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Monotonicity of a Quadratic Lienard Equation

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Abstract— **we study the period function of the quadratic Lienard equation of a certain type in order to give necessary and sufficient conditions for monotonicity and isochronicty of the period function. We apply this result to identify the region of monotonicity of the period function of particular cases.**

Key Words: center, isochronous center, Lienard equation, monotonicity, period function.

I. Introduction

IN this paper we study the monotonicity property of the period function of the quadratic Lienard equation of the type

$$x'' + f(x)x'^2 + g(x)x' + h(x) = 0 \quad \text{..........} \ (1)$$

We shall look for an appropriate equivalent differential system such that the required computations can be actually performed. This equivalent differential system is of the form

$$\begin{aligned} x' &= y - a(x)x - b(x)xy \\ y' &= -c(x) - a(x)y - b(x)y^2 \end{aligned} \quad \text{...............} \ (2)$$

where $a, b,$ and c are functions of class C^1 defined on an open neighborhood I of the origin.

A singular point of system (2) is called a center if there is a deleted neighborhood of the point which consists entirely of closed trajectories surrounding that point. We say that equation (1) has a center if any one of the equivalent systems, and then all, has a center. If system (2) has a center at the origin O, we call U_0 the largest open connected region covered with cycles surrounding O. Define a function $P: U_0 \to R$, by associating to every $(x, y) \in U_0$ the minimal period of the cycle passing through (x, y). P is called the period function of O. The period function has been extensively studied by number of different authors see [1], [2], [3], [5], [7], [8], and references therein. Let N be an invariant connected subset of U_0. We say that P is increasing (strictly increasing) in N if, for every couple of cycles $\delta_1, \delta_2 \in N$, with δ_1 contained in the interior of δ_2, we have $P(\delta_1) \le P(\delta_2)$ ($P(\delta_1) < P(\delta_2)$). We say that O is an isochronous center if P is constant in a neighborhood of O. The general approach of the article has been introduced before by Marco Sabatini in his work with the case $f(x) = 0$, and accordingly with a different planar system, see [6].

Since we are applying Theorem 1 of Sabatini [6] in our work, and for the sake of competence, we state that Theorem here.

Theorem 1 of [6]: let (2) have a center at O. Assume that there exist a star-shaped set $\Delta \subset R^2$ such that $\omega(r,\theta) \ne 0$ for all $(r,\theta) \in \Delta_P$, then

1) If there exists a zero-measure set $Z \subset [0, 2\pi)$ such that, for all $\theta \in [0, 2\pi) \setminus Z$, the function $r \to |\omega(r, \theta)|$ is increasing (decreasing) in $(0, r_\Delta(\theta))$, then P is decreasing (increasing) in N_Δ;

2) If point 1) holds, and for every orbit δ in a neighborhood V of O there exists a point $(r_\delta, \theta_\delta) \in \delta$ such that $r \to |\omega(r, \theta_\delta)|$ is strictly increasing (strictly decreasing) at r_δ, then P is strictly decreasing (strictly increasing) in N_Δ;

3) If there exists a zero-measure set $Z \subset [0, 2\pi)$, such that, for all $\theta \in [0, 2\pi) \setminus Z$, then $r \to |\omega(r, \theta)|$ is constant in $(0, r_\Delta(\theta))$, then P is constant in N_Δ.

$C^0(I, R)$ defined as follows:

$$a(x) = \begin{cases} \dfrac{\delta(x)}{x^2 e^{F(x)} Q(x)}, & x \ne 0 \\ \dfrac{1}{2} g(0), & x = 0 \end{cases}$$

And

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$$b(x)=\begin{cases}\dfrac{\beta(x)}{x^2e^{F(x)}}, & x\neq 0\\ \dfrac{1}{2}f(0), & x=0\end{cases},$$

Where

$$F(x)=\int_0^x g(s)ds,\quad \beta(x)=\int_0^x e^{F(s)}ds,$$

$$\gamma(x)=\int_0^x sf(s)e^{F(s)}ds \quad ,$$

$$=xe^{F(x)}-\beta(x)\quad \text{By integrating by parts}$$

$$Q(x)=\frac{1}{1-xb(x)},\quad \Omega(x)=\int_0^x Q(s)b(s)ds \quad \text{and}$$

$$\delta(x)=\int_0^x sg(s)\Omega(s)e^{F(s)}ds.$$

The following Lemmas are stated without proofs since the details of the proofs are long, we skip it and can be sent on a request.

Lemma 1.
If $f\in C^0(I,R), I\subseteq R,$ then b is continuous, and if $f\in C^1(I,R),$ then

$$b\in C^1(I,R), b'(0)=\frac{2f'(0)-f^2(0)}{6}.$$

Lemma 2.
If $f,g\in C^0(I,R), I\subseteq R,$ then a is continuous, and if $g\in C^1(I,R),\ f\in C^1$, then

$$a\in C^1(I,R), a'(0)=\frac{g'(0)}{3}.$$

Lemma 3.
Let $f,g,h\in C^0(I,R),$ then the function,

$$c(x)=Q(x)[h(x)-xa^2(x)Q(x)]$$

is continuous, and if $f\in C^3$, $g\in C^2$, $h\in C^1$, then $c\in C^1$ and $c(0)=h(0)=0,\ c'(0)=h'(0).$

Lemma 4.
$Q(x)>0,$ for every $x\in R$.

Utilizing these Lemmas, one can prove the next Lemma. Since
Lemma 5.
If $f,\,g,\,h\in C^1(I,R),\quad h(0)=0$, then system (2) is of class C^1 in a neighborhood of the origin and equivalent to the equation (1).

II. The Main Results

In order to state the first theorem, we define the following function σ as:

$$\sigma(x)=x^3h(x)-x^4h'(x)-x^4f(x)h(x)+[b(x)h(x)+2g(x)a(x)]x^4Q(x)-4x^4a^2(x)Q^2(x)$$

Theorem 1.
If $f,\,g,\,h\in C^1(I,R),\quad g(0)=h(0)=0$. Let the origin be a center of (1). If $xc(x)>0$ for $x\in I_0$, and:

1) $\sigma(x)\le 0\ (\sigma(x)\ge 0)$ *for* $x\in I$, then P is decreasing (increasing) in U_I;

2) $\sigma(x)\equiv 0$ *in* I, Then P is constant in U_I.

Proof.

The angular speed of (2) has the form $\dfrac{-xc(x)-y^2}{x^2+y^2}$.

Since $xc(x)>0$ for $x\in I_0$, then the angular speed is negative in I_0. In polar coordinates, for almost all values of, $\theta\in[0,2\pi)$ namely $\theta\neq\frac{\pi}{2},\frac{3\pi}{2}$, the angular speed is

$$\theta'=\omega(r,\theta)=-\frac{\cos\theta c(r\cos\theta)}{r^2}-\sin^2\theta$$

Hence

$$-\frac{\partial\omega}{\partial r}=\frac{r\cos^2\theta c'(r\cos\theta)-\cos\theta c(r\cos\theta)}{r^2}=\frac{x^2c'(x)-xc(x)}{(x^2+y^2)^{\frac{3}{2}}}$$

This yield

$$x^2c'(x)-xc(x)=-\frac{Q(x)}{x^2}\sigma(x)$$

Therefore

$$\frac{\partial\omega}{\partial r}=\frac{Q(x)}{x^2}\sigma(x).$$

Since $\omega(r,\theta)<0$ for $(r\cos\theta,r\sin\theta)\in I_0$, then $|\omega(r,\theta)|=-\omega(r,\theta)$, so

$$\frac{\partial(\omega(r,\theta)}{\partial r}=\frac{-Q(x)}{x^2(x^2+y^2)^{\frac{3}{2}}}\sigma(x)$$

From Lemma 4, we have $Q(x)>0$ for all x, hence if $\sigma(x)\le 0$ then $\dfrac{\partial|\omega(r,\theta)|}{\partial r}\ge 0$, and consequently, the function $r\to|\omega(r,\theta)|$ is increasing. Then applying Theorem 1 of [6] completes the proof.

Theorem 2.
If the origin of system (1) is center, and $f,\,g,\,h\in C^1(I,R),\quad g(0)=h(0)=0$, $xc(x)>0$

for $x \in I_0$, and $\sigma(x) \le 0\ (\sigma(x) \ge 0)$ for $x \in I$, and there exists a sequence $x_n \in I,\quad x_n \to 0$ with $\sigma(x_n) < 0\ (\sigma(x_n) > 0)$, then P is strictly decreasing (strictly increasing) in U_I.

Proof.

Let Λ be a cycle of the system (2) contained in W_I. Then Λ meets the line $x = x_n$ at some points (x_n, y_n) corresponding to (r_n, θ_n) in polar coordinates.

Since $\sigma(x_n) < 0\ (\sigma(x_n) > 0)$, we have, from the detailed computation in the proof of Theorem 1, that $\frac{\partial|\omega(r,\theta)|}{\partial r} > 0\ (<)$ at (r_n, θ_n). Then applying Theorem 1 of [6] completes the proof.

Corollary 1.

If the origin of system (1) is center, and $f, g, h \in C^1(I, R),\quad g(0) = h(0) = 0$, and $h'(0) > 0$. Then the statement of the Theorem holds in a suitable subinterval of I.

Remark 1.

With some computation one can find

$$\sigma(x) = \frac{x^5}{Q(x)}[\frac{\tau(x)}{x^4}]'$$

where

$$\tau(x) = x^4 a^2(x) Q^2(x) - x^3 Q(x) h(x) + h'(0) x$$

Corollary 2.

Let $f, g, h \in C^1(I, R),\quad g(0) = h(0) = 0, h'(0) > 0$, $\tau(x) \equiv 0$, then the origin is the unique singular point of system (2).

Define a function $\mu(x)$ as,

$$\mu(x) = \begin{cases} a^2(x)Q^2(x) - Q(x)\frac{h(x)}{x} + g'(0), & x \ne 0 \\ 0 & x = 0 \end{cases}$$

Lemma 6.

1) For $x \ne 0$, $\mu(x) = h'(x) - \frac{1}{x}c(x)$.
2) If $f, g, h \in C^2(I, R)$, then $\mu(x) \in C^1(I, R)$.

Theorem 3.

Let the origin be a center, f, g, h be odd analytic functions on $(-\eta, \eta)$ for some $\eta > 0$, and $xc(x) > 0$ for $x \ne 0$, then

1) P is strictly decreasing (strictly increasing) at the origin if and only of $\tau(x)$ has a proper maximum (Proper minimum) at the origin.
2) The origin is an isochronous center if and only if $\tau(x) \equiv 0$ in a neighborhood of the origin.

Proof.

We have $\sigma(x) = \frac{x^5}{Q(x)}\mu'(x)$, and $Q(x) > 0$ for all x, then $\sigma(x) > 0$, if and only if $x\mu'(x) > 0$. On the other hand, we have $\tau(x) = x^4 \mu(x)$, then $\tau(x) > 0$ if and only if $\mu(x) > 0$. In fact, μ, σ are even analytic functions, then the origin is either proper minimum of σ , or it is proper maximum, or is identically zero. By Theorem 2, P is strictly increasing, strictly decreasing, or constant, respectively. Vice –versa, if P is strictly increasing, then σ cannot be constant, otherwise, by Theorem 2, the center would be isochronous. Moreover, if P is strictly increasing, then σ cannot have a proper maximum at 0, that would imply P to be strictly decreasing by Theorem 2. Hence, if P is strictly increasing, then σ has a proper minimum. The other cases can be treated similarly. Therefore, P is strictly increasing at 0 (strictly decreasing at 0, constant), if and only if, $\sigma(x)$ has a proper minimum at 0 (has a proper maximum at 0, constant).

Now we shall show that σ has a proper maximum at 0 (has a proper minimum at 0, constant) if and only if τ has a proper maximum at 0 (has a proper minimum at 0, constant) as follows:

Since τ is even, analytic and $\tau(0) = \mu(0) = 0$, then there are only three possibilities for τ can occur in a neighborhood of 0 : A proper maximum at 0, a proper minimum at 0, or identically zero. If 0 is a proper minimum of σ, then $\sigma(x) = x\mu'(x) > 0$ for small $x \ne 0$, that is, also τ has a proper minimum at 0. Similarly we can prove that if 0 is a proper maximum of σ, and then 0 is proper maximum of τ. If $\sigma(x) \equiv 0$, then $\mu'(x) \equiv 0$, hence $\mu(x) \equiv 0$, and $\tau(x) \equiv 0$. Vice-versa, if τ has a proper minimum at 0, then by what above, σ cannot have a proper maximum, nor be constant, hence it has a proper minimum at 0. The other cases follow similarly. The proof completes.

Corollary 3.

Let the origin be a center, f, g, h be odd analytic functions on $(-\eta, \eta)$ for some $\eta > 0$, and $xc(x) > 0$ for $x \ne 0$, then

1) P is strictly decreasing (strictly increasing) in N_I if and only of $\sigma(x)$ has a maximum at the origin.
2) P is constant in N_I, if and only if $\sigma(x) \equiv 0$.

Now we consider the equation (1) with linear restoring term,

$$x'' + f(x)x'^2 + g(x)x' + x = 0 \;\ldots\ldots\ldots\; (3)$$

This equation with the case $f(x) \equiv 0$, has been considered in [4], means the Lienard equation with linear restoring term,

$$x'' + g(x)x' + x = 0 \;\ldots\ldots\ldots\; (4)$$

where its monotonicity at the origin is proved by computing the constant period function. In [6], the author provides an estimate of the region of monotonicity of P for the equation (4). The next Corollary provides an estimate of the region of monotonicity of P but with $f(x) \neq 0$, means the equation (3).

Corollary 4.

Let the origin be a center of (3). If f, g, are analytic, $g(x) \neq 0$ then P is strictly increasing in N_I, where $I = (\alpha, \beta)$, $\alpha = \inf\{x : 1 - xb(x) - a^2(x) > 0\}$, $\beta = \sup\{x : 1 - xb(x) - a^2(x) > 0\}$.

III. Example

The following is an example exhibiting an application of the work for equations with linear restoring term of a Rayleigh equation type of the form

$$x'' + nx' + mx'^3 + x = 0$$

where $n, m \neq 0$ constant to be determent and accordingly find out the interval I containing the origin on which the period function P is increasing. Consider the slandered system

$$x' = y,$$
$$y' = -x - ny - my^3$$

This system has a unique singular point (0, 0).
Exchanging variables and multiplying the vector field by -1, the system becomes

$$x' = y + nx + mx^3,$$
$$y' = -x$$

This is a system of the type of equation (1) with

$$h(x) = x, \quad f(x) \equiv 0, \quad g(x) = -n - 3mx^2$$

As a consequence of eigenvalue analysis, the origin is a center if $|n| < 2$.

Now, we apply Corollary 4.

Since $f(x) = 0$, then $F(x) \equiv 0$.

Therefore $b(x) = 0, Q(x) = 1$. Hence

$$a(x) = -\frac{n}{2} - \frac{3}{4}mx^2$$

As a consequence of Corollary 4, P is strictly increasing in N_I, $I = (\alpha, \beta)$

$$\alpha = \inf\left\{x : 1 - \left(-\frac{n}{2} - \frac{4}{3}mx^2\right)^2 > 0\right\},$$

$$\beta = \sup\left\{x : 1 - \left(-\frac{n}{2} - \frac{4}{3}mx^2\right)^2 > 0\right\}$$

But $1 - (-\frac{n}{2} - \frac{4}{3}mx^2)^2 > 0$, is equivalent to $\left|\frac{n}{2} + \frac{4}{3}mx^2\right| < 1$. Then

$$\frac{-4 - 2n}{3m} < x^2 < \frac{4 - 2n}{3m}$$

So, if $m < 0$, then there is on interval on which P is increasing. This means that the system has no increasing period function. If $m > 0$, then P is increasing on the set of cycles contained in the vertical strip (x, y) defined by the inequality $|x| < \dfrac{4 - 2n}{m}, \quad m > 0, \quad |n| < 2$.

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