Frequency transform basis with even symmetry elements

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Abstract—The frequency transform modification is proposed. It calculated with different basis functions. This approach considered on the digital cosine transform as simplest example. We compared results of the transform in forward and reverse data index orders. The differences are presented. The proposed algorithm in which a input data set is divided into two equal parts and transform is performed independently in both directions from the center. It allows to eliminate changing values of amplitude and sign. This approach applies to two-dimensional signal. It simplifies a calculation for different applications.

Keywords—digital signal processing, frequency representation, discrete cosine transform, even symmetry, reverse order.

I. Introduction

A serious disadvantage of the discrete Fourier transform for significant number of applications is that when converting complex values are obtained even if the data has only real part. Devoid of this disadvantage, in particular, the discrete cosine transform. This cosine transform is a separate transformation, not just the real part of the Fourier transform [1]. It is widely used in graphic and video applications for data compression coding systems, such as JPEG, MPEG, etc. There are a number of transformations are similar in essence to the DCT [1, 2].

II. The blocked kind of digital cosine transform

The determination DCT is used y (n) - symmetrical expansion of signal samples x (n). This is not the only way to expand the signal, but this method leads to the form most widely used DCT (DCT-2). It has connection with the best results on the energy performance [1, 4]. Consequently, the direct discrete cosine transform will have the form

$$X_c(k) = \sqrt{2/N}\,C(k)\sum_{n=0}^{N-1} x(n)\cos\bigl(\pi(2n+1)k/2N\bigr) \qquad (1)$$

and inverse transform

$$x(n) = \sqrt{2/N}\sum_{k=0}^{N-1} C(k)\,X_c(k)\cos\bigl(\pi(2n+1)k/2N\bigr), \qquad (2)$$

where

$$C(k) = \begin{cases} 1 & ,k \neq 0 \\ 1/\sqrt{2} & ,k = 0 \end{cases}.$$

The expressions (1) and (2) shows each even basis function is symmetric relatively the mid-range, and each is an odd anti-symmetric. The one-dimensional signal processing direction change leads to change sign of each odd element to the opposite value. The expression (1) takes the following form.

$$X_c(k) = \sqrt{2/N}\,C(k)\sum_{n=0}^{N-1} x(N-(n+1))\cos\bigl(\pi(2n+1)k/2N\bigr) \qquad (3)$$

The frequency components indexes k corresponds to the expressions (1) and (3) in direct k_F and reverse k_R placements are related

$$k_F = N - (k_R + 1)\,.$$

The change direction process effect can be achieved also by changing the direction of basic functions. It derived from the convolution properties or matrix multiplication properties.

$$X_c(k) = \sqrt{2/N}\,C(k)\sum_{n=0}^{N-1} x(n)\cos\bigl(\pi(2N-(2n+1))k/2N\bigr) \qquad (4)$$

There are important only antisymmetric components, because of which the sign of the received DCT components.

A full signal processing in parts (blocks processing) demands in practice.

Two blocks DCT may be calculated as follows:

$$\begin{aligned} X1_c(k) &= \sqrt{4/N}\,C(k)\sum_{n=0}^{N/2-1} x(n)\cos\bigl(\pi(2n+1)k/N\bigr) \\ X2_c(k) &= \sqrt{4/N}\,C(k)\sum_{n=N/2}^{N-1} x(n)\cos\bigl(\pi((2n+1)-N)k/N\bigr) \end{aligned} \qquad (5)$$

где *k=0, …, N/2-1*.

These components are combined in the resulting sequence

$$X_c = \bigl(X1_c, X2_c\bigr)\,. \qquad (6)$$

III. The blocked kind of digital cosine transform in modified forms

Using the approach proposed [4, 5], for two blocks of the processed signal consider two groups of basic functions. There are 2 types – time-reverse the first or second set of basis functions.

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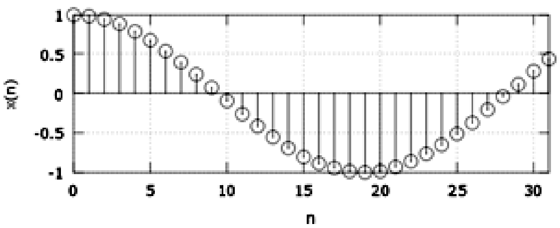

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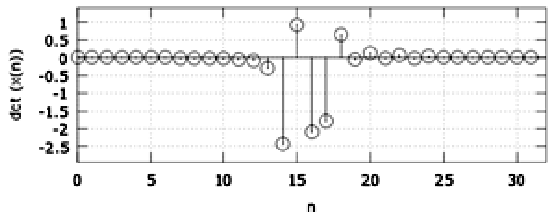

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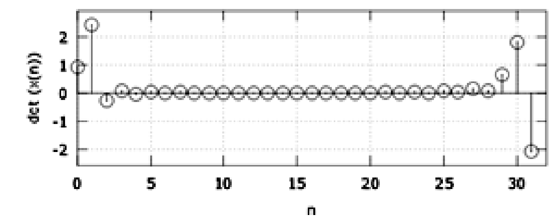


c)

Fig. 1. The original signal (a) and the results of discrete cosine transform (b, c – 1 and 2 types, respectively)

We had got the symmetrical (even) blocks pair relative set's bounds. Here the proposed transform modification method is very demonstrable. The transform execution is carried out separately for each block. All transform properties is saved in this case. The change of transform direction execution (one block from pair) allows work with symmetrical (even) basis functions of pair blocks in fact.

The pair of blocks transform the total length *N* and *k* from *0* to *N/2-1* and from *N/2* to *N-1* is calculated by the following expressions. The first kind corresponds to

$$\begin{aligned} Xl_c(N/2-(k+1)) &= \sqrt{4/N}\,C(k)\sum_{n=0}^{N/2-1} x(n)\cos\big(\pi\big((N-(2n+1)k/N\big)\\ Xr_c(k) &= \sqrt{4/N}\,C(k)\sum_{n=N/2}^{N-1} x(n)\cos\big(\pi\big(2(n-N/2)+1\big)k/N\big) \end{aligned}, \quad (7)$$

and the second type

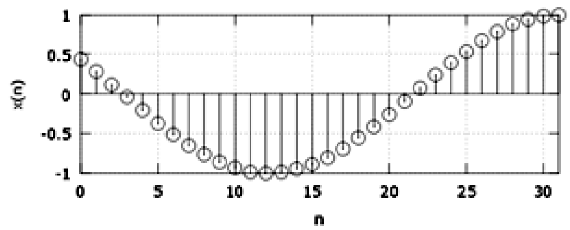
$$\begin{aligned} Xl_c(k) &= \sqrt{4/N}\,C(k)\sum_{n=0}^{N/2-1} x(n)\cos\big(\pi\big(2n+1\big)k/N\big)\\ Xr_c(N-(k+1)) &= \sqrt{4/N}\,C(k)\sum_{n=N/2}^{N-1} x(n)\cos\big(\pi\big((N-(2n+1)k/N\big) \end{aligned}. \quad (8)$$

The transform is a consistent association of two vectors

$$X_c = \left(Xl_c, Xr_c\right) \quad (9)$$

Frequency composition based on digital cosine transform for the forward and reverse signal samples order is identical, but the components are placed in reverse order in each case

The simulation results shows on Fig. 1. and Fig. 2. Forward and reverse signal forms Fig. 1. a, Fig. 2. a have identical kind in spatial and frequency domains. The signal components values are equal in amplitude and sign on both domains.



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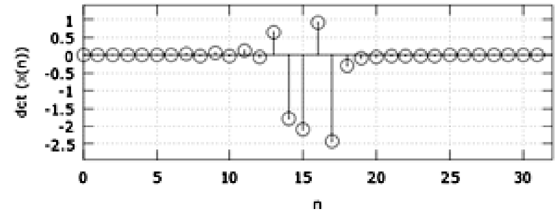

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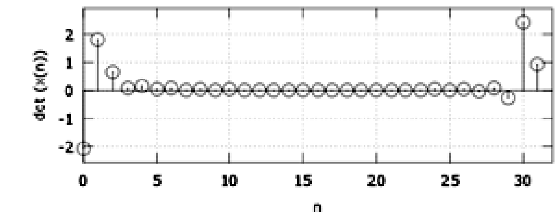


c)

Fig. 2. The original signal in reverse form (a) and the results of discrete cosine transform (b, c – 1 and 2 types, respectively)

IV. The Blocked Frequency Transform in Modified Forms

Expressions (7) and (8) and (9) may be considered to *2M* blocks each of length *N*.

The expression for the first type

$$\begin{aligned} Xl((2m+1)N-(k+1)) &= f(k)\sum_{n=0}^{N-1} x((2m+1)N-(n+1))\,f_b(n,k)\\ Xr((2m+1)N+k) &= f(k)\sum_{n=0}^{N-1} x((2m+1)N+n)\,f_b(n,k) \end{aligned} .(10)$$

Second type is

$$\begin{aligned} Xl(2mN+k) &= f(k)\sum_{n=0}^{N-1} x(2mN+n)\,f_b(n,k)\\ Xr(2N(m+1)-(k+1)) &= f(k)\sum_{n=0}^{N-1} x(2N(m+1)-(n+1))\,f_b(n,k) \end{aligned} ,(11)$$

where $f_b(n,k)$ - basis function dependent from *n, k* ;

$f(k)$ - function dependent *k* ;

k=0, …, N-1, m=0, …, M-1 for each case.

The frequency components sequence is prepared analogously (9)

$$X = (Xl_0, Xr_0, Xl_1, Xr_1, ..., Xl_{M-1}, Xr_{M-1})\,. \quad (12)$$

The inverse transformation of the first embodiment is as follows

$$\begin{aligned} xl((2m+1)N-(n+1)) &= \sum_{k=0}^{N-1} f(k)\,Xl((2m+1)N-(k+1))\,f_b(n,k)\\ xr((2m+1)N+n) &= \sum_{k=0}^{N-1} f(k)\,Xr((2m+1)N+k)\,f_b(n,k) \end{aligned} .(13)$$

The second type is

$$xl(2mN+n)=\sum_{k=0}^{N-1} f(k)Xl(2mN+k)\,f_b(n,k)$$
$$xr(2N(m+1)-(n+1))=\sum_{k=0}^{N-1} f(k)Xr(2N(m+1)-(k+1))\,f_b(n,k) \quad (14)$$

The restored original data sequence x consists of $2M$ blocks with length N each

$$x=(xl_0, xr_0, xl_1, xr_1, \ldots, xl_{M-1}, xr_{M-1})\,. \quad (15)$$

It should be noted that there may be some placements set of direct and inverse order basis functions for obtaining elements even symmetry. Expressions (10) - (15) describe variants of the maximum number of such elements. This is due to the fact that the range of options to place with symmetry will be from *2* to *M/N-1*, where *M* - total length of the sequence, and *N* - block length processed separately.

V. The two-dimensional blocked frequency transform in modified forms

The basis functions system in the form is used in (5) can be represented in two-dimensional form. A blocks number increase generates a block (partitioned) transform matrix. All the units constituting the block matrix are oriented identically in traditional form. Its amount can be arbitrary and is determined only by a lines length or the samples number in general

The block transform matrix (4 blocks) in traditional form has a next kind

$$A=\begin{pmatrix} F_b(n,k) & F_b(n,k) \\ F_b(n,k) & F_b(n,k) \end{pmatrix}, \quad (16)$$

where A- partitioned transform matrix;

$F_b(n,k)$ - matrix of basis functions values.

The coefficients set DCT block matrix (2*32x2*32) show on Fig. 3.

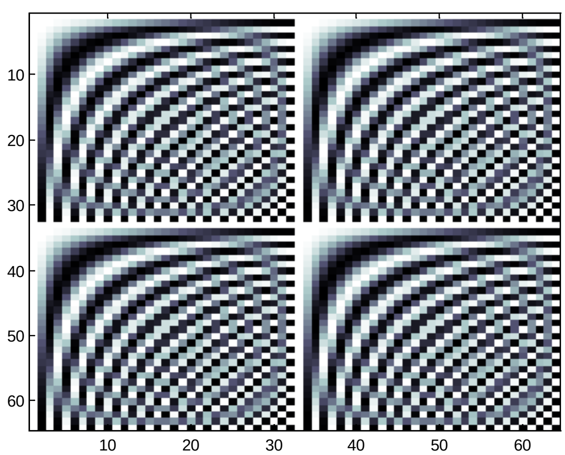


Fig. 3. The basis function sets in traditional form

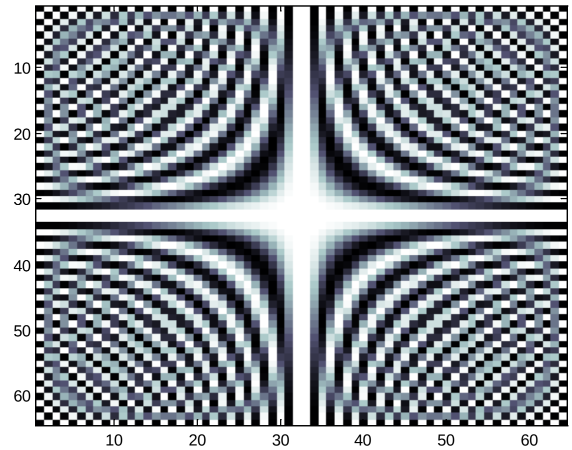
As seen in Fig. 3, it used in traditional arrangement, has no symmetry elements on vertical or horizontal axes that could be useful in an image processing, when values placed in reverse order.

The proposed approach was described above use partially changing orders rows and columns in the following schemes [7].

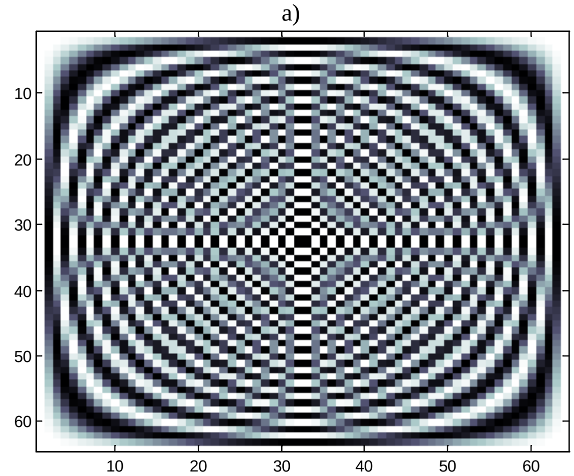
The expression for first type A_{m1} symmetry transform matrix is

$$A_{m1}=\begin{pmatrix} F_b(n,k) & F_b(N-n,k) \\ F_b(n,K-k) & F_b(N-n,K-k) \end{pmatrix}. \quad (17)$$

Blocks in expression (17) have the next differences: 1 - without changes; 2- reverse columns order; 3 - reverse rows order; 4 - reverse rows and columns order. The graphic representation DCT partitioned transform matrix A_{m1} shows on Fig. 4 a.



a)



b)

Fig. 4. The basis function sets in modificated form (a – first type; b – second type)

The expression for first type A_{m2} symmetry transform matrix is

$$A_{m2}=\begin{pmatrix} F_b(N-n,K-k) & F_b(n,K-k) \\ F_b(N-n,k) & F_b(n,k) \end{pmatrix}. \quad (18)$$

Blocks in expression (18) have the next differences: 1 - reverse rows and columns order; 2- reverse rows order; 3 - reverse columns order; 4 - without changes. The graphic representation DCT partitioned transform matrix A_{m2} shows on Fig. 4 b.

The energy maximum A_{m1} places in matrix center. The energy maximum A_{m2} places on matrix corners.

Results of matrixes usage in frequency transform will be differences. The presence of the same values about 25% in common case.

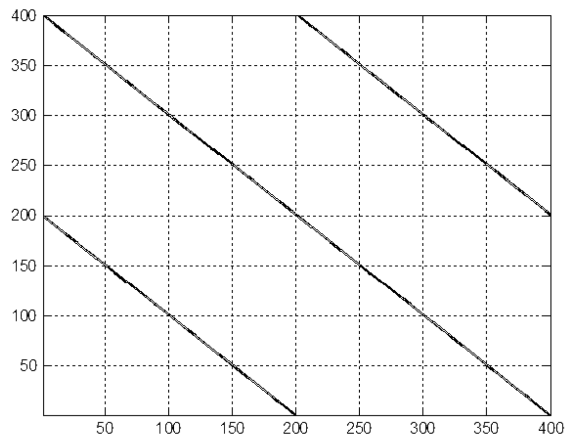
In general terms, the orthogonality described [1, 2] by equation

$$A^* = A^{-1}$$

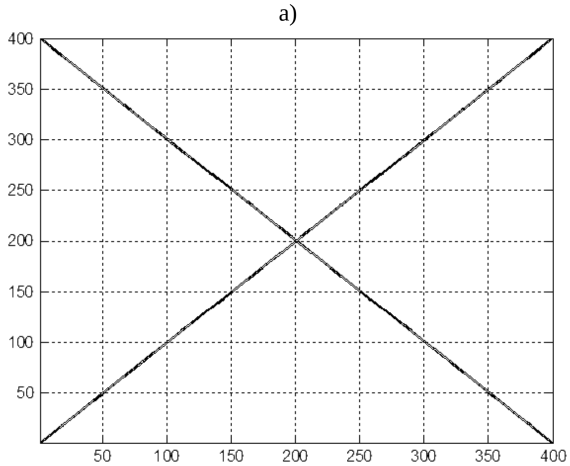
and correspondingly

$$A^* A^T = E \;. \tag{19}$$

The matrix multiplication results for verification orthogonality properties show on Fig. 5. We have four identity matrix blocks (Fig. 5 a) for traditional form described expression (16). It should be noted identity matrix blocks multiplied on constant in result.



a)



b)

Fig. 5. The identity matrixes placement (a - traditional and b - proposed approaches)

The equations (17) and (18) allow to obtain result based on matrix multiplication (19) shown on Fig. 5. b. Here are the results of simulation with the following parameters. Each basis functions block - 200x200. Block 2x2 matrix consisting of four blocks, multiplied by its inverse (transposed) matrix. The result is a matrix of size 400x400 block consisting of four matrices, each 200x200 size. We have two blocks identity matrix (main diagonal) and two blocks identity matrix (auxiliary diagonal). The traditional identity matrix obtains by reversing columns order in matrix with auxiliary diagonal. Thus the orthogonality conditions are met up to an columns order and normalization by constant. Its graphic representation shows the some combinations have central symmetry properties. It is useful in digital signal and image processing and other areas.

VI. Conclusion

When we use untreated (without distorting) a direct transform results then a complete reconstruction will be carried out in all cases. The proposed analysis scheme has a slightly more computational complexity then the traditional scheme. This is due to the fact that the blocks are processed independently. This effect can be reduced by using parallel computation at the same time. A run-time difference between traditional and modified algorithms insignificant. The symmetric scheme allows reconstruct original signal with direction process change. It works with multidimensional signals. Extension of this approach to various basic function types is also working. The most noticeable difference between the traditional and the proposed approach for asymmetric basis functions. This approach can be used to data compression, search for images in the media databases and the Internet.

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