (2.882500E-04)	34.70936563 (2.882500E-04)
0.60	38.49046552	38.49079121 (3.256900E-04)	38.49079121 (3.256900E-04)
0.70	42.87824424	42.87861417 (3.699300E-04)	42.87861417 (3.699300E-04)
0.80	47.87595352	47.87637451 (4.209900E-04)	47.87637451 (4.209900E-04)
0.90	53.50489338	53.50537223 (4.788500E-04)	53.50537223 (4.788500E-04)
1.00	59.80952381	59.81006733 (5.435200E-04)	59.81006733 (5.435200E-04)

(Value in the bracket shows an error of numerical solution)

TABLE XV. NUMERICAL SOLUTIONS FOR CASE $N = 960$ OF TEST PROBLEM 2

x	*Exact Solution*	$2-EGGS$	$2-EGSOR$
0.00	25.48809524	25.48814603 (5.079000E-05)	25.48814603 (5.079000E-05)
0.10	25.95631052	25.95636216 (5.164000E-05)	25.95636216 (5.164000E-05)
0.20	27.14101638	27.14107057 (5.419000E-05)	27.14107057 (5.419000E-05)
0.30	29.01275281	29.01281126 (5.845000E-05)	29.01281126 (5.845000E-05)
0.40	31.54361981	31.54368421 (6.440000E-05)	31.54368421 (6.440000E-05)
0.50	34.70907738	34.70914944 (7.206000E-05)	34.70914944 (7.206000E-05)
0.60	38.49046552	38.49054695 (8.143000E-05)	38.49054695 (8.143000E-05)
0.70	42.87824424	42.87833672 (9.248000E-05)	42.87833672 (9.248000E-05)
0.80	47.87595352	47.87605877 (1.052500E-04)	47.87605877 (1.052500E-04)
0.90	53.50489338	53.50501309 (1.197100E-04)	53.50501309 (1.197100E-04)
1.00	59.80952381	59.80965969 (1.358800E-04)	59.80965969 (1.358800E-04)

(Value in the bracket shows an error of numerical solution)

TABLE XVI. NUMERICAL SOLUTIONS FOR CASE $N = 1920$ OF TEST PROBLEM 2

x	*Exact Solution*	$2-EGGS$	$2-EGSOR$
0.00	25.48809524	25.48810794 (1.270000E-05)	25.48810794 (1.270000E-05)
0.10	25.95631052	25.95632343 (1.291000E-05)	25.95632343 (1.291000E-05)
0.20	27.14101638	27.14102993 (1.355000E-05)	27.14102993 (1.355000E-05)
0.30	29.01275281	29.01276742 (1.461000E-05)	29.01276742 (1.461000E-05)
0.40	31.54361981	31.54363591 (1.610000E-05)	31.54363591 (1.610000E-05)
0.50	34.70907738	34.70909540 (1.802000E-05)	34.70909540 (1.802000E-05)
0.60	38.49046552	38.49048588 (2.036000E-05)	38.49048588 (2.036000E-05)
0.70	42.87824424	42.87826736 (2.312000E-05)	42.87826736 (2.312000E-05)
0.80	47.87595352	47.87597984 (2.632000E-05)	47.87597984 (2.632000E-05)
0.90	53.50489338	53.50492331 (2.993000E-05)	53.50492331 (2.993000E-05)
1.00	59.80952381	59.80955778 (3.397000E-05)	59.80955778 (3.397000E-05)

(Value in the bracket shows an error of numerical solution)

TABLE XVII. NUMERICAL SOLUTIONS FOR CASE $N = 3840$ OF TEST PROBLEM 2

x	*Exact Solution*	$2-EGGS$	$2-EGSOR$
0.00	25.48809524	25.48809841 (3.170000E-06)	25.48809841 (3.170000E-06)
0.10	25.95631052	25.95631375 (3.230000E-06)	25.95631375 (3.230000E-06)
0.20	27.14101638	27.14101977 (3.390000E-06)	27.14101977 (3.390000E-06)
0.30	29.01275281	29.01275646 (3.650000E-06)	29.01275646 (3.650000E-06)
0.40	31.54361981	31.54362383 (4.020000E-06)	31.54362383 (4.020000E-06)
0.50	34.70907738	34.70908188 (4.500000E-06)	34.70908188 (4.500000E-06)
0.60	38.49046552	38.49047061 (5.090000E-06)	38.49047061 (5.090000E-06)
0.70	42.87824424	42.87825002 (5.780000E-06)	42.87825002 (5.780000E-06)
0.80	47.87595352	47.87596010 (6.580000E-06)	47.87596010 (6.580000E-06)
0.90	53.50489338	53.50490086 (7.480000E-06)	53.50490086 (7.480000E-06)
1.00	59.80952381	59.80953230 (8.490000E-06)	59.80953230 (8.490000E-06)

(Value in the bracket shows an error of numerical solution)

TABLE XVIII. NUMERICAL SOLUTIONS FOR CASE $N = 7680$ OF TEST PROBLEM 2

x	*Exact Solution*	$2-EGGS$	$2-EGSOR$
0.00	25.48809524	25.48809603 (7.900000E-07)	25.48809603 (7.900000E-07)
0.10	25.95631052	25.95631133 (8.100000E-07)	25.95631133 (8.100000E-07)
0.20	27.14101638	27.14101723 (8.500000E-07)	27.14101723 (8.500000E-07)
0.30	29.01275281	29.01275372 (9.100000E-07)	29.01275372 (9.100000E-07)
0.40	31.54361981	31.54362082 (1.010000E-06)	31.54362082 (1.010000E-06)
0.50	34.70907738	34.70907851 (1.130000E-06)	34.70907851 (1.130000E-06)
0.60	38.49046552	38.49046680 (1.280000E-06)	38.49046680 (1.280000E-06)
0.70	42.87824424	42.87824568 (1.440000E-06)	42.87824568 (1.440000E-06)
0.80	47.87595352	47.87595517 (1.650000E-06)	47.87595517 (1.650000E-06)
0.90	53.50489338	53.50489525 (1.870000E-06)	53.50489525 (1.870000E-06)
1.00	59.80952381	59.80952593 (2.120000E-06)	59.80952593 (2.120000E-06)

(Value in the bracket shows an error of numerical solution)

VI. Conclusion

In this paper, 2-EGSOR method has been successfully applied in solving linear Fredholm integral equations of the second kind. By referring Tables I and II, the numerical results show that implementation of the 2-EGSOR method solved the both test problems with minimum number of iterations and fastest CPU time. In terms of accuracy, numerical solutions obtained via 2-EGSOR method are in good agreement compared to the 2-EGGS method. Through the observation in Tables III to XVIII, increment of N improved the accuracy of numerical solutions and maximum error of the numerical solution occurred at point $x = 1.00$ for both test problems. Finally, it can be summarized that the 2-EGSOR method is better than 2-EGGS method, especially in the aspect of number of iterations and CPU time.

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