Robust Stability Analysis for Affine Linear Plants with Time-Delay Using the Value Set Concept

Silvia Florida, Gerardo Romero, Ramiro Ibarra, David Lara, Irma Pérez, Aldo Méndez and Alberto Reyna

***Abstract*—In this paper we present a new computational algorithm for the robust stability analysis of linear systems with time delay with affine linear uncertainty. This method evaluates a family of polynomials through slope tracing whose extreme points will form the value set. With this value set and the zero exclusion principle we can verify the robuts stability of the system.**

***Index Terms*—Robust stability, time-delay systems, affine lineal uncertainty, value set.**

I. INTRODUCTION

THE majority of the control systems usually have components that generate time delay. These time delays along with uncertain parameters may affect the stability of the system. In the last years, the researchers has been more interested in the analysis of robust stability for linear systems with the characteristics mentioned before, see [9][15][16][17]. The criteria of system stability with time delay are classified in two types: those of independent delay that tend to be conservative, specially when the size of a delay is small, see [10][11]. And those of dependent delay which are worried by the size of the delay, see [5][8][14].

The analysis of a system's stability with time delay can be applied made in time domain (variables of state) or in the domain of frequency (functions of transfer). Many important results are presented in the domain of frequency for linear systems considering uncertainty, the uncertainty as well can be of two types: dynamic uncertainty [6] or parametric uncertainty[1][2]. Inside the parametric uncertainty we can find uncertainty of type interval, multilinear, polynomial and affine linear uncertainty. For each of these uncertainties there are methods of analyses that guarantee the property of robust stability.

The robust stability of linear systems with affine linear uncertainty was solved by [3] using the theorem of edges; another method to solve these systems was prsented by [4] who developed a computational algorithm to generate the value set formed by segments of line or elliptical arches, In [13] they proposed a new approach based on the analysis of Nyquist's stability to find the margins of robust stability for these systems, in [7] they preseted a lema condition kwons critical condition which presented a sufficient and necessary condition for the group of roots of a convex combination of complex polynomials, in [12] they evaluated the stability and sensibility radii, in [18] shows a computational procedure for the calculation of the Bode's convex hull, in [19] it shows a computational technology for the obtaining of the value set using the theorem of edges.

In this paper we will developped a computational algorithm to guarantee the property of robust stability in linear systems with affine linear uncertainty considering time delay. This result is based on the characterization of the value set. The present work is organized as follows: in section 2 the problem statement will be presented; in section 3 we include some preliminary results; the main result will be presented in section 4;a example in section 5 and finally in secction 6 the conclusion.

The authors are with the Department of Electrical and Electronic Engineering, Universidad Autonoma de Tamaulipas, Reynosa, Tamaulipas, 88779 Mexico e-mails: silvia.florida@uat.edu.mx; gromero@uat.edu.mx

II. PROBLEM STATEMENT

The analysis presented in this paper is performed for systems with affine linear uncertainty and time delay define as follows:

Definition 1: An uncertain polynomial $p(s,q) = \sum_{i=0}^{n} a_i(q)s^i$ to have an affine linear uncertainty structure if each coefficient function $a_i(q)$ is a linear function of q; for each $i \in 0,1,2,...,n$ there exists a column vector α_i and a scalar β_i, so such that

$$a_i(q) = \alpha_i^T + \beta_i \tag{1}$$

transfer function affine linear uncertainty has the following structure: [2]

$$P(s,q) = \frac{N_0(s) + \sum_{i=1}^{l} q_i N_i(s)}{D_0(s) + \sum_{i=1}^{l} q_i D_i(s)}, \quad q \in Q \tag{2}$$

were $Q = q \in R^l \mid \underline{q_i} \leq q_i \leq \overline{q_i},\ i = 1,2,...,l$ and $q_i = [q_1, q_2, ..., q_l]^T$. Her $N_i(s)$ and l denotes the number of uncertainties.

Definition 2: A system with affine linear uncertainty and time delay has the following structure:

$$P(s,q,e^{-\tau s}) = \frac{[N_0(s) + \sum_{i=1}^{l} q_i N_i(s)]e^{-\tau s}}{D_0(s) + \sum_{i=1}^{l} q_i D_i(s)} \tag{3}$$

$\forall\ q \in Q;\ \tau \in [0, \tau_{max}]Q$

This work focussed in the analysis of robust stability of control systems represented with the followins block diagram.

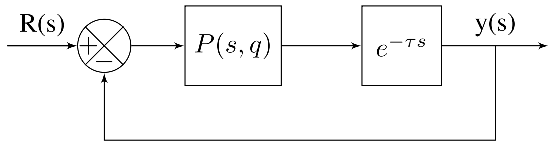

Fig. 1. Closed-loop system for the affine linear system with time-delay.

The property of robust stability is determined in terms of the following typical equation:

$$P(s,q,e^{-\tau s}) = d(s,q) + n(s,q)e^{-\tau s} \quad (4)$$

The former characteristic equation (4) represents an infinite amount of characteristics equations that have to be considered to verify the robust stability property; this family will be defined as follows:

$$P_\tau \equiv \{p(s,q,e^{-\tau s}) : q \in Q; \tau \in [0, \tau_{max}]\} \quad (5)$$

The objective of this work is to present an algorithm to verify the robust stability property of time delay systems as the ones shown in figure 1.

III. Preliminary results

The result presented in this paper is based on the value set characterization of the family of polynomials with time delay affine linear uncertainty, the value set can be defined as:

Definition 3: The value set of P_τ, noted by $V_\tau(\omega)$, is the graph in the complex plane of $p(s,q,e^{-\tau s})$ when $s = j\omega$ is substituted; this is:

$$V_{\tau(\omega)} = \{p(s,q,e^{-\tau s}) : q \in Q\}; \quad (6)$$

$\tau \in [0, \tau_m ax]; \omega \in R$

The value set of a polynomial with affine linear uncertainty is not a convex polygon. Therefore, the value set is contained inside the convex hull hence we consider the following statement:

A politope P in R^k is the convex hull of a set of points $p^1, p^2, ..., p^m$. we write

$$P = conv p^i$$

To form the value set we need to consider the extreme points of $P = conv\{p^i\}$, if P results an extreme point then does not exist $p^a, p^b \in P$ with $p^a \neq p^b$ and $\lambda \in (0,1)$ such that $\lambda p^a + (1-\lambda)p^b = p$.

Given a family of polynomials $P = p(\cdot,q) : q \in Q$ is said to be a polytope of polynomials if $p(s,q)$ has an affine linear uncertainty structure and Q is a polytope. If $Q = conv\{q^i\}$ then we call $p(s,q^i)$ the i-th generator for P.

The value set of P_τ is a set of complex numbers plotted on the complex plane. We can apply equation (4) to get the value set which forms next figure:

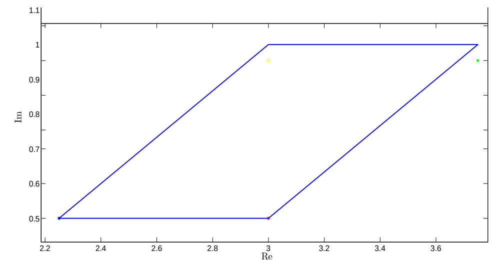

Fig. 2. Value set when the time-delay is equal to zero.

Let as consider two values set N(s) and D(s) when they aligned, they be have as shown in Fig. 3.

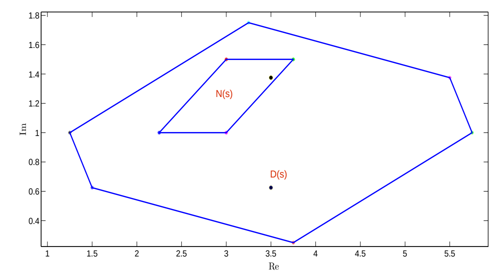

Fig. 3. Value set alineados.

The sum of two aligned value sets N(s) and D(s) it is shown in the Fig. 4.

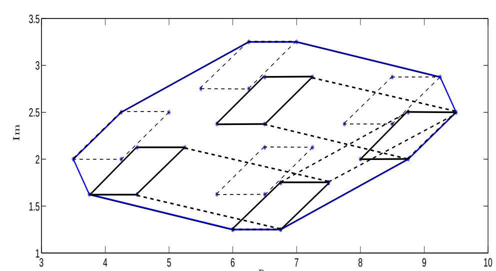

Fig. 4. Sum of two aligned Value set.

When the value set is affected by a time delay it rotates as we can see in Fig. 5.

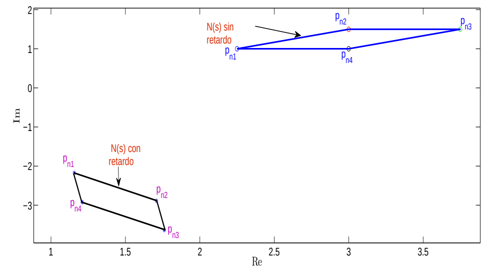

Fig. 5. Value set with time-delay.

To verify the robust stability property we applied the zero exclusion principle, which we can be define as:

Theorem 1: An family of polynomials $P(\cdot, Q)$ of invariant degree with continuous coefficient $a_i(q)$ for $i = 0, 1, 2, \cdots, n$ and at least one stable member, were Q is an convex of arco-convex. Then the family $P(\cdot, Q)$ is robustly stable if and only if $P(j\omega, Q)$ it contains no zero for all $\omega \geq 0$. Then the robust stability property of the control system shown in Fig. 1 is guaranteed if and only if:

$$0 \neq V_\tau(\omega); \quad \forall \omega \geq 0 \tag{7}$$

IV. Main result

In this section we will present a computational algorithm to create for a value set. This method evaluates the polynomial of Eq.(4) for $\omega \geq 0$ and $\tau \geq 0$ obtaining 2^q Polynomials considering q as an uncertain parameters.

The algorithm starts identifying the set point that is formed by the smallest and largest real part as follows:

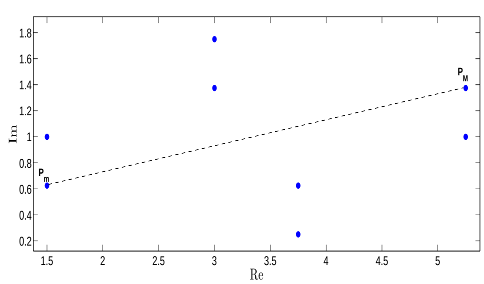

Fig. 6. Location of points smallest and largest real part.

After this, the starting point is going to be the point with the smallest real part P_m, and by drawing line segments we find the point with the greatest slope P_v, this point becomes the new starting point and the process is repeated in order to find the extreme points over and under the imaginary line between P_m and P_M, see Fig. 6 and Fig. 7.

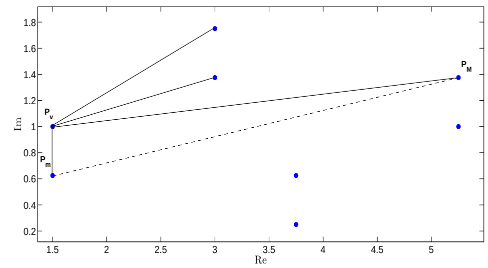

Fig. 7. Slope trace from the point of minor real part.

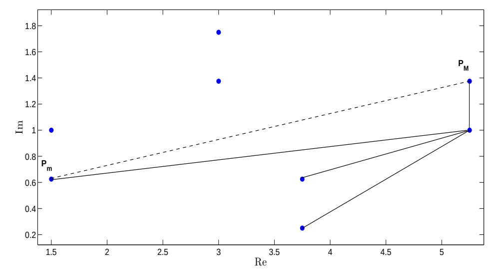

Fig. 8. Slope trace from the point of major real part.

When there are located the extreme points of set points the value set is formed, see Fig. 9.

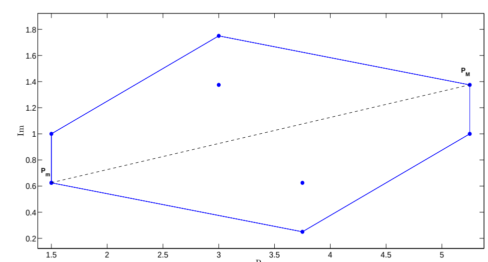

Fig. 9. Value set $V(\omega)$.

When the value set is affected with time delay the value set rotates as mentioned earlier, which generates that the extreme points are not always the same, nevertheless means of slope trace is possible find these extreme points.

V. Numerical example

The intention of this example is to show the obtaining of the value set for the following system with related linear uncertainty considering three uncertain parameters to be given for:

$$P(s, q, e^{-\tau s}) = \frac{(q_1 + q_2)s^2 + (q_1)s + (q_1 + q_2)e^{-\tau s}}{(q_1 + q_2 + q_3)s^3 + (q_2 + q_3)s^2 + (q_1 + q_2)s + (q_2 + q_3)}$$

where

$q_1 \in [1, 2], \quad q_2 \in [1, 3], \quad q_3 \in [1, 4]$

evaluating $N(s)$ and $D(s)$ with $\omega = (0.5 : 0.1 : 5)$ With time delay of $\tau \in [0, 2]$ is obtained:

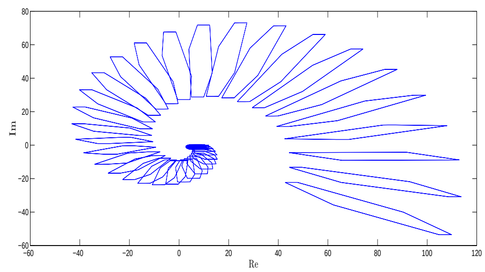

Fig. 10. Value set with time delay fixed.

Applying the beginning zero exclusion condition it is possible to prove the robust stability. In the figure 1.5 can observe that this beginning is valid since " value set " it does not touch to the zero.

Now let's consider the same frequency of $\omega \in [0, 5]$ with intervals of 0.1 And time delay of $\tau \in [0, 2]$ with intervals of 0.1 the following figure is obtained: Applying

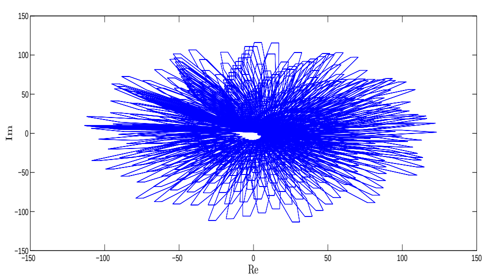

Fig. 11. Value set with variant time delay.

the zero exclusion principe it can be determined by visual inspection that the system is robustly stable and the robust stability property is garantized.

VI. Conclusion

This paper presented an algorithm that evaluates the affine linear plant of order n and constructs the value set by means of slope trace to locate the extreme points that form the value set, having considered time delay in the plant allows us to guarantee the property of robust stability of systems.

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