Analyzing the set of uncontrollable second order generalized linear systems

M. ISABEL GARCÍA-PLANAS, J. CLOTET
Departament de Matemática Aplicada I
Universitat Politècnica de Catalunya,
C. Minería 1, Esc C, 1º-3ª
08038 Barcelona, Spain
E-mail: maria.isabel.garcia@upc.edu

Abstract:- In this paper we consider the space of second order generalized time-invariant linear systems, $E\ddot{x} = A_1\dot{x} + A_2x + Bu$, where E, A_1, $A_2 \in M_n(\mathbb{C})$, $B \in M_{n\times m}(\mathbb{C})$.

We study the controllability of second order generalized systems by means the rank of a certain constant matrix that we will call "the controllability matrix" of second order generalized linear systems. We use this matrix to study the geometry of the set of uncontrollable systems and we explicit the subset contained in the set of standardizable ones.

Key-Words:- Second order linear systems, Feedback equivalence, Bundles.

1 Introduction

Generalized linear systems have been widely studied in recent years. First order are commonly applied in engineering for example they are used in modelling a three-link planar manipulator by M. Hou [9]. Second order generalized systems are applied in many fields, such as vibration an structural analysis spacecraft control, robotics control as well to power systems (see [1] and [4] for example).

A second order generalized linear system is described by the following state space equation

$$E\ddot{x} = A_1\dot{x} + A_2x + Bu, \qquad (1)$$

where $x \in \mathbb{C}^n$, $u \in \mathbb{C}^m$ are the state vector and the control vector respectively, E, A_i are n-square complex matrices and B a rectangular complex matrix of appropriate size. In certain applications the matrices E, A_1, A_2 are called the mass matrix, the structural damping matrix and stiffness matrix respectively. We denote this type of systems by quadruples of matrices (E, A_1, A_2, B), and the space of all quadruples by $\mathcal{M}_{n,m}$:

$$\mathcal{M}_{n,m} = \{(E, A_1, A_2, B) \mid E, A_1, A_2 \in M_n(\mathbb{C}), B \in M_{n\times m}(\mathbb{C})\}.$$

When $E = I_n$ the system is called standard.

One of the problems for a control theory is to maintain stability and controllability of the system. If the system is not stable and/or not controllable then ones would like to choose the control variables u in such a way that the resulting system is stable and controllable. If the chosen control variables are $u = -F_3\ddot{x} + F_1\dot{x} + F_2 + v$, then the system becomes $(E + BF_3)\ddot{x} = (A_1 + BF_1)\dot{x} + (A_2 + BF_2)x + v$. This system is called "close-loop system" whereas the system (1) is called "open-loop system".

Controllability is a widely studied qualitative property of second order linear dynamical systems (see [6], [8], [10] for example).

It is well known the following result (see [2], for example): a second order generalized linear system (E, A_1, A_2, B), is controllable if and only if

$i)$ $\quad \operatorname{rank}\begin{pmatrix} E & B \end{pmatrix} = n$
$ii)$ $\quad \operatorname{rank}\begin{pmatrix} s^2E - sA_1 - A_2 & B \end{pmatrix} = n \ \forall s \in \mathbb{C}$.

We observe that the first of these conditions ensures that there the is a second order derivative feedback F such that $E + BF$ is regular. If the system is then premultiplied by $(E + BF)^{-1}$ the new system is standard. The systems that verify this property are called standardizable systems.

In this paper we present a necessary and sufficient controllability condition for second order generalized linear systems in terms of a rank of a certain constant matrix that only depends on the matrices E, A_1, A_2 and B.

This condition can be used to study the geometry of the set of uncontrollable systems in the open set of standardizable systems.

A standard approach to study controllability is to use the generalized first order realization of equation (1)

$$\mathbb{E}\dot{X} = \mathbb{A}X + \mathbb{B}u, \qquad (2)$$

where $X = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$, $\mathbb{E} = \begin{pmatrix} I_n & 0 \\ 0 & E \end{pmatrix}$, $\mathbb{A} = \begin{pmatrix} 0 & I_n \\ A_2 & A_1 \end{pmatrix}$ and $\mathbb{B} = \begin{pmatrix} 0 \\ B \end{pmatrix}$, we will call this system "reduced system".

Note that controllability of the second order generalized system (1) is equivalent to the controllability of the reduced system (2):

i) $\quad \operatorname{rank}\begin{pmatrix} \mathbb{E} & \mathbb{B} \end{pmatrix} = n + \operatorname{rank}\begin{pmatrix} E & B \end{pmatrix}$.

ii) $\quad \operatorname{rank}\begin{pmatrix} s\mathbb{E} - \mathbb{A} & \mathbb{B} \end{pmatrix} =$

$$\operatorname{rank}\left(s\begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix} - \begin{pmatrix} 0 & I \\ A_2 & A_1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ B \end{pmatrix} \right) =$$
$$\operatorname{rank}\begin{pmatrix} 0 & I & 0 \\ s^2E - sA_1 - A_2 & 0 & B \end{pmatrix} =$$
$$n + \operatorname{rank}\begin{pmatrix} s^2E - sA_1 - A_2 & B \end{pmatrix}.$$

Therefore, in this case, we use the criterion given in [3] and [6], for singular systems based on the rank of a constant matrix.

This well-known controllability criterion is difficult to use when the quadruple of matrices depends on parameters $(E(\lambda), A_1(\lambda), A_2(\lambda), B(\lambda))$ with the parameter vector $\lambda \in \mathbb{C}^k$. In this paper we generalize the result for singular systems and present a criterion that depends only on the matrices of the quadruple so that the controllability analysis is simpler and more systematic.

We introduce an equivalence relation that preserves the controllability character to obtain a reduced form for standardizable systems that is canonical for one-input generic case. This enables us to describe the set of standardizable systems as a bundle over $\mathbb{C}^{n(n-1)}$. Consequently, we can reduce the study of geometry of the uncontrollability set by analyzing the projection of the set over the base of the bundle.

Knowing the geometric structure of the set of uncontrollable systems, given a parametric family of systems we can choose a change of parameters in order to obtain a good controllable second order linear system.

2 Controllability

In this section we show how to study the controllability character of a second order generalized linear system by computing the rank of a certain matrix.

We consider the following $2n^2 \times ((2n-2)n + 2nm)$-matrix which we will call controllability matrix.

$$\mathcal{C} = \underbrace{\begin{pmatrix} -E & 0 & \dots & 0 \\ -A_1 & -E & \dots & 0 \\ A_2 & -A_1 & \dots & 0 \\ & \ddots & & \\ 0 & 0 & \dots & -E \\ 0 & 0 & \dots & -A_1 \\ 0 & 0 & \dots & A_2 \end{pmatrix}}_{(2n-2)n}\!\!\!\!\underbrace{\begin{matrix} B & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & B & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & B & \dots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \dots & B & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & B & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & B \end{matrix}}_{2nm}$$

Remark 1

i) If $n = 1$, $\mathcal{C} = \begin{pmatrix} B & \\ & B \end{pmatrix} \in M_{2\times 2m}(\mathbb{C})$,

ii) If $n = 2$, $\mathcal{C} = \begin{pmatrix} -E & 0 & B & 0 & 0 & 0 \\ -A_1 & -E & 0 & B & 0 & 0 \\ A_2 & -A_1 & 0 & 0 & B & 0 \\ 0 & A_2 & 0 & 0 & 0 & B \end{pmatrix} \in M_{8\times(4+4m)}(\mathbb{C})$,

iii) If $m = 1$, the matrix $\mathcal{C}$ is square.

The controllability of a system is related to the rank of this matrix, as shown in the following proposition.

Proposition 1([5]) *A second order generalized linear system $(E, A_1, A_2, B) \in \mathcal{M}_{n,m}$, is controllable if and only if the controllability matrix $\mathcal{C}$, has full rank:*

$$\operatorname{rank} \mathcal{C} = 2n^2.$$

Proof.

It is sufficient to recall that a generalized linear system is controllable if and only if the generalized controllability matrix for generalized linear systems

$$M = \begin{pmatrix} \mathbb{E} & \mathbb{B} & 0 & 0 & 0 & 0 & \\ \mathbb{A} & 0 & \mathbb{E} & \mathbb{B} & 0 & 0 & \\ 0 & 0 & \mathbb{A} & 0 & \mathbb{E} & \mathbb{B} & \\ 0 & 0 & 0 & 0 & \mathbb{A} & 0 & \\ & & & & & & \ddots \end{pmatrix} \in M_{4n^2 \times (4n^2 - 2n + 2nm)}(\mathbb{C})$$

has full rank, (see [6] for more details).

By making block-elementary row transformations to matrix M

$$\operatorname{rk} \begin{pmatrix} I_n & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & E & \dots & 0 & 0 & B & 0 & \dots & 0 & 0 \\ 0 & I_n & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ A_2 & A_1 & \dots & 0 & 0 & 0 & B & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ & & \ddots & & & & & \ddots & & \\ 0 & 0 & \dots & I_n & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & E & 0 & 0 & \dots & B & 0 \\ 0 & 0 & \dots & 0 & I_n & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & A_2 & A_1 & 0 & 0 & \dots & 0 & B \end{pmatrix} =$$

$$\operatorname{rk} \begin{pmatrix} I_{2n^2} & & & & & & & & \\ & -E & & & B & & & & \\ & -A_1 & \ddots & & & & & & \\ & A_2 & & & & \ddots & & & \\ & & & -E & & & B & & \\ & & & -A_1 & & & & B & \\ & & & A_2 & & & & & B \end{pmatrix} =$$

$$= 2n^2 + \operatorname{rk} \begin{pmatrix} -E & \dots & 0 & B & 0 & \dots & 0 \\ -A_1 & \dots & 0 & & \ddots & & \\ A_2 & \dots & 0 & & & & \\ & \ddots & & & & & \\ 0 & \dots & -E & 0 & \dots & & \\ 0 & \dots & -A_1 & 0 & \dots & \ddots & \\ 0 & \dots & A_2 & 0 & & \dots & B \end{pmatrix} = 4n^2$$

if and only if

$$\operatorname{rk} \begin{pmatrix} -E & \dots & 0 & B & 0 & \dots & 0 \\ -A_1 & \dots & 0 & & \ddots & & \\ A_2 & \dots & 0 & & & & \\ & \ddots & & & & & \\ 0 & \dots & -E & 0 & \dots & & \\ 0 & \dots & -A_1 & 0 & \dots & \ddots & \\ 0 & \dots & A_2 & 0 & & \dots & B \end{pmatrix} = 2n^2$$

$\square$

Example 1. Let $(E, A_1, A_2, B) \in \mathcal{M}_{n,m}$ be a two-parametric family of quadruples of matrices where

$$E = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; A_1 = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} \lambda & 3\lambda & \lambda \\ 3\lambda + \mu & \lambda + \mu & \lambda + 3\mu \\ 0 & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The matrix $\mathcal{C}$:

$$\begin{pmatrix} -E & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 \\ -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 \\ A_2 & -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 & 0 \\ 0 & A_2 & -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 \\ 0 & 0 & A_2 & -A_1 & 0 & 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & A_2 & 0 & 0 & 0 & 0 & 0 & B \end{pmatrix},$$

has full rank if and only if $\lambda \neq 0$. That is to say, the quadruples of the given family are controllable if and only if $\lambda \neq 0$.

Note that it is easier to compute $\operatorname{rank} \mathcal{C}$ than $\operatorname{rank} \begin{pmatrix} s^2 E - sA_1 - A_2 & B \end{pmatrix}$ for all $s \in \mathbb{C}$.

Corollary 1 *A necessary condition for controllability is the system being standardizable.*

3 Equivalence relation in $\mathcal{M}_{n,m}$

In order to determine the controllability properties of the systems we can define an equivalence relation that preserves controllability character permitting in this way, to consider equivalent quadruples in a simpler form.

Taking into account that a system can only be controllable if it standardizable, we can consider the following definition.

Definition 1 *Two second order generalized linear systems* (E, A_1, A_2, B),

$(E', A'_1, A'_2, B') \in \mathcal{M}_{n,m}$ are equivalent if and only if the second system can be obtained from the first by means one or more of the following elementary transformations.

a) basis change in the state space:

$$(E', A'_1, A'_2, B') = (EP, A_1P, A_2P, B),$$

b) basis change in the input space:

$$(E', A'_1, A'_2, B') = (E, A_1, A_2, BR),$$

c) feedback:

$$(E', A'_1, A'_2, B') = (E, A_1, A_2 + BF_2, B),$$

d) derivative feedback:

$$(E', A'_1, A'_2, B') = (E, A_1 + BF_1, A_2, B),$$

e) second order derivative feedback:

$$(E', A'_1, A'_2, B') = (E + BF_3, A_1, A_2, B)$$

f) premultiplication by an invertible matrix:

$$(E', A'_1, A'_2, B') = (QE, QA_1, QA_2, QB).$$

where $P, Q \in Gl(n; \mathbb{C})$, $F_i \in M_{m\times n}(\mathbb{C})$, $R \in Gl(m; \mathbb{C})$.

This can be Written in matrix form as:

$$\begin{pmatrix} E' & A'_1 & A'_2 & B' \end{pmatrix} = Q \begin{pmatrix} E & A_1 & A_2 & B \end{pmatrix} \begin{pmatrix} P & & & \\ & P & & \\ & & P & \\ F_3 & F_1 & F_2 & R \end{pmatrix}, \quad (3)$$

for some $P, Q \in Gl(n; \mathbb{C})$, $F_i \in M_{m\times n}(\mathbb{C})$, $R \in Gl(m; \mathbb{C})$.

It is straightforward that the relation is an equivalence relation and we will therefore refer to it as feedback equivalence.

Note that, all close-loop systems $(E + BF_3, A_1 + BF_1, A_2 + BF_2, B)$ for all $F_1, F_2, F_3 \in M_{m\times n}(\mathbb{C})$, derived from a given open-loop system (E, A_1, A_2, B) are in the same equivalence class than (E, A_1, A_2, B).

We observe that if two systems (E, A_1, A_2, B), $(E', A'_1, A'_2, B') \in \mathcal{M}_{n,m}$ are equivalent, then the first order realizations are equivalent under feedback and derivative feedback equivalence considered for linear systems: The equality (3) is verified if and only if

$$\begin{pmatrix} I_n & 0 & 0 & I_n & 0 \\ 0 & E' & A'_2 & A'_1 & B' \end{pmatrix} = \mathbf{Q} \cdot \begin{pmatrix} I_n & 0 & 0 & I_n & 0 \\ 0 & E & A_2 & A_1 & B \end{pmatrix} \cdot \mathbf{P}$$

where

$$\mathbf{Q} = \begin{pmatrix} P^{-1} & \\ & Q \end{pmatrix}$$

and

$$\mathbf{P} = \begin{pmatrix} P & & & & \\ & P & & & \\ & & P & & \\ & & & P & \\ 0 & F_3 & F_2 & F_1 & R \end{pmatrix}.$$

Canonical forms under feedback equivalence are only knowing for triples of matrices corresponding to generalized linear systems $E\dot{x} = Ax + Bu$ (see [12], [6] for example). It remains difficult to obtain a canonical form for quadruples of matrices or larger n-ples. We present a reduced form for standardizable one input systems. Notice that we could take the canonical form for generalized linear systems but we want to preserve the structure of the second order generalized systems.

We observe that if (E, A_1, A_2, B) and $(E', A'_1, A'_2, B') \in \mathcal{M}_{n,m}$ are two equivalent quadruples, the triple (E, A_1, B) is feedback equivalent to (E', A'_1, B'):

$$\begin{pmatrix} E' & A'_1 & B' \end{pmatrix} = Q \begin{pmatrix} E & A_1 & B \end{pmatrix} \begin{pmatrix} P & & \\ & P & \\ F_3 & F_1 & R \end{pmatrix}.$$

(Analogously (E, A_2, B) is feedback equivalent to the triple (E', A'_2, B')).

Therefore, we can reduce the quadruple (E, A_1, A_2, B), to (E', A'_1, A'_2, B'), where (E', A'_1, B') is the triple equivalent to

(E, A_1, B) in its canonical form (see [6] for details).

In particular, let $(E, A_1, A_2, B) \in \mathcal{M}_{n,m}$ be a quadruple for which we suppose that the first controllability condition is verified. The quadruple can therefore be reduced to $(I_n, \overline{A}_1, \overline{A}_2, \overline{B})$, where $(I_n, \overline{A}_1, \overline{B})$ is a triple in its Kronecker canonical form (see [6]).

It is well known that the triple (E, A_1, B) is controllable in the most generic case. In the most generic case, when $m = 1$, the quadruple can be reduced to $(I_n, \overline{A}_1, \overline{A}_2, \overline{B})$ where:

$$\overline{A}_1 = \begin{pmatrix} 0 & I_{n-1} \\ 0 & 0 \end{pmatrix},$$
$$\overline{A}_2 = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n-11} & \dots & a_{n-1n} \\ 0 & \dots & 0 \end{pmatrix} \quad \overline{B} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{4}$$

The quadruple $(I_n, \overline{A}_1, \overline{A}_2, \overline{B})$ is univocally determined by the a_{ij}-numbers as shown in the following proposition

Proposition 2 *Let (E, A_1, A_2, B), $(E, A'_1, A'_2, B') \in \mathcal{M}_{n,1}$ be equivalent quadruples where $E = E' = I_n$, $A_1 = A'_1 = \overline{A}_1$, $B = B' = \overline{B}$. Consequently, $A_2 = A'_2 = \overline{A}_2$.*

In other words, a system $(E, A_1, A_2, B) \in \mathcal{M}_{n,1}(\mathbb{C})$ in which (E, A_1, B) is controllable is univocally determined, by the collection of $n(n-1)$ numbers a_{ij}.

Proof. Let $(I_n, \overline{A}_1, \overline{A}_2, \overline{B})$, $(I_n, \overline{A}_1, A'_2, \overline{B}) \in \mathcal{M}_{n,m}$ be two equivalent quadruples, therefore $Q^{-1} = P + Q\overline{B}F_3$, $Q^{-1}\overline{A}_1 = \overline{A}_1 P + \overline{B}F_1$, $Q^{-1}A'_2 = \overline{A}_2 P + BF_2$, $Q^{-1}\overline{B} = \overline{B}R$, and that is only possible if $P = \alpha I_n$ $\alpha \neq 0$, $R = \alpha I_m$, and $Q^{-1} = \left(\begin{smallmatrix} \alpha I_{n-1} & 0 \\ q_1 & q_2 \end{smallmatrix}\right)$ with (q_1, q_2) a row matrix with last term being non-zero. Consequently $A'_2 = \overline{A}_2$ taking adequate feedback matrices. □

The canonical reduced form is useful for to study qualitative properties of the systems.

Transformations c), d) and e) are carried out when the equivalence relation is applied, ensure that the controllability is invariant under equivalence considered. Therefore, we have the following proposition.

Proposition 3 *The rank of the matrix $\mathcal{C}$ is invariant under equivalence relation considered.*

Proof. Let (E, A_1, A_2, B), $(E', A'_1, A'_2, B') \in \mathcal{M}_{n,m}$ be two equivalent quadruples, therefore $E' = QEP + QBF_3$, $A'_1 = QA_1P + QBF_1$, $A'_2 = QA_2P + QBF_2$, $B' = QBR$. Consequently,

$$\begin{pmatrix} -E' & \dots & 0 & B' & 0 & \dots & 0 \\ -A'_1 & \dots & 0 & & \ddots & & \\ A'_2 & \dots & 0 & & & & \\ & \ddots & & & & & \\ 0 & \dots & -E' & 0 & \dots & & \\ 0 & \dots & -A'_1 & 0 & \dots & \ddots & \\ 0 & \dots & A'_2 & 0 & & \dots & B' \end{pmatrix} =$$

$$\mathbf{Q} \begin{pmatrix} -E & \dots & 0 & B & 0 & \dots & 0 \\ -A_1 & \dots & 0 & & \ddots & & \\ A_2 & \dots & 0 & & & & \\ & \ddots & & & & & \\ 0 & \dots & -E & 0 & \dots & & \\ 0 & \dots & -A_1 & 0 & \dots & \ddots & \\ 0 & \dots & A_2 & 0 & & \dots & B \end{pmatrix} \mathbf{P}$$

where

$$\mathbf{Q} = \begin{pmatrix} Q & & \\ & \ddots & \\ & & Q \end{pmatrix},$$

and

$$\mathbf{P} = \begin{pmatrix} P & & & & & & \\ 0 & P & & & & & \\ \vdots & \vdots & \ddots & & & & \\ 0 & 0 & \dots & P & & & \\ -F_3 & 0 & \dots & 0 & R & & \\ -F_1 & -F_3 & \dots & 0 & 0 & \ddots & \\ F_2 & -F_1 & \dots & 0 & & & \\ & & \ddots & & & & \ddots \\ 0 & 0 & & -F_3 & 0 & \dots & \\ 0 & 0 & & -F_1 & 0 & \dots & \\ 0 & 0 & & F_2 & 0 & \dots & R \end{pmatrix}$$

□

Therefore, we can consider an equivalent quadruple in a simpler form in order to compute the controllability condition.

Example 2 The family of quadruples given in example 1 can be reduced to $E = I_n$,

$$\overline{A}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ \overline{A}_2 = \begin{pmatrix} \lambda & 0 & \mu \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \overline{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

It is easier to compute rank $\mathcal{C}$ in this form than the initial one.

4 Geometry of uncontrollable set

As stated at the introduction, we are interested in the geometry of the set of non-controllable quadruples in the open and dense set $\mathcal{A}_{n,m}$ of quadruples where (E, A_1, B) is controllable: $\mathcal{A}_{n,m} = \{(E, A_1, A_2, B) \in \mathcal{M}_{n,m} \mid (E, A_1, B) \text{ controllable}\}$. The other non-controllable quadruples constitutes a set of higher codimension contained in the frontier $\mathcal{F}\mathcal{A}_{n,m} \subset \mathcal{M}_{n,m}$.

Firstly, we observe that $\mathcal{A}_{n,m}$ is closed by the equivalence relation considered.

Proposition 4 *Let $(E, A_1, A_2, B) \in \mathcal{A}_{n,m}$ be a quadruple. Thus, for all quadruples (E', A'_1, A'_2, B') equivalent to it, is $(E', A'_1, A'_2, B') \in \mathcal{A}_{n,m}$.*

Proof. It suffices to observe that the controllability character of generalized linear systems is invariant under feedback equivalence. □

The equivalence relation defined in §3, can be seen as an action α by a Lie group $\mathcal{G} = GL(n;\mathbb{C}) \times GL(n;\mathbb{C}) \times GL(m;\mathbb{C}) \times M_{m\times n}(\mathbb{C}) \times M_{m\times n}(\mathbb{C}) \times M_{m\times n}(\mathbb{C})$ acting over $\mathcal{M}_{n,m}$ in this form, let $g = (P, Q, R, F_1, F_2, F_3) \in \mathcal{G}$ and $x = (E, A_1, A_2, B) \in \mathcal{M}_{n,m}$, then $\alpha(g, x) = (QEP + QBF_3, QA_1P + QBF_1, QA_2P + QBF_2, QBR)$. Therefore, $\mathcal{M}_{n,m}$ is a $\mathcal{G}$-space provided because the map α verifies:

$$\begin{aligned} &\alpha(g_1g_2, x) = \alpha(g_1, \alpha(g_2x)) \\ &\alpha(e, x) = x \end{aligned}$$

where $e \in \mathcal{G}$ is the unit element. (See [11] for more details about bundles).

We now consider the projection $\pi : \mathcal{M}_{n,m} \longrightarrow \mathcal{M}_{n,m}/\mathcal{G}$, which describes $\mathcal{M}_{n,m}$ as a $\mathcal{G}$-bundle $\xi = (\mathcal{M}_{n,m}, \pi, \mathcal{M}_{n,m}/\mathcal{G})$.

Remember that a $\mathcal{G}$-bundle is a bundle with an additional structure derived from the action of a topological (differentiable in our case) group on the fibres.

The existence of a canonical form for quadruples in $\mathcal{A}_{n,m}$ with $m = 1$, induces us to study the uncontrollable set of systems contained in the set $\mathcal{A}_{n,1} = \{(E, A_1, A_2, B) \in \mathcal{M}_{n,1} \mid (E, A_1, B) \text{ controllable}\}$. At the sequel we consider $m = 1$ and if confusion is not possible we will write the set simply $\mathcal{A}$.

Proposition before, shows that $(\mathcal{A}, \pi, \mathcal{A}/\mathcal{G})$ is a $\mathcal{G}$-subbundle of ξ, in fact we have the following theorem. Denoting $\widetilde{\mathcal{A}} = \mathcal{A}/\mathcal{G}$.

Theorem 1 *$\mathcal{A}$ is a $\mathcal{G}$-bundle $(\mathcal{A}, \pi, \mathbb{C}^{n(n-1)})$, over $\mathbb{C}^{n(n-1)}$.*

Proof. It is sufficient to prove that there exists a bijection:

$$\varphi : \widetilde{\mathcal{A}} \longrightarrow \mathbb{C}^{n(n-1)}$$

Let $\overline{x}$ be an element in $\widetilde{\mathcal{A}}$. Proposition 2 ensures that this element is univocally determined by $(a_{1,1}, \dots, a_{n-1,n}) \in \mathbb{C}^{n(n-1)}$. We therefore define $\varphi(\overline{x}) = (a_{1,1}, \dots, a_{n-1,n})$. This map is obviously, a bijection. □

A global section $\sigma : \mathbb{C}^{n(n-1)} \longrightarrow \mathcal{A}$ can be defined as $\sigma(x_1, \dots, x_{n(n-1)}) = (I_n, \overline{A}_1, \overline{A}_2, \overline{B})$ with $(I_n, \overline{A}_1, \overline{A}_2, \overline{B})$ as (4). Specifically,

$$\overline{A}_2 = \begin{pmatrix} x_1 & \dots & x_n \\ \vdots & & \vdots \\ x_{(n-1)^2} & \dots & x_{n(n-1)} \\ 0 & \dots & 0 \end{pmatrix}$$

The $\mathcal{G}$-bundle $(\mathcal{A}, \pi, \mathbb{C}^{n(n-1)})$, can be used to determine the set of non controllable quadruples in $\mathcal{A}$. We will denote this set by un$\mathcal{C} \subset \mathcal{A}$.

Proposition 5 *The set of no controllable quadruples in $\mathcal{A}$ is un$\mathcal{C} = \sigma(\Lambda) \times \mathcal{G}$ where Λ is the differentiable manifold in codimension one determined by the set of zeros of a polynomial with $n(n-1)$-variables.*

Proof. We consider matrix $\mathcal{C}$ associated with $\sigma(x_1, \dots, x_{n(n-1)})$, therefore

$$\Lambda = \{(x_1, \dots, x_{n(n-1)}) \in \mathbb{C}^{n(n-1)} \mid \det \mathcal{C} = 0\}$$

(Note that $\det \mathcal{C}$ is not identically zero). $\square$

Let now consider the $2n$-square matrix C:

$$\begin{pmatrix} 0 & B & A_1B & \cdots & X_i & \ldots & X_{2n} \\ B & 0 & A_2B & \cdots & Y_i & \ldots & Y_{2n} \end{pmatrix}$$

constructed inductively in the following manner, the i-column denoted by $\begin{pmatrix} X_i \\ Y_i \end{pmatrix}$, the $i+1$-column is $\begin{pmatrix} Y_i + A_1X_i \\ A_2X_i \end{pmatrix} = \begin{pmatrix} X_{i+1} \\ Y_{i+1} \end{pmatrix}$.

Corollary 2

$$\begin{aligned} \Lambda \ &= \{(x_1, \ldots, x_{n(n-1)}) \in \mathbb{C}^{n(n-1)} \mid \\ &\quad P_{n(n-1)}(x_1, \ldots, x_{n(n-1)}) = 0\} \end{aligned}$$

where $P_{n(n-1)}(x_1, \ldots, x_{n(n-1)}) = \det C$ *is a* $n(n-1)$*-degree polynomial with* $n(n-1)$*-variables.*

Proof. Let (E, A_1, A_2, B) a quadruple in $\mathcal{A}$, we can use the equivalent quadruple in a reduced form and now it suffices to make the elementary block row transformations in the controllability matrix corresponding to the reduced form, obtaining the following rank-equivalent matrix:

$$\begin{pmatrix} I_{2n(n-1)} & & & & & & & \\ & 0 & B & A_1B & \cdots & X_i & \ldots & X_{2n} \\ & B & 0 & A_2B & \cdots & Y_i & \ldots & Y_{2n} \end{pmatrix}.$$

So,

$$\begin{aligned} \operatorname{rk} \mathcal{C} &= 2n(n-1) + \\ &\quad \operatorname{rk} \begin{pmatrix} 0 & B & A_1B & \cdots & X_i & \ldots & X_{2n} \\ B & 0 & A_2B & \cdots & Y_i & \ldots & Y_{2n} \end{pmatrix}, \end{aligned}$$

and $\operatorname{rank} \mathcal{C} = 2n^2$ if and only if

$$\det \begin{pmatrix} 0 & B & A_1B & \cdots & X_i & \ldots & X_{2n} \\ B & 0 & A_2B & \cdots & Y_i & \ldots & Y_{2n} \end{pmatrix} \neq 0$$

$\square$

We describe the specific case of the uncontrollable set for $n = 2$ and $n = 3$.
For $n = 2$, we have

Example 3 The uncontrollability set $\text{un}\mathcal{C} = \sigma(\Lambda) \times \mathcal{G}$, is determined by the following differentiable 1-manifold

$$\Lambda = \{(x_1, x_2) \in \mathbb{C}^2 \mid P_2(x_1, x_2) = x_1 - x_2^2 = 0\}.$$

For $n = 3$, we have,

Example 4 The uncontrollability set $\text{un}\mathcal{C} = \sigma(\Lambda) \times \mathcal{G}$, is determined by the following differentiable manifold

$$\begin{aligned} &\Lambda = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{C}^6 \mid \\ &P_6(x_1, x_2, x_3, x_4, x_5, x_6) = \\ &-2x_1x_3x_4x_2 + 4x_1x_3x_4x_6 + 3x_2x_5x_4x_3 + \\ &3x_2x_5x_4x_3^2 - 4x_3x_2^2x_6x_4 - x_3x_2x_6^2x_4 + \\ &2x_3x_1x_6^2x_5 - x_2x_1x_3^2x_4 + x_4x_3x_5x_6 - \\ &x_4x_3^2x_5x_6 + 5x_6x_1x_3^2x_4 - 2x_1x_3x_5^2 \\ &-x_2^2x_6x_4 - x_1x_4x_2 + x_1x_4x_6 - x_1x_6^2x_5 \\ &+x_4x_2^3 - 2x_1x_3^2x_5^2 + x_2^2x_6^2x_5 - x_5x_1x_2^2 \\ &+x_5x_1^2x_3^2 + 2x_5x_3x_1^2 + 2x_3x_1x_2x_5x_6 + \\ &x_5x_1^2 - 3x_4^2x_3^2 - x_4^2x_3 - 3x_4^2x_3^3 - x_3^2x_5^2x_6^2 \\ &-x_3x_4x_6^3 - x_6^2x_1^2x_3^2 + x_6^2x_1x_2^2 - 2x_6^2x_3x_1^2 + \\ &2x_3^2x_2x_6^2x_4 + 2x_3^2x_1x_6^2x_5 - 2x_4x_3^3x_5x_6 + \\ &2x_6x_1x_3^3x_4 - 2x_3x_2x_6x_5^2 + 2x_3x_2x_6^3x_5 - \\ &2x_1x_3x_2x_6^3 + x_3^2x_5^3 \\ &-x_4^2x_3^4 + x_1x_6^4 - x_6^2x_1^2 - x_2^2x_6^4 = 0\}. \end{aligned}$$

5 Conclusion

In this paper a geometric study of set of non-controllable second order generalized linear systems is presented. The used method is to see the set of one input second order generalized systems as a bundle.

References

[1] S.L. Campbell, N.J. Rose, *A second order singular system arising in electric power analysis.* Int. J. Systems Science, **13**, pp. 101-108, (1982).

[2] J. Clotet, Mª I. García-Planas, *Second order generalized linear systems. A geometric approach.* Int. Journal of Pure and Applied Maths. vol. 21, (2), pp. 269-276, (2005).

[3] L. Dai, *Singular Control Systems*, Springer-Verlag New York, Inc., NJ, (1989).

[4] B. N. Datta, F. Rincon, *Feedback stabilization of a second order system: a non-modal approach*. Linear Algebra and its Applications, **188**, pp. 138-161, (1993).

[5] Mª I. García-Planas, *Controllability matrix of second order generalized linear systems*. System theory and Scientific computation. WSEAS Press. pp.278-281, (2007).

[6] M. I. García-Planas, *Controllability indices for multi-input singular systems*. ICM06. European Mathematical Society, pp. 542-542, (2006).

[7] R. George, J. Sharma, *Controllability of Matrix Second Order Systems - A Trigonometric Matrix Approach*. Fifth International Conference on Dynamic Systems and Applications. (2007).

[8] A.M.A. Hamdan, A.H. Nayfeh, *Measures and modal controllability and observability for first and second order linear systems*. AIAA Journal of guidance control and dynamics, vol. 2, (3), pp. 421-429, (1989).

[9] M. Hou, *Descriptor Systems: Observer and Fault Diagnosis*. Fortschr-Ber. VDI Reihe 8, Nr. 482. VDI Verlag, Düsseldorf, FRG (1995).

[10] P.C. Hughes, R.E. Skelton, *Controllability and observability of linear matrix second order systems*. Journal of Applied Mechanics, vol. 47, (1980).

[11] D. Husmoller, “Fibre bundles”, Springer Verlag, New York, (1975).

[12] J. J. Loiseau, K. Özaldiram, M. Malabre, *Feedback canonical forms of singular systems*. Kybernetica 27, (4), pp289-305, (1991).

[13] P. Losse, V. Mehrmann, *Controllability and Observability of Second Order Descriptor Systems*. SICON **47** (3), pp. 1351-1379, (2008).