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Simulation model of Heat Transfer through the Wall

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***Abstract*—This paper deals with describing of mathematical model of heat transfer through the wall and simulations, which were obtained by MATLAB Simulink. Model is a part of complex model of heating system. During our model design research we solve partial differential equation system and problem with inverse Laplace transform occurs, because of function of real argument from image function of complex argument is not define.**

***Keywords*—heat transfer through the wall, Laplace transform, application of partial differential equations system**

I. Introduction

The goals of the sustainable development in the field of heat production and supply can only be achieved by automatic process control using digital controllers and others control devices, which perform advanced control algorithms base on the intelligent control methods.

The classical verified approach to the heating process control is using of outdoor temperature compensation, which provides for the optimal temperature of supply water to heating bodies according to outdoor temperature. To achieve this requirement, it is necessary to find a equilibrium between the supplied heat output and heat losses, i. e. to ensure optimum temperature of heating water. This is realized by such methods to the water temperature in the heating system was controlled by so-called equithermic curves. [1, 2] For that reason it is necessary to design partial models of heating system, including model of heat transfer through the wall, model of heating body and equithermic curves. Sub-models implemented into the complex model will be the basis for heating control and different controller can be implemented and compared.

The model of heat transfer dynamics through the wall was designed on the base of mathematical describing of the energy balance for the elementary layer of plane wall and problem of solving system of partial differential equations by Laplace transform occured.

II. The basis for model design

A. Heat transfer dynamic throught the wall

For design of the heat transfer dynamic model through the wall we consider a plane wall, where the wall is considered as continuum with continuously distributed thermal resistance and capacity. We choose elementary layer with thickness dy in the plane wall with thickness d_w at distance y from the heated surface (Fig. 1).

Let's temperature of the heated wall surface is θ_{w1}, temperature of the refrigerated wall surface is θ_{w2} and temperature of elementary layer is θ_w. The heat flow supplied into heated wall surface is Φ_1 and the heat flow taken away from refrigerated wall surface is Φ_2. The heat flow Φ inputs into unit surface of layer dy and the heat flow $\Phi + d\Phi$ outputs from it.

B. Mathematical description of energy balance

According to [3] the heat energy doesn't originate either doesn't dissolve in considering elementary layer of the wall. Then difference of input heat and output heat in the layer has to be equal to the time variation of the energy in layer. Let's c is specific heat capacity (specific heat) and ρ is volume weight of the wall material, then:

$$\Phi - (\Phi + d\Phi) = \frac{\partial}{\partial t}(c \cdot \rho \cdot \theta_w \cdot dy). \tag{1}$$

Considering that heat flow $d\Phi$ is:

$$d\Phi = \frac{\partial \Phi}{\partial y} dy . \tag{2}$$

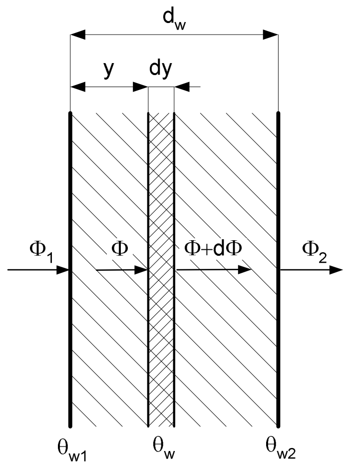

Fig. 1. Plane wall

If specific heat capacity and volume weight of the wall material are constant, then:

$$-\frac{\partial \Phi}{\partial y} = c \cdot \rho \cdot \frac{\partial \theta_w}{\partial t}. \tag{3}$$

According to Fourier's law the heat flow is directly proportional to the temperature gradient

$$\Phi = -\lambda \frac{\partial \theta_w}{\partial y}, \tag{4}$$

where λ is heat conductivity coefficient of the wall material.

Partial differential equations (3) and (4) with relevant initial and border conditions completely describe non-stationary one-dimensional heat flow.

It is considerable for automatic control the dynamic dependence of the control deviation according to changes of variables which have effect on the deviation. For that reason, we express dependent variables by their values and their increments as:

$$\Phi = \Phi_0 + \Delta\Phi, \quad \theta_w = \theta_{w0} + \Delta\theta_w. \tag{5}$$

By substitution (5) to (3) and (4) we get:

$$-\frac{\partial \Phi_0}{\partial y} - \frac{\partial \Delta\Phi}{\partial y} = c \cdot \rho \cdot \left(\frac{\partial \theta_{w0}}{\partial t} + \frac{\partial \Delta\theta_w}{\partial t} \right) = c \cdot \rho \cdot \frac{\partial \Delta\theta_w}{\partial t}, \tag{6}$$

$$\Phi_0 + \Delta\Phi = -\lambda \cdot \left(\frac{\partial \theta_{w0}}{\partial y} + \frac{\partial \Delta\theta_w}{\partial y} \right). \tag{7}$$

For initial steady-state is valid:

$$\frac{\partial \Phi_0}{\partial y} = 0, \tag{8}$$

$$\Phi_0 = -\lambda \frac{\partial \theta_{w0}}{\partial y} = -\lambda \frac{\theta_{w20} - \theta_{w10}}{d_w} = \lambda \frac{\theta_{w10} - \theta_{w20}}{d_w}, \tag{9}$$

and then by substitution (8) to (6) and subtraction (9) from (7) we get partial differential equation system of heat transfer dynamic through the wall:

$$-\frac{\partial \Delta\Phi}{\partial y} = c \cdot \rho \cdot \frac{\partial \Delta\theta_w}{\partial t}, \tag{10}$$

$$\Delta\Phi = -\lambda \frac{\partial \Delta\theta_w}{\partial y}. \tag{11}$$

To simplify computation, we express each of dependences in non-dimensional form (relative changes of variables, i. e. changes compared with initial values of variables):

$$x_\Phi = \frac{\Delta\Phi}{\Phi_0}, \quad x_{\theta w} = \frac{\Delta\theta_w}{\theta_{w10} - \theta_{w20}}. \tag{12}$$

Then equations (10) and (11) will be:

$$-\frac{\partial x_\Phi}{\partial y} = c \cdot \rho \cdot \frac{\left(\theta_{w10} - \theta_{w20}\right)}{\Phi_0} \frac{\partial x_{\theta w}}{\partial t}, \tag{13}$$

$$x_\Phi = -\lambda \frac{\left(\theta_{w10} - \theta_{w20}\right)}{\Phi_0} \frac{\partial x_{\theta w}}{\partial y}. \tag{14}$$

According (9) it is possible to express:

$$\frac{\theta_{w10} - \theta_{w20}}{\Phi_0} = \frac{d_w}{\lambda}, \tag{15}$$

and then

$$\frac{\partial x_\Phi}{\partial y} + c \cdot \rho \cdot \frac{d_w}{\lambda} \frac{\partial x_{\theta w}}{\partial t} = 0, \tag{16}$$

$$x_\Phi + d_w \frac{\partial x_{\theta w}}{\partial y} = 0. \tag{17}$$

C. Laplace transform

By the Laplace transform of partial differential equations (16) and (17) and the other mathematical operations it is possible to get a system of equations, which describes dependence of non-dimensional variables for heat flows Φ_1, Φ_2 and temperatures θ_{w1}, θ_{w2}. [4]

We substitute d_w and T_s as thickness η and time τ for exclusion of constants:

$$\eta = \frac{y}{d_w}, \tag{18}$$

$$\tau = \frac{t}{T_s}, \; T_s = \frac{d_w^2.\rho.c}{\lambda} \tag{19}$$

Let system of partial differential equations (16) and (17) is system of partial differential equations of variables η, τ:

$$\frac{\partial x_{\Phi}}{\partial \eta}+\frac{\partial x_{\theta w}}{\partial \tau}=0 \tag{20}$$

$$x_{\Phi}+\frac{\partial x_{\theta w}}{\partial \eta}=0 \tag{21}$$

Using Laplace transform defined with complex argument p:

$$\overline{X}(\eta,p)=\int_0^{\infty} x(\eta,p).e^{-p\tau}d\tau \tag{22}$$

we find image of function of real argument τ with initial condition $x_{\theta w}(\eta,0)=0$ (see substitution (19)) and we get:

$$\frac{d\overline{X}_{\Phi}}{d\eta}+p.\overline{X}_{\theta w}(p)=0$$

$$\overline{X}_{\Phi}(p)+\frac{d\overline{X}_{\theta w}}{d\eta}=0$$

Secondly we use Laplace transform defined as

$$\overline{\overline{X}}(q,p)=\int_0^{\infty} \overline{X}(\eta,p).e^{-q\eta}d\eta \tag{23}$$

of real argument η and complex argument q, initial condition $\overline{X}_{\theta w}(0,p)=\overline{X}_{\theta w1}(p)$, $\overline{X}_{\Phi}(0,p)=\overline{X}_{\Phi 1}(p)$:

$$q.\overline{\overline{X}}_{\Phi}-\overline{X}_{\Phi 1}+p.\overline{\overline{X}}_{\theta w}=0$$

$$\overline{\overline{X}}_{\Phi}+p.\overline{\overline{X}}_{\theta w}-\overline{X}_{\theta w1}=0$$

Laplace transform image of partial differential equations system is :

$$\overline{\overline{X}}_{\Phi}=\frac{q}{q^2-p}.\overline{X}_{\Phi 1}-\frac{p}{q^2-p}.\overline{X}_{\theta w1}$$

$$\overline{\overline{X}}_{\theta w}=-\frac{1}{q^2-p}.\overline{X}_{\Phi 1}+\frac{q}{q^2-p}.\overline{X}_{\theta w1}$$

According to inverse Laplace transform (23) complex argument q we get:

$$\overline{X}_{\Phi}=\cosh\sqrt{p}.\overline{X}_{\Phi 1}-\sqrt{p}.\sinh\sqrt{p}.\overline{X}_{\theta w1}$$

$$\overline{X}_{\theta w}=-\frac{\sinh\sqrt{p}}{\sqrt{p}}.\overline{X}_{\Phi 1}+\cosh\sqrt{p}.\overline{X}_{\theta w1},$$

Let denote $\overline{X}_{\Phi}=\overline{X}_{\Phi 2}$ and $\overline{X}_{\theta w}=\overline{X}_{\theta w2}$. And simultaneously due to the fact that the unknown is heat flow taken away from refrigerated wall surface Φ_2 and temperature of the heated wall surface θ_{w1}, we express $\overline{X}_{\Phi 2}$ a $\overline{X}_{\theta w1}$. We get:

$$\overline{X}_{\Phi 2}=\frac{1}{\cosh\sqrt{p}}.\overline{X}_{\Phi 1}-\sqrt{p}.tgh\sqrt{p}.\overline{X}_{\theta w2} \tag{24}$$

$$\overline{X}_{\theta w1}=-\frac{tgh\sqrt{p}}{\sqrt{p}}.\overline{X}_{\Phi 1}+\frac{1}{\cosh\sqrt{p}}\overline{X}_{\theta w2} \tag{25}$$

If we denote:

$$G_1(p)=\frac{1}{\cosh\sqrt{(p)}}, \tag{26}$$

$$G_2(p)=\sqrt{(p)}\cdot tgh\sqrt{(p)}, \tag{27}$$

$$G_3(p)=\frac{tgh\sqrt{(p)}}{\sqrt{(p)}}. \tag{28}$$

Equations (24) and (25) will be: [3, 5]:

$$\overline{X}_{\Phi 2}(p)=G_1(p)\cdot\overline{X}_{\Phi 1}(p)-G_2(p)\cdot\overline{X}_{\theta w2}(p) \tag{29}$$

$$\overline{X}_{\theta w1}(p)=G_3(p)\cdot\overline{X}_{\Phi 1}(p)-G_1(p)\cdot\overline{X}_{\theta w2}(p) \tag{30}$$

Next problem is to find function which correspond to (29) and (30) using inverse Laplace transform with complex argument p. View of the fact, that transfers function (26) - (28) are not as images defined we have to use Taylor series:

$$\sinh x = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}...,$$

$$cosh\,x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} ...,$$

and we get transfer functions:

$$G_1(p) = \frac{1}{\cosh\sqrt{p}} = \frac{1}{1 + \frac{\sqrt{p^2}}{2} + \frac{\sqrt{p^4}}{24}} = \frac{24}{24 + 12p + p^2},$$

$$G_2(p) = \sqrt{p}\,\frac{sinh\sqrt{p}}{cosh\sqrt{p}} = \sqrt{p}\,\frac{\left(\sqrt{p} + \frac{\sqrt{p}^3}{6}\right)}{1 + \frac{\sqrt{p^2}}{2} + \frac{\sqrt{p^4}}{24}} = \frac{4.\left(6p + p^2\right)}{24 + 12p + p^2},$$

$$G_3(p) = \frac{\sinh\sqrt{p}}{\cosh\sqrt{p}}\frac{1}{\sqrt{(p)}} = \frac{\sqrt{(p)}}{\sqrt{(p)}}.\frac{1 + \frac{p}{6} + \frac{p^2}{120}}{1 + \frac{p}{2} + \frac{p^2}{24}} =$$

$$= \frac{120 + 20.p + p^2}{\left(24 + 12.p + p^2\right).5}$$

Using substitution according (19) $p = T_s.s$

$$G_1(p) = \frac{24}{24 + 12T_s s + T_s s^2}, \tag{31}$$

$$G_2(p) = = \frac{4.\left(6T_s s + T_s s^2\right)}{24 + 12T_s s + T_s s^2}, \tag{32}$$

$$G_3(p) = \frac{120 + 20.T_s s + T_s s^2}{\left(24 + 12.T_s s + T_s s^2\right).5}, \tag{33}$$

where $T_s = \frac{d_w^2 \cdot \rho \cdot c}{\lambda}$ is constant depended on wall properties.

III. Application of solution of partial differential equations for simulation model

Relations expressed by system of partial differential equations (29) and (30) can be shows as a block diagram in fig. 2. [3, 7]:

Equations (16) - (17) or (20) - (21) still need to be supplemented by the equations of heat transfer on both sides of the wall surfaces.

In dimensionless form for internal surface area of the equation:

$$x_{\Phi 1} = x_{\Phi 1}^* - \kappa_1 x_{\theta w1}, \tag{34}$$

where $x_{\Phi 1}^*$ includes external conditions on heat flow changes and $\kappa_1 = \frac{\alpha_1.d_w}{\lambda}$.

For external surface area will be valid equation

$$x_{\Phi 2} = x_{\Phi 2}^* + \kappa_2 x_{\theta w2}, \tag{35}$$

where $x_{\Phi 2}^*$ includes external conditions on heat flow changes and $\kappa_2 = \frac{\alpha_2.d_w}{\lambda}$.

Finally, simulation model of heat transfer dynamic through the wall was created based on the block diagrams in fig. 2 and 3. in Matlab Simulink. Problem of transfer functions (31), (32), (33) specifying was to system (29) - (30) was stable. If we used more terms of Taylor series system was unstable. Transfer function in the forms (31) - (33) have roots negative and therefore is stable. [7, 8, 9,10]

Block diagram in fig. 2 will be extended using (34), (35) to ensure impact of heat flow supplied into heated wall surface Φ_1 and Φ_2 and impact of external conditions.

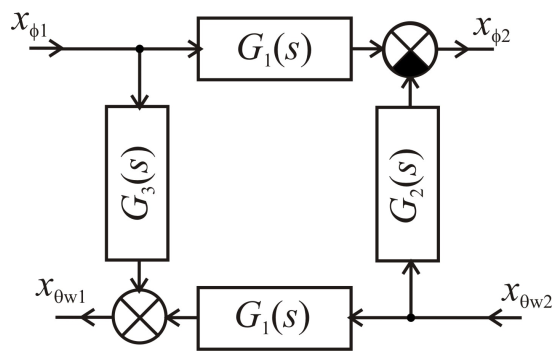

Fig. 2. Block diagram based on the (31) - (33)

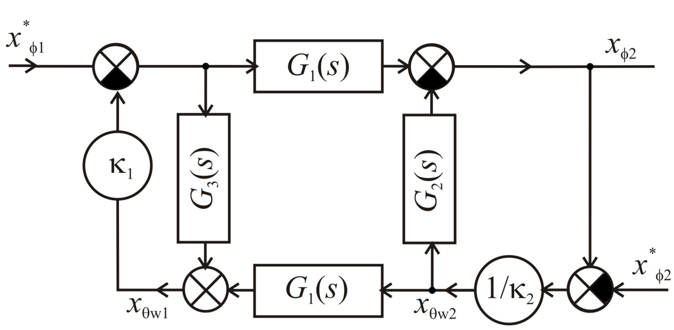

Fig. 3. Block diagram with $x_{\Phi 1}^*$, $x_{\Phi 2}^*$

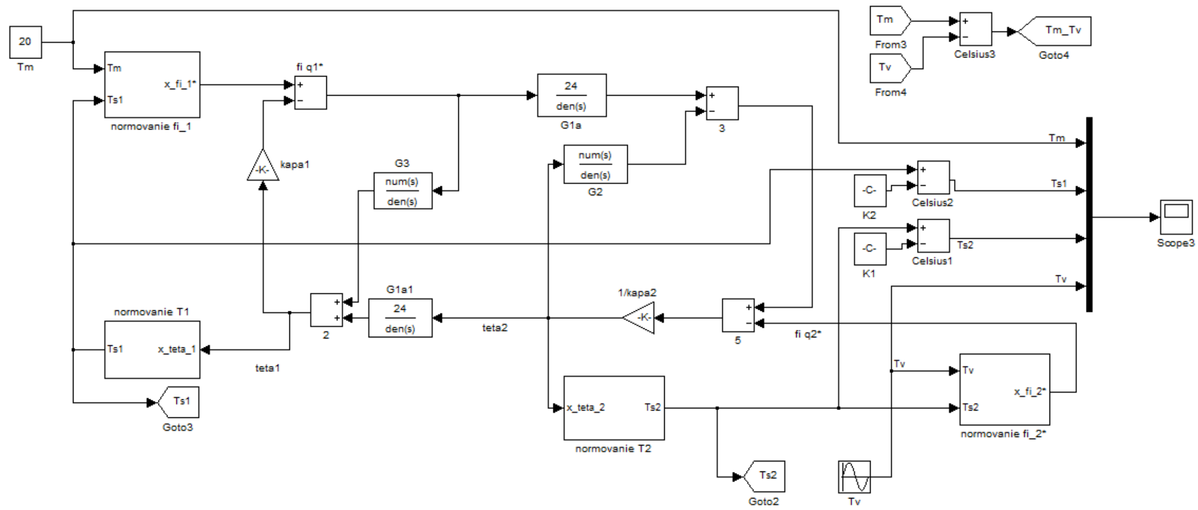

Fig. 4. Simulation model heat transfer dynamic through the wall.

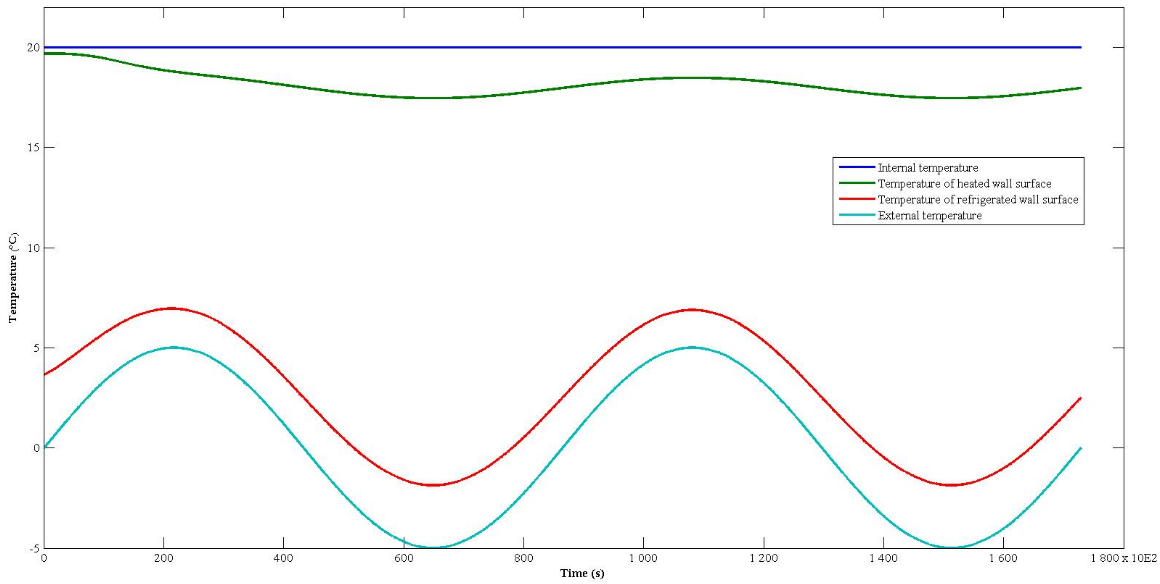

Fig. 5. Simulation results: internal temperature θ_{w10}, external temperature θ_{w20}, temperature of the heated wall surface θ_{w1}, temperature of the refrigerated wall surface θ_{w2} with input conditions parameters c, ρ, d_w.

IV. Discussion and Results

Next we can present some results of simulations. For the simulation model in Matlab (see Fig. 4) were used the following real parameter values measured on typical wall of building: wall thickness $d_w = 0{,}52$ m, where wall consists of internal plaster with thickness 0,015 m, external plaster with thickness 0,015 m, internal isolation 0,15 m and external isolation 0,05 m and finally thickness of brick 0,29 m. Volume of wall material $\rho = 1\,400$ kg/m^3 and specific heat capacity $c = 840$ J/(kg.K). Thermal conductivity is $\lambda = 0{,}2$ W/(m.K) and also consists of thermal conductivities of individual wall layers. Outdoor temperature is simulated as sine wave with frequency 2π/86400 rad/s and amplitude 5°C. Internal requested temperature is constant 20°C. Fig. 5 captures 2 days, i. e. 172800 sec. Behavior of temperatures corresponds to simulated sine wave, but internal wall layer is refrigerated by about 3 degrees of Celsius.

In Fig. 6 is similar situation, but wall consist of wood layer without isolation layers. As we can see of heat losses appeared. Refrigerated wall surface has higher temperature due to higher heat transfer through the wall. And on the other hand, heated wall surface has lower temperature as requested internal temperature is.

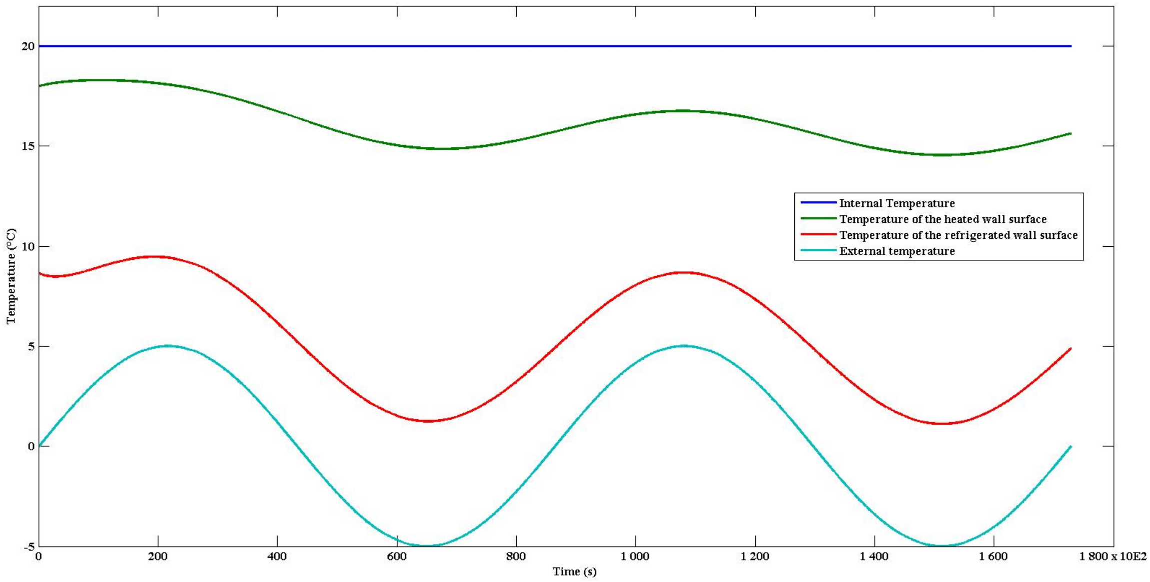

Fig. 6. Simulation results: internal temperature θ_{w10}, external temperature θ_{w20}, temperature of the heated wall surface θ_{w1}, temperature of the refrigerated wall surface θ_{w2} with input conditions parameters c, ρ, d_w.

Conclusion

To summarize, the first part of paper deals with solving of partial differential equations system, where problem of solution inverse Laplace transform appears, because transfers function (26) - (28) are not as images defined. To solve this problem we have to find Taylor series of image function as a result of first inverse Laplace transform. These results are in the used in second part of paper where were used as transfer functions for simulation model. Next problem was to find such number of terms of Taylor series to system was stable.

There is described creation of model by Matlab Simulink and there are presented obtained simulation results. Basic model in Fig. 3 had to be extended by for example standardization of input temperature variables or unit conversion from Celsius to Kelvin scale, etc.

The designed model of heat transfer dynamic through the wall together with model of heating body and with model of equithermic curves will be implemented into control system, will be tested and will be compared with real heating system in future work.

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