Numerical solutions of non linear parabolic PDE by discrete Adomian decomposition method

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Abstract

In this paper, a discrete Adomian decomposition method (DADM) is developed in avoid to find the solution of non linear parabolic partial differential equation (PDE) with Dirichlet boundary conditions. The method converts the nonlinear boundary value problem into a system of ordinary differential equations. By solving the system by Adomian method, the solution can be determined. Compary the methodology for some examples where the exact solution are known and with the classical technique : finite difference method shows that the present approach is easy to use and reliable.

   Key-word : Discrete Adomian decomposition method- PDE- Heat equation-Fisher equation.

1   Introduction

Many phenomena (physical, mechanics, biological, ...) governed by the partial differential equations (PDE) was solved by the discrete Adomian decomposition method (DADM). Bratsos and al. applied the DADM to discrete nonlinear Schrödinger equations. Zhu and al. [2] have developed the DADM to 2D Burgers' difference equations. In [3], the DADM is implemented to nonlinear difference scheme of generalized Burger's–Huxley equation.

   In this context, many algorithm of Adomian decomposition method (ADM) was proposed in the literature ([11],[16], [4],[7],). In [4] a numerical algorithm, based on the Adomian decomposition method, is presented for solving heat equation with an initial condition and non local boundary condition. Khan Marwat et al.[7] have solved two problems dealing with the heat conduction with variable physical properties by ADM , the compression of the first problem with eigenfunc-

tion expansion is also made. The two analytical solution agree exactly with each other. Although, the two methods arrive at the same result. To apply this method, the authors have shown that generalized Fourier series is required to build up the solution instead of trigonometric Fourier series.

   The main disadvantage of the Adomian decomposition method (ADM) is do not take into account the boundary conditions for solving most partial differential equations ([9] , [10]). In order to avoid this problem the DADM is proposed to solve the linear and non linear parabolic PDE with the Dirichlet boundary conditions. This approch needs the use of two steps : first , the space variable is discretised , then the differential system is solved by the ADM.

   The paper is organized as follows. In section 2, a discrete Adomian decomposition method for non linear parabolic PDE is presented. In section 3, the Adomian decomposition method is given to solve an ordinary differential system and the analysis of the proposed discrete ADM is also made . The illustrative examples are given and developed with numerical results in Section 4. Section 5 concludes the paper.

2   Proposed discrete Adomian decomposition method to non linear parabolic PDE

Let us consider the non linear parabolic PDE for one space variable :

(1)
$$\begin{cases} \partial_t u(x,t) = \alpha \partial_{xx} u(x,t) + N(u) + R(u) + h & ,(x,t) \in Q \\ \quad u(x,0) = f(x); \quad x \in \bar{\Omega} \\ \quad u(0,t) = g_1(t), t \geqslant 0 \\ \quad u(l,t) = g_2(t), \quad t \geqslant 0 \end{cases}$$

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Where :
$\Omega = [0, l], l > 0, Q =]0, l[\times]0, T[$.
$u(x, t)$ is a function of two variables x and t. Here x is the space variable, so $x \in [0, l]$, where l is the length.
$N(u)$ is non linear function of u,
$R(u)$ is a linear function of u,
$h(x, t)$ is a known function,
f, g_1 and g_2 are known functions, and α is a positive constant.

The second derivative by the central difference approximation is given as follow :

$$(2) \qquad \frac{\partial^2 u(x, t)}{\partial x^2} = \frac{u_{i+1}(t) - u_i(t) + u_{i-1}(t)}{\Delta x^2}$$

The interval $0 \leqslant x \leqslant l$ is uniformly spaced such that :

$$x_i = i\Delta x$$

where $\Delta x = \frac{l}{M}$, M is a discretization number of the interval $[0, l]$.

The function $u(x, t) = u(i\Delta x, t)$ is the discrete function denoted by $u_i(t)$. Similarly the function $u(x, 0) = f(x) = f(i\Delta x)$ is the discrete function designed by f_i.

The time derivation is approximated at point (x_i, t) as follow :

$$(3) \qquad \frac{\partial u(x_i, t)}{\partial t} = \dot{u}_i(t)$$

Substituting the approximation (2) and (3) in equation (1), we get :
(4)
$$\begin{cases} \dot{u}_i(t) = \alpha \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{\Delta x^2} + N(u_i) + R(u_i) + h_i(t), \\ u_i(0) = f_i, \quad i = 1 \ldots M \\ \text{Boundary conditions are :} \\ u_0(t) = g_1(t), t \in]0, T[\\ u_{M+1}(t) = g_2(t), \quad t \in]0, T[\end{cases}$$

The ordinary differential equations (ODEs) (4) can be written in matrix form as follows :
(5)
$$\begin{cases} \dot{U}(t) = BU(t) + N(U) + R(U) + \frac{1}{\Delta x^2}C + H(t), \\ \text{with initial condition } u_i(0) = f_i; \quad i = 1, .., M \end{cases}$$

It is the discrete form of initial value problem (1).
Where :
$U(t) = (u_1(t), \ldots, u_M(t))^T$
$N(U) = (N(u_1), \ldots, N(u_M))^T$
$R(U) = (R(u_1), \ldots, R(u_M))^T$
$H(t) = (h_1, \ldots, h_M))^T$
B is tridiagonal matrix of $(M \times M)$:

$$B = \frac{-\alpha}{\Delta x^2} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & . & . & 0 & 0 \\ 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix};$$

The vector :
$C = \begin{bmatrix} g_1(t) & 0 & 0 \ldots \ldots 0 & 0 & g_2(t) \end{bmatrix}^T$

To solve the differential system (5) with initial conditions, we propose the use of Adomian decomposition method ([11],[9],[8]). We have shown, that the boundary conditions are introduced in the system (4) without any difficulty.

3 Applying Adomian decomposition method to solve the ODEs

Adomian decomposition method is applied to solve the ODEs (5), we assume that (5) has series solution :

$$(6) \qquad u_i(t) = \sum_{n=0}^{\infty} u_{in}, i = 1, .., M$$

where $u_{in}(t)(n \geq 0)$ is the approximation of $u_i(t)$.
The nonlinear function is decomposed as follow :

$$(7) \qquad N_i(u_1(t), u_2(t), \ldots, u_M(t)) = \sum_{n=0}^{\infty} A_{in}$$

where A_{in} are the Adomian polynomials ([9]) depends on $w = (u_{01}, \ldots, u_{n1}, \ldots, u_{0M}, \ldots, u_{nM})$, and obtained via the following formula, $i = 0, .., M$:

$$(8) \quad A_{in}(w) = \frac{1}{n!} \frac{d^n}{d\lambda^n} N_i \left(\sum_{j=0}^{n} \lambda^j u_{ji} \right) \Bigg|_{\lambda=0}, i = 0, .., M$$

Put the equations (6) and (7) and applying the inverse operator $L_t^{-1}(.) = \int_0^t (.)d\tau$ on both the sides of equation (5), yields :

$$\sum_{n=0}^{\infty} u_{in} = f_i - \frac{\alpha}{\Delta x^2} L_t^{-1} b_i U_n^T(t) + L_t^{-1} \sum_{n=0}^{\infty} A_{in} +$$
$$L_t^{-1} R(\sum_{n=0}^{\infty} u_{in}) + \frac{1}{\Delta x^2} L_t^{-1} C_i + L_t^{-1} h_i \quad (9)$$

where b_i is the i th row of matrix B and $U_n^T(t) = (u_{1n}(t), \ldots, u_{Mn}(t))^T$.

The termes u_{in} ,$i = 1, .., M$ are given as follow :

(10)

$$
\begin{cases}
u_{10}(t) = f_1 + L_t^{-1}h_1(t) + \frac{1}{\Delta x^2}L_t^{-1}g_1(t) \\
u_{1n}(t) = \frac{\alpha}{\Delta x^2}L_t^{-1}\left(u_{2n} - 2u_{1n}\right) + L_t^{-1}A_{10} + \\
L_t^{-1}R(u_{10}), \qquad\qquad n \geq 0 \\
\\
u_{i0}(t) = f_1 + L_t^{-1}h_i(t), i = 2, .., M - 1 \\
u_{in}(t) = \frac{\alpha}{\Delta x^2}L_t^{-1}\left(u_{i+1n} - 2u_{in} + u_{i-1n}\right) + L_t^{-1}A_{in} + \\
L_t^{-1}R(u_{in}), \qquad\quad i = 2, .., M - 1 \\
\\
u_{M0}(t) = f_M + L_t^{-1}h_1(t) + \frac{1}{\Delta x^2}L_t^{-1}g_2(t) \\
u_{Mn+1}(t) = \frac{\alpha}{\Delta x^2}L_t^{-1}\left(u_{M-1n} - 2u_{Mn}\right) + L_t^{-1}A_{Mn} + \\
L_t^{-1}R(u_{Mn}), \qquad\qquad n \geq 0.
\end{cases}
$$

In practice the solution will be approximated by series form to order "s":

(11)
$$
u_i(t) = \sum_{n=0}^{s} u_{in} \ , i = 1, .., M
$$

The convergence results of ADM for ODEs exist in the literature (see [9], [10]).

4   Illustrative examples

We use an example where the exact solution is known : the heat equation and an application of Fisher equation are made to illustrate the proposed approach.

4.1   Example 1: Heat equation Consider the heat equation ([12]) :

(12)
$$
\begin{cases}
\partial_t u(x,t) = 0.01\partial_{xx}u(x,t) & , (x,t) \in\ ]0,l[\ \times]0,+\infty[. \\
u(x,0) = 300; & 0 < x < l \\
u(0,t) = u(l,t) = 0; & t \geqslant 0
\end{cases}
$$

• **The exact solution :**

The theorical solution is given as follow :

$$
u(x,t) = \sum_{m=1}^{\infty} b_m \sin\left(\lambda_m x\right) e^{-\alpha\lambda_m^2 t} \qquad (13)
$$
$$
= \frac{2f}{l}\sum_{m=1}^{\infty}\frac{(1 - \cos\lambda_m l)}{\lambda_m}\sin\left(\lambda_m x\right)e^{-\alpha\lambda_m^2 t}
$$

where $\lambda_m = \frac{m\pi}{l}$ and b_m is Fourier series coefficients of $u(x,0)$ given by the following formula :

(14) $b_m = \frac{2}{l}\displaystyle\int_0^l f(\xi)\sin(\lambda_m\xi)d\xi = \frac{2f(1 - \cos\lambda_m l)}{l\lambda_m}$

• **The ADM solution**

The classical form of ADM yields the solution of equation (12) as follows :

(15) $u(x,t) = \displaystyle\sum_{i=0}^{\infty} u_i = u_0 + u_1 + ... = 300$

where :

(16)
$$
\begin{cases}
u_0 = u(x,0) = 300 \\
u_1 = 0.01L_t^{-1}L_{xx}\ u_0 = 0 \\
... \\
u_{n+1} = 0.01L_t^{-1}L_{xx}\ u_n = 0
\end{cases}
$$

The solution (15) dosen't satisfaying the boundary conditions, this is a disadvantage of the ADM.

• **The proposed approach**

We propose the use of the method presented in section (2) to solve the boundary value problem (12). We subdivide the interval $[0,1]$ in 10 points, with $\Delta x = 0.1$, we get the ODEs :

(17)
$$
\begin{cases}
\dot{u}_i(t) = 0.01\frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{\Delta x^2}; i = 1, .., 10, t \in]0,+\infty[\\
u_i(0) = 300\ , \quad i = 1....10 \\
\text{Boundary conditions are :} \\
u_0(t) = 0, t \geqslant 0 \\
u_{11}(t) = 0, \quad t \geqslant 0
\end{cases}
$$

Which can be easily solved by ADM.

The following figures presents the superposition curves of solutions given by DADM (line) and the exact solution (points) for $(m = 30)$ at time $t = 1$ and the relative error for different values of "s" :
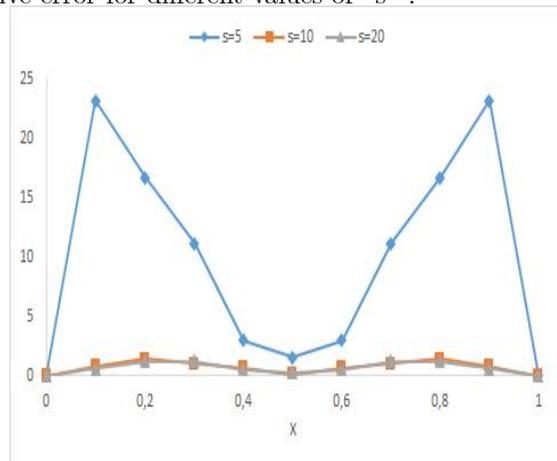

Fig 1. Error curves between DADM
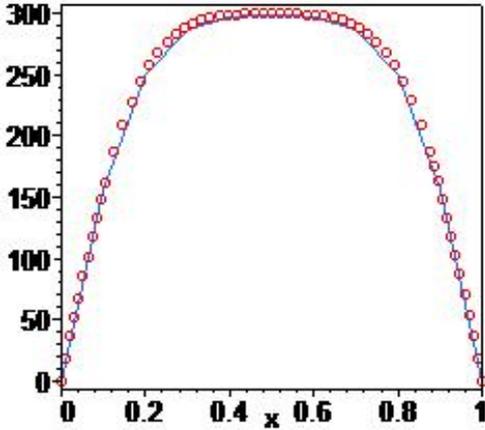and exacte solution for
different value of "s"

Fig 2. Curves of solution obtained
by DADM (ligne) with exact
solution (points) at $t = 1$

It is noteworthy that the greater the value of order series "s", the closer we are to the exact solution and the error is 1.2% for s=20.

4.2 Example 2: Application to Fisher equation

Consider the nonlinear diffusion equation is the so-called unforced Fisher's equation in one dimension [14] :

$$(18) \quad \begin{cases} \partial_t u(x,t) = \alpha \partial_{xx} u(x,t) + \lambda u(1-u) \ , \ (x,t) \in]0,l[\times]0,T[\\ u(x,0) = f(x); \quad 0 < x < l \\ u(0,t) = g_1(t), t \geqslant 0 \\ u(l,t) = g_2(t), \quad t \geqslant 0 \end{cases}$$

Which provides a simple model for gene selection-migration, with u denoting the frequency of an advantageous gene and the constant λ measuring the intensity of selection ([15] ,[6], [5]).This equation gives a deterministic approximation to a model for the spread of an advantageous gene in a population (see Fisher [15]). It is also referred to as the Kolmogorov, Petrovsky and Piskunov (KPP) equation ([17]).

We assume that the solution satisfy the condition (see [6]) : $0 \leqslant u(x,t) \leqslant 1$.

By approximating the second derivation of equation (18) by finite difference scheme yields :

$$(19) \quad \begin{cases} \dot{u}_i(t) = \alpha \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{\Delta x^2} + \lambda u_i(t) - \lambda u_i^2(t) \ ; \\ u_i(0) = f_i \ , i = 1,..,M \\ \text{Boundary conditions are :} \\ u_0(t) = g_1(t), t \in]0,T[\\ u_{M+1}(t) = g_2(t), \quad t \in]0,T[\end{cases}$$

The differential system (19) are M non linear equations and it's can be solved by the ADM. It's consist of find the solutions into series form :

$$(20) \quad u_i = \sum_{n=0}^{\infty} u_{in} \ , i = 1,...,M$$

and decompose the non linear termes $N(u_i) = u_i^2$, in series :

$$(21) \quad N(u_i) = \sum_{n=0}^{\infty} A_{in}$$

Where A_i are the Adomian polynomials calculated by the formula (8), yields :

$$(22) \quad \begin{cases} A_{i0} = u_{i0}^2 \\ A_{i1} = 2u_{i0}u_{i1} \\ A_{i2} = 2u_{i0}u_{i2} + u_{i1}^2 \\ ...... \text{ and so on} \end{cases}$$

We deduce a formulate to calculate this polynomials :

$$(23) \quad A_{in} = \sum_{j=o}^{n} u_{in-j} \ u_{ij} \ , i = 1,..M, \ n \in \mathbb{N}$$

Subsitute the series (20) and (21) in (19) and integrate the equation (19) between 0 and t, yields :

$$\sum_{n=0}^{\infty} u_{in} = f_i + \frac{\alpha}{\Delta x^2} \ L_t^{-1} \ (u_{i+1n} - 2u_{in} + u_{i-1n}) +$$
$$\lambda \sum_{n=0}^{\infty} u_{in} - \lambda L_t^{-1} \sum_{n=0}^{\infty} A_{in} \qquad (24)$$

where the u_{in} , $n \in \mathbb{N}$ are given as follow :

$$(25) \quad \begin{cases} i = 1: \\ u_{10}(t) = f_1 + \frac{\alpha}{\Delta x^2} L_t^{-1} u_{00} = f_1 + \frac{\alpha}{\Delta x^2} L_t^{-1} g_1(t) \\ u_{1n+1}(t) = \frac{\alpha}{\Delta x^2} L_t^{-1} \ (u_{2n} - 2u_{1n}) + \lambda L_t^{-1} u_{1n} + \\ \lambda L_t^{-1} A_{1n} \ . \ i = 2,...,M-1 \\ u_{i0}(t) = f_i \\ u_{in+1}(t) = \frac{\alpha}{\Delta x^2} L_t^{-1} \ (u_{i+1n} - 2u_{in} + u_{i-1n}) + \lambda L_t^{-1} u_{in} + \\ \lambda L_t^{-1} A_{in} \\ i = M: \\ u_{M0}(t) = f_M + \frac{\alpha}{\Delta x^2} L_t^{-1} u_{M+10} = f_M + \frac{\alpha}{\Delta x^2} L_t^{-1} g_2(t) \\ u_{Mn+1}(t) = \frac{\alpha}{\Delta x^2} L_t^{-1} \ (u_{M-1n} - 2u_{Mn}) + \lambda L_t^{-1} u_{Mn} + \\ \lambda L_t^{-1} A_{Mn} \end{cases}$$

4.2.1 Numerical results

We solve the Fisher equation by the discrete ADM and compare the solutions with those given by explicit scheme in finite differences (FDM) for various choices of the discretization parameters Δx and Δt.

We take the intial value function $u(x,0)$ and the Dirichlet condition as follow [6] :

$$
\begin{aligned}
\lambda &= \pi^2 \\
u(x,0) &= sin^2(2\pi x), 0 \leqslant x \leqslant 1 \\
u(0,t) &= 0, \quad 0 < t < 1 \\
u(1,t) &= 0, \quad 0 < t < 1.
\end{aligned}
$$

Case 1 : $\Delta x = 0.1$. The following figure presents the absolute error (%) between the FDM and DADM for differents values of Δt :
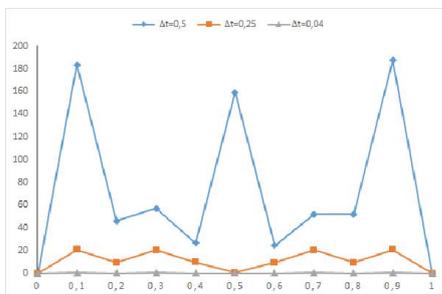

Fig 3. Error curves between DADM and FDM for different values of Δt

The figure 4, show the superposition curves of solutions obtained by the DADM (ligne) and DFM (points) for $\Delta t = 0.25$ (fig 4.a) and $\Delta t = 0.04$ (fig 4.b).
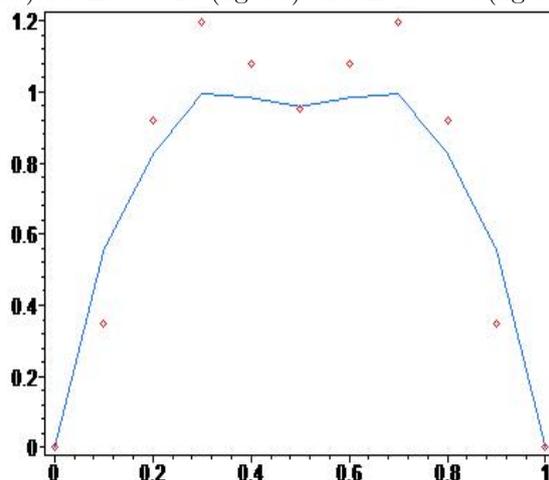

Fig 4.(a) : Superposition curves of solutions obtained by DADM (ligne) and FDM (points)
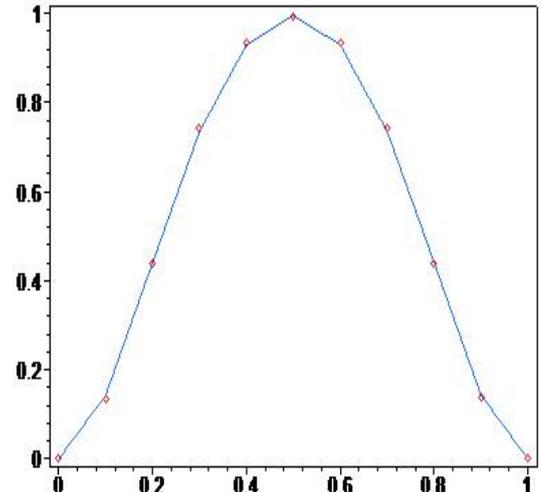

Fig 4.(b) : Superposition curves of solutions obtained by DADM (ligne) and FDM (points)

Discussion :
It is found that the absolute error increase for $\Delta t = 0.5$ which varies from 75% to 187% and for $\Delta t = 0.25$ the error varies from 0.6% to 20%, and it was smallest value of $\Delta t = 0.04$ from 0.2% to 0.7% (see Figure 3).

We deduce that the DADM gives good solutions over the time interval $[0, 0.2]$. The FDM gives solutions for small Δt, and for any instant $t \in [0, T]$, it suffices to satisfy the stability condition of the explicit scheme $(\frac{\alpha \Delta t}{\Delta x^2} \leqslant \frac{1}{2})$. In order to increase the time interval we will use the pasting together the solutions of a differentials equations by the Adomian method.

Figure 5, gives a superposition of the curves of solutions obtained by the DADM with pasting of the solutions (line) for $\Delta x = 0.1$ and the finite difference method (points) with a time step $\Delta t = 0.04$ at time $t = 1$:
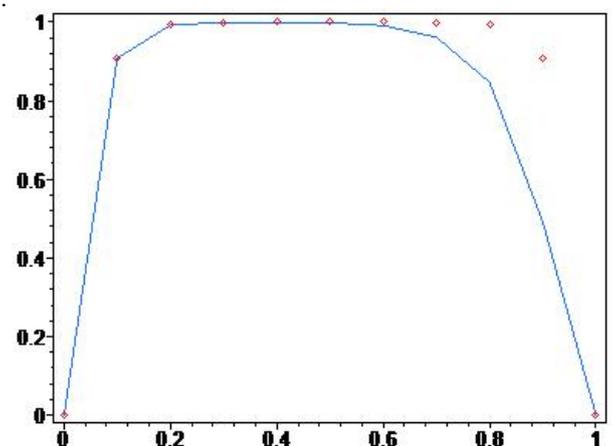

Fig 5. Curves of solutions given by the DADM with pasting of the solutions (s = 8) and the FDM at $\Delta t = 0.04$

The absolute error varies from 0.01% to 46%. The curve is not smooth at $x = 1$.

Case 2 : $\Delta x = 0.04$. The following figure shows the variation of the absolute error between the FDM and DADM for different time step values :
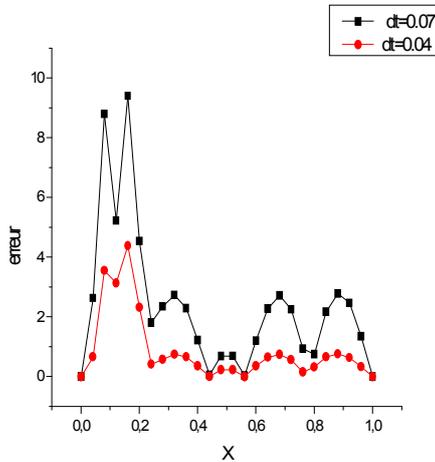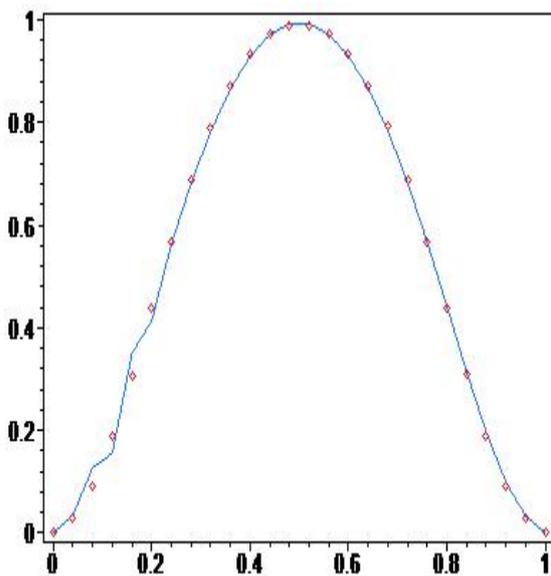

Fig 6. Error curve between DADM and FDM
for different time steps

The following figure show the superposition of the solutions given by DADM (line) with those of FDM (points) for $\Delta t = 0.04$:

$(a)\Delta t = 0.04$

Fig 7. Superposition curves of solutions obtained
by DADM (ligne) and FDM (points)
for different values of Δt

We notice in figure 6, that the error is minimal and varied from 0.002% to 4% at $\Delta t = 0.04$, and the error increase from 0.03% to 9% at time step $\Delta t = 0.07$, this is interpreted as the smaller of the step, the error becomes minimal.

We found that DADM gives good solutions over small time interval $[0, 0, 2]$, for this we will use the DADM with the pasting of solutions to have the solutions on large intervalles.

The superposition of the solutions obtained by the DADM with pasting of the solutions (line) and the FDM to $\Delta t = 0.04$ (points) at time $t = 1$ is given by the following figure :
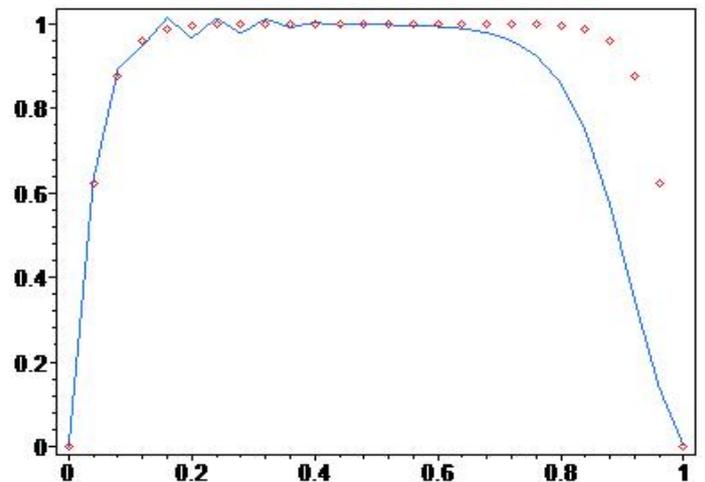

Fig 8. Superposition of curves solutions given by
the DADM with pasting of the solutions (s = 5)
and the FDM at $t = 1$

Note that the absolute error varies from 0.003% to 53% for the truncated-order $s = 5$, this is due to the fact that the solution given by the DADM is not smooth at the boundary $x = 1$. In order to reduce this error, it is sufficient to increase the order of the truncated series or to reduce the step of pasting of the solutions. This requires too many calculations and the execution time increase of the DADM with pasting of the solutions.

5   Conclusion

The discrete Adomian decomposition method is successfully applied to find the solutions of linear or non linear partial differential equations. The efficiency of the proposed method is demonstrated by two examples: Heat and Fisher equation. It may also be a promising method to solve other nonlinear partial differential equations and with other kinds of boundary conditions : Neumman or Cauchy.

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