

A complete system of structural invariants for regularizable singular systems

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Abstract:- Let (E, A, B, C) be a quadruple of matrices with $E, A \in M_n(\mathbb{C})$, $B \in M_{n \times m}(\mathbb{C})$, $C \in M_{p \times n}(\mathbb{C})$ representing a singular time-invariant linear system, $E\dot{x} = Ax + Bu$, $y = Cx$.

In this paper we present a collection of invariants for singular systems in terms of ranks of certain matrices, that permit us to reduce the system in a canonical form in such a way that the system is decomposed in following five independent subsystems: i) controllable and observable system, ii) controllable non observable system, iii) observable non controllable system, iv) Jordan system v) completely singular system.

Key- Words:- Singular systems, proportional and derivative feedback, proportional and derivative output injection, canonical forms.

1 Introduction

Let us consider a finite-dimensional singular linear time-invariant system $E\dot{x}(t) = Ax(t) + Bu(t)$, $y = Cx$, where $(E, A, B, C) \in M = M_n(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{n \times m}(\mathbb{C}) \times M_{p \times n}(\mathbb{C})$.

Different useful and interesting equivalence relations between singular systems have been defined. We deal with the equivalence relation $(E', A', B', C') = (QEP + QBF_E^B + F_E^C CP, QAP + QBF_A^B + F_A^C CP, QBR, SCP)$, with $P, Q \in Gl(n; \mathbb{C})$, $R \in Gl(m; \mathbb{C})$, $S \in Gl(p; \mathbb{C})$, $F_A^B, F_E^B \in M_{m \times n}(\mathbb{C})$, $F_A^C, F_E^C \in M_{n \times p}(\mathbb{C})$, that is to say the equivalence relation accepting one or more, of the following standard transformations: basis change in the state space, input space, proportional and derivative feedback, proportional and derivative output injection and premultiplication by an invertible matrix.

It is easy to check that this relation is an

equivalence relation.

Systems $(E, A, B, C) \in M$, for which there exist matrices F_E^B and F_E^C such that the matrix $E + BF_E^B + F_E^C C$ is invertible are called standardizable because of by means a proportional and derivative feedback, the system is reduced to an standard system.

Systems $(E, A, B, C) \in M$, for which there exist matrices F_E^B, F_E^C, F_A^B and F_A^C such that the pencil $s(E + BF_E^B + F_E^C C) + (A + BF_A^B + F_A^C C)$ is regular (i.e. $\det s(E + BF_E^B + F_E^C C) + (A + BF_A^B + F_A^C C) \neq 0$ for some $s \in \mathbb{C}$), are called regularizable. Remember that regular systems are those such that there exists a unique solution for some consistent initial condition.

Obviously, regularizable character is invariant under equivalence relation considered.

The equivalence relation permit us to reduce regularizable systems to the following re-

duced form.

Proposition 1 *Let $(E, A, B, C) \in M$ be a n -dimensional m -input regularizable singular system. Then, it can be reduced to (E', A', B', C') with $E' = \begin{pmatrix} I_1 & \\ & N_1 \end{pmatrix}$, $A' = \begin{pmatrix} A_c & \\ & I_2 \end{pmatrix}$, $B' = \begin{pmatrix} B_c \\ 0 \end{pmatrix}$, $C' = (C_c \ 0)$ where (A_c, B_c, C_c) is in its Kronecker canonical form as a triple representing a standard system (see [3]), and N_1 is a nilpotent matrix in its Jordan reduced form.*

Loiseau, Ölcadiram and Malabre in [4] in the case of triples of matrices, consider the restricted pencil $s\pi E - \pi A$ where π is the projection of state space over $\text{Im } B$, and they prove that two triples are equivalent if and only if the associated restricted pencils are strictly equivalent, consequently a singular system (E, A, B) , can be reduced to

$$\left(\begin{pmatrix} 0 \\ E'_1 \end{pmatrix}, \begin{pmatrix} 0 \\ A'_1 \end{pmatrix}, \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \right)$$

where (E'_1, A'_1) is the Kronecker canonical reduced form of the pencil $s\pi E + \pi A$. García-Planas and Magret in [2] obtain the same result using polynomial matrices.

Remember that a standard system in its Kronecker reduced form is partitioned in three independent subsystems (see [1] for example):

- i) $\dot{x}_1 = A_1x_1 + B_1u_1$, $y_1 = C_1x_1$ controllable and observable,
- ii) $\dot{x}_2 = A_2x_2 + B_2u_2$ controllable,
- iii) $\dot{x}_3 = A_3x_3$, $y_2 = C_2x_3$ observable,
- iv) $x_4 = A_4x_4$,

all of them in its corresponding canonical reduced form.

In this paper, we present a collection of invariants that permit us to obtain the canonical form for all regularizable systems

2 Collection of invariants

First of all, we remember the equivalence relation considered over the space M of quadruples of matrices.

Definition 1 *Two quadruples (E', A', B', C') and (E, A, B, C) in M are called equivalent if, and only if, there exist matrices $P, Q \in Gl(n; \mathbb{C})$, $R \in Gl(m; \mathbb{C})$, $S \in Gl(p; \mathbb{C})$, $F_E^B, F_A^B \in M_{m \times n}(\mathbb{C})$, $F_E^C, F_A^C \in M_{n \times p}(\mathbb{C})$, such that $(E', A', B', C') = (QEP + QBF_E^B + F_E^C CP, QAP + QBF_A^B + F_A^C CP, QBR, SCP)$. with $P, Q \in Gl(n; \mathbb{C})$, $R \in Gl(m; \mathbb{C})$, $S \in Gl(p; \mathbb{C})$, $F_A^B, F_E^B \in M_{m \times n}(\mathbb{C})$, $F_A^C, F_E^C \in M_{n \times p}(\mathbb{C})$, or in a matrix form*

$$\begin{pmatrix} E' & B' \\ C' & \end{pmatrix} = \begin{pmatrix} Q & F_E^C \\ & S \end{pmatrix} \begin{pmatrix} E & B \\ C & \end{pmatrix} \begin{pmatrix} P & \\ & R \\ & & P \\ & & & F_A^B \\ & & & & R \end{pmatrix}$$

It is easy to check that this relation is an equivalence relation.

Now, we consider a list of ranks of a certain matrices associated to the matrices E, A, B, C in the triple $(E, A, B, C) \in M$.

Proposition 2 *Two quadruples (E, A, B, C) , (E', A', B, C') are equivalent under equivalence relation considered, if and only if the pencils $s \begin{pmatrix} E & B & 0 \\ C & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ C & 0 & 0 \end{pmatrix}$ and $s \begin{pmatrix} E' & B' & 0 \\ C' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} A' & 0 & B' \\ 0 & 0 & 0 \\ C' & 0 & 0 \end{pmatrix}$ are strictly equivalent.*

So, the collection of Kronecker-invariants classifies these kind of pencils (see [3, 5, 6], for example). Nevertheless, we are interested in emphasizes the structure of the system. And as in the standard systems we are interested in to obtain a decomposition of the system in the following independent subsystems:

- i) $\dot{x}_1 = A_1x_1 + B_1u_1$, $y_1 = C_1x_1$ controllable and observable,
- ii) $\dot{x}_2 = A_2x_2 + B_2u_2$ controllable,
- iii) $\dot{x}_3 = A_3x_3$, $y_2 = C_2x_3$ observable,
- iv) $x_4 = A_4x_4$,
- v) $N_1\dot{x}_5 = x_5$,

all of them in its canonical reduced form, that is to say matrices A_c, B_c, C_c in proposition 1,

are $A_c = \text{diag}(A_1, A_2, A_3, A_4)$, $B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$,
 $C_c = \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 \end{pmatrix}$.

Now, we consider a list of ranks (denoted by rk), of a certain matrices associated to the matrices E, A, B, C in the quadruple $(E, A, B, C) \in M$.

Definition 2 We consider the following numbers

$$1) r_0^{co} = \text{rk} \begin{pmatrix} E & B \\ C & 0 \end{pmatrix}$$

$$r_\ell^{co} = \text{rk} \begin{pmatrix} E & B & 0 & 0 \\ C & 0 & 0 & 0 \\ A & 0 & E & B \\ 0 & 0 & C & 0 \\ & & \dots & \\ & & & E & B & 0 & 0 \\ & & & C & 0 & 0 & 0 \\ & & & A & 0 & E & B \\ & & & 0 & 0 & C & 0 \end{pmatrix}, \forall \ell \geq 1$$

$$2) r_0^c = \text{rk } B$$

$$r_\ell^c = \text{rk} \begin{pmatrix} E & B & 0 & 0 & 0 \\ C & 0 & 0 & 0 & 0 \\ A & 0 & E & B & 0 \\ 0 & 0 & C & 0 & 0 \\ 0 & 0 & A & 0 & B \\ & & \dots & \\ & & & E & B & 0 \\ & & & C & 0 & 0 \\ & & & A & 0 & B \end{pmatrix}, \forall \ell \geq 1$$

$$3) r_0^o = \text{rk } C$$

$$r_\ell^o = \text{rk} \begin{pmatrix} E & B & A & 0 & 0 \\ C & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 \\ & & \dots & \\ & & & E & B & A \\ & & & C & 0 & 0 \\ & & & 0 & 0 & C \end{pmatrix}, \forall \ell \geq 1$$

$$4) r_\ell^J = \text{rk} \begin{pmatrix} C & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \lambda E - A & B & \dots & 0 & 0 & 0 \\ 0 & E & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda E - A & B & 0 \\ 0 & \lambda E - A & 0 & 0 & \dots & C & 0 & 0 \\ \lambda E - A & 0 & 0 & \dots & E & 0 & B \end{pmatrix},$$

$\forall \ell \geq 1$ and for each $\lambda \in \mathbb{C}$ such that $\text{rk} \begin{pmatrix} \lambda E - A & B \\ C & 0 \end{pmatrix} < \text{rk} \begin{pmatrix} sE - A & B \\ C & 0 \end{pmatrix}$.

Notice that $\text{rk} \begin{pmatrix} sE - A & B \\ C & 0 \end{pmatrix}$ is the rank of the associate pencil considered in proposition 2.

Proposition 3 In the set M of singular systems, the r -numbers defined, as well the $\lambda \in \mathbb{C}$ considered, are invariant under the equivalence relation considered.

Proof. Let (E_1, A_1, B_1, C_1) be a quadruple in M equivalent to (E, A, B, C) , then there exist matrices $Q, P \in \text{Gl}(n; \mathbb{C}), R \in \text{Gl}(m; \mathbb{C}), S \in \text{Gl}(p; \mathbb{C}), F_E^B, F_A^B \in M_{m \times n}(\mathbb{C})$ y $F_E^C, F_A^C \in M_{n \times p}(\mathbb{C})$ such that

$$\begin{aligned} E_1 &= QEP + F_E^C CP + QBF_E^B \\ A_1 &= QAP + F_A^C CP + QBF_A^B \\ B_1 &= QBR \\ C_1 &= SCP. \end{aligned}$$

So,

$$\text{rk} \begin{pmatrix} E_1 & B_1 \\ C_1 & 0 \end{pmatrix} = \text{rk} \begin{pmatrix} Q & F_E^C \\ 0 & S \end{pmatrix} \begin{pmatrix} E & B \\ C & 0 \end{pmatrix} \begin{pmatrix} P & 0 \\ F_E^B & R \end{pmatrix}.$$

We denote by

$$Q = \begin{pmatrix} Q & F_E^C & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & F_A^C & Q & F_E^C \\ 0 & 0 & 0 & S \\ & & \dots & \\ & & & Q & F_E^C & 0 & 0 \\ & & & 0 & S & 0 & 0 \\ & & & 0 & F_A^C & Q & F_E^C \\ & & & 0 & 0 & 0 & S \end{pmatrix}$$

and

$$P = \begin{pmatrix} P & 0 & 0 & 0 \\ F_E^B & R & 0 & 0 \\ 0 & 0 & P & 0 \\ F_A^B & 0 & F_E^B & R \\ & & \dots & \\ & & & P & 0 & 0 & 0 \\ & & & F_E^B & R & 0 & 0 \\ & & & 0 & 0 & P & 0 \\ & & & F_A^B & 0 & F_E^B & R \end{pmatrix},$$

we have

$$\text{rk } Q \cdot \begin{pmatrix} E & B & 0 & 0 \\ C & 0 & 0 & 0 \\ A & 0 & E & B \\ 0 & 0 & C & 0 \\ & & \dots & \\ & & & E & B & 0 & 0 \\ & & & C & 0 & 0 & 0 \\ & & & A & 0 & E & B \\ & & & 0 & 0 & C & 0 \end{pmatrix} \cdot P = \text{rk} \begin{pmatrix} E_1 & B_1 & 0 & 0 \\ C_1 & 0 & 0 & 0 \\ A_1 & 0 & E_1 & B_1 \\ 0 & 0 & C_1 & 0 \\ & & \dots & \\ & & & E_1 & B_1 & 0 & 0 \\ & & & C_1 & 0 & 0 & 0 \\ & & & A_1 & 0 & E_1 & B_1 \\ & & & 0 & 0 & C_1 & 0 \end{pmatrix}.$$

Analogously we can proof the invariance of the other numbers. \square

The collection of r -numbers is called the system of discrete invariants and the collection of $\lambda \in \mathbb{C}$ are called the system of continuous invariants or the set of eigenvalues of the system.

Theorem 1 *The r -numbers are a complete system of invariants and they permit us to describe the canonical reduced form for regularizable singular systems.*

Proof. We are to show how we can deduce the canonical form from these collection of invariants.

We are going to distinguish two cases,

a) If $r_0^{co} \geq n$ the system is standardizable then there are not subsystem v), and in this case we construct the following numbers, that they are invariant.

$$\bar{r}_\ell^{co} = r_\ell^{co} - n(\ell + 1), \forall \ell \geq 0$$

and

$$\begin{aligned} \rho_0^{co} &= \bar{r}_0^{co} \\ \rho_1^{co} &= \bar{r}_1^{co} - 2\bar{r}_0^{co} \\ \rho_2^{co} &= \bar{r}_2^{co} - \bar{r}_1^{co} - (\bar{r}_1^{co} - \bar{r}_0^{co}) \\ &\vdots \\ \rho_n^{co} &= \bar{r}_n^{co} - \bar{r}_{n-1}^{co} - (\bar{r}_{n-1}^{co} - \bar{r}_{n-2}^{co}) \end{aligned}$$

The ρ_ℓ^{co} -numbers give the quantity of $(\ell + 1)$ -blocks appearing in the controllable and observable subsystem.

b) If $r_0^{co} < n$ the system is not standardizable and there are subsystem v), in fact there are $n - r_0^{co}$ -blocks in N_1 .

In this case we construct the following invariant numbers

$$\bar{r}_\ell^{co} = r_\ell^{co} - r_0^{co}(\ell + 1), \forall \ell \geq 0$$

and

$$\begin{aligned} \rho_0^{co} &= \bar{r}_0^{co} = 0 \\ \rho_1^{co} &= \bar{r}_1^{co} - 2\bar{r}_0^{co} \\ \rho_2^{co} &= \bar{r}_2^{co} - \bar{r}_1^{co} - (\bar{r}_1^{co} - \bar{r}_0^{co}) \\ &\vdots \\ \rho_n^{co} &= \bar{r}_n^{co} - \bar{r}_{n-1}^{co} - (\bar{r}_{n-1}^{co} - \bar{r}_{n-2}^{co}) \end{aligned}$$

Let ν be the least integer such that $\rho_1 + \dots + \rho_{\nu-1} < n - r_0^{co}$ and $\rho_1 + \dots + \rho_\nu \geq n - r_0^{co}$, partitioning $\rho_\nu^{co} = \rho_{\nu_1}^{co} + \rho_{\nu_2}^{co}$ such that $\rho_1 + \dots + \rho_{\nu_1} = n - r_0^{co}$, so there are ρ_i -blocks of size i in the subsystem v), for $1 \leq i \leq r$, and ρ_i -blocks of size $i + 1$ in the controllable and observable subsystem for $r + 1 \leq i \leq n$.

The number of blocks in the controllable and observable subsystem is

$$t = \rho_{\nu_2} + \rho_{\nu_1+1} + \dots + \rho_n.$$

As a consequence we have that the ρ_i^{co} -numbers give us the canonical form for subsystems i) and v).

We observe that the size of the subsystem v) is $n_2 = \rho_1 + 2\rho_2 + \dots + (\nu - 1)\rho_{\nu-1}^{co} + \nu_1\rho_{\nu_1}^{co}$ and the size of the subsystem i) is $n_1 = n - n_2$.

Now we are going to analyze the r_i^c -numbers. They give us the canonical form for subsystem ii)

We consider the following (invariant) numbers $\bar{r}_0^c = r_0^c$, $\bar{r}_\ell^c = r_\ell^c - \ell n$, $\forall \ell \geq 1$ and

$$\begin{aligned} \rho_0^c &= \bar{r}_0^c - t \\ \rho_1^c &= \bar{r}_1^c - \bar{r}_1^c - t, \\ &\vdots \\ \rho_n^c &= \bar{r}_n^c - \bar{r}_{n-1}^c - t, \end{aligned}$$

This numbers verify

$$\rho_0^c \geq \dots \geq \rho_\tau^c > \rho_\tau^c = \dots = \rho_0^c = 0.$$

The size of the blocks in the subsystem ii) are the controllability indices of the system ii) and they are numbers of the conjugate partition of

$$[\rho_0^c, \dots, \rho_\tau^c].$$

The r_i^o -numbers give us the canonical form for subsystem iii)

We consider the following (invariant) numbers $\bar{r}_0^o = r_0^o$, $\bar{r}_\ell^o = r_\ell^o - \ell n$, $\forall \ell \geq 1$ and

$$\begin{aligned} \rho_0^o &= \bar{r}_0^o - t \\ \rho_1^o &= \bar{r}_1^o - \bar{r}_0^o - t, \\ &\vdots \\ \rho_n^o &= \bar{r}_n^o - \bar{r}_{n-1}^o - t, \end{aligned}$$

This numbers verify

$$\rho_0^o \geq \dots \geq \rho_\sigma^o > \rho_\sigma^o = \dots = \rho_0^o = 0.$$

The size of the blocks in the subsystem iii) are the observability indices of the system iii) and they are numbers of the conjugate partition of

$$[\rho_0^o, \dots, \rho_\sigma^o].$$

Finally, for each eigenvalue λ the r_i^J -numbers give the Segre characteristic defining the canonical form for the subsystem iv).

We consider for each eigenvalue λ_i the following (invariant) numbers

$$\bar{r}_\ell^J(\lambda_i) = r_\ell^J - (\ell - 1)n - n_2$$

and

$$\begin{aligned} \rho_1^J(\lambda_i) &= n_1 - \bar{r}_1(\lambda_i) + t \\ \rho_2^J(\lambda_i) &= \bar{r}_1(\lambda_i) - \bar{r}_2(\lambda_i) + t \\ &\vdots \\ \rho_n^J(\lambda_i) &= \bar{r}_{n-1}(\lambda_i) - \bar{r}_n(\lambda_i) + t. \end{aligned}$$

The Segre characyeristic corresponding to the eigenvalue λ_i is the conjugate partition of

$$[\rho_1^J(\lambda_i), \dots, \rho_n^J(\lambda_i)].$$

□

Example 1 Let (E, A, B, C) a quadruple

$$\begin{aligned} \text{with } E &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & 1 & 4 & 3 & 1 & 7 & -2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & -3 & 1 & 1 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 1 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \end{pmatrix} \\ A &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ -1 & 5 & 0 & 3 & -3 & 1 & 7 & 0 & 0 & 9 & -2 \\ 0 & 0 & 0 & 1 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 6 & 3 & -1 & 1 & 1 & 0 & 4 & 0 & 3 & -1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 6 & 3 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{pmatrix} \\ B &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ C &= \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

So, computing the r -numbers we deduce the ρ -numbers obtaining

$$\begin{aligned} \rho_1^{co} &= 0, \rho_2 = \rho_{2_1} + \rho_{2_2} = 1 + 1, \rho_i^{co} = 0, \forall i > 2 \\ t &= 1, n_2 = 2 \\ \rho_0^c &= 1, \rho_1^c = 1 \\ \rho_0^o &= 1, \rho_1^o = 1 \\ \rho_1^J(3) &= 1, \rho_2^J(3) = 1 \end{aligned}$$

Consequently the canonical reduced form is (E', A', B', C') with

$$\begin{aligned} E' &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ A &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{pmatrix} \\ B &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ C &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

- The subsystems :
- i) $\dot{x}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_1, y_1 = (1 \ 0 \ 0) x_1$
 - ii) $\dot{x}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x_2 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_2,$
 - iii) $\dot{x}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x_3, y_2 = (1 \ 0) x_3$
 - iv) $\dot{x}_4 = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} x_4$
 - v) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \dot{x}_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x_5.$

3 Conclusion

In this paper a complete system of invariants for regularizable systems is presented. This collection is easily computerizable as a ranks of a certain matrices and permit us to describe the independent subsystems characterizing the control properties of the system.

References

- [1] M^a I. García-Planas, M.D. Magret, *An alternative System of Structural Invariants for Quadruples of Matrices*, Linear Algebra and its Applications **291**, (1-3), pp. 83-102 (1999).
- [2] M^a I. García-Planas, M. D. Magret, *Polynomial Matrices and generalized linear multivariable dynamical systems*. IRBS of Wseas Press, pp. 17-22 (2000).
- [3] I. Gohberg, P. Lancaster, L. Rodman, "Invariant Subspaces of matrices with applications," Wiley, (1986).
- [4] J.J. Loiseau, K. Ölçadım , M. Malabre, *Feedback canonical forms of singular systems*. Kybernetika 27, (4), pp289-305, (1991).
- [5] A.S. Morse, *Structural invariants of linear multivariable systems*, SIAM J. Contr. **11**, pp. 446-465 (1973).
- [6] J.S. Thorp, *The Singular Pencil of Linear Dynamical System*, System. Int. J. Control, 18 (3) (1973), pp. 557-596.