A numerical algorithm for a one-dimensional nonlinear Timoshenko system

Jemal Peradze and Vladimir Odisharia

***Abstract*—**The boundary value problem for a system of nonlinear ordinary differential equations is considered for the functions u, w and ψ which describe the static behavior of a plate. The functions u and ψ are expressed explicitly through the function w for which a nonlinear integro-differential equation with a boundary condition is written. For the approximate solution of the problem for w we apply the algorithm that includes the Galerkin method and the nonlinear Jacobi iteration. The error of the obtained value of w is estimated. This value is used for constructing approximations for the functions u and ψ and the respective errors are estimated.

***Keywords*—**Error estimate, Jacobi iteration process, nonlinear Timoshenko system, numerical algorithm.

I. Introduction

The nonlinear system of Timoshenko plate equations is important from the theoretical and applied standpoints. I. Vorovich [7] attributed the topic of the solvability of the system of Timoshenko equations and construction for it of approximate algorithms to the range of unsolved problems of the mathematical theory of plates and shells. After the publication of the monograph [7], the above-mentioned problems, as far as we know, still wait for their solution. In this context, it seems to us that the study of one-dimensional variants of the Timoshenko system will help get a better insight into the nature of nonlinearity inherent in these models and will make it easier to proceed to the investigation of two-dimensional cases.

II. Problem Formulation

If from the system of Timoshenko equations for a shell given in [6], p. 42, we discard the variables t and y and assume $k_x = k_y = 0$, then we obtain the one-dimensional system of equations which characterizes the static state of the plate under the action of axially symmetric load. It has the form

$$\begin{aligned} &N' = 0, \\ &Q' + (Nw')' + f = 0, \\ &M' - Q = 0, \end{aligned} \tag{1}$$

where

$$\begin{aligned} &N = \frac{Eh}{1-\nu^2}\left(u' + \frac{1}{2}w'^2\right), \\ &Q = k_0^2\,\frac{Eh}{2(1+\nu)}(\psi + w'), \\ &M = D\psi'. \end{aligned} \tag{2}$$

Here $u = u(x)$, $w = w(x)$, $\psi = \psi(x)$ are the functions we want to define, $f = f(x)$ is given function, $x \in [0,1]$, ν, E, h, D and k_0^2 are the given positive constants, and

$$D = \frac{Eh^3}{12(1-\nu^2)}, \quad 0 < \nu < 0{,}5 .$$

Note that system (1) can also be obtained from the system of Timoshenko equations for a plate presented in [1], p. 24.

Using (2) together with the formula for D, (1) can be rewritten as a system

$$\begin{aligned} &u'' + \frac{1}{2}\left(w'^2\right)' = 0, \quad k_0^2\,\frac{Eh}{2(1+\nu)}(\psi' + w'') \\ &+ \frac{Eh}{1-\nu^2}\left[\left(u' + \frac{1}{2}w'^2\right)w'\right]' + f = 0, \\ &\frac{h^2}{6(1-\nu)}\psi'' - k_0^2(\psi + w') = 0 . \end{aligned} \tag{3}$$

Suppose that the following boundary conditions are fulfilled

$$u(0) = u(1) = 0, \quad w(0) = w(1) = 0, \quad \psi'(0) = \psi'(1) = 0. \tag{4}$$

Manuscript received February 28, 2009.

J. Peradze is with the Faculty of Exact and Natural Sciences of Ivane Javakhishvili Tbilisi State University, 1, I. Chavchavadze Ave., Tbilisi 0128, GEORGIA (e-mail: j_peradze@ yahoo.com).

V. Odisharia is with the Faculty of Exact and Natural Sciences of Ivane Javakhishvili Tbilisi State University, 1, I. Chavchavadze Ave., Tbilisi 0128, GEORGIA (corresponding author to provide phone: 995 32 32 27 64; e-mail: vodisharia@yahoo.com).

III. REDUCTION OF THE PROBLEM

Using the first and the third equation from (3) and taking into account the respective boundary conditions from (4), the functions $u(x)$ and $\psi(x)$ can be expressed through the function $w(x)$ as follows

$$
\begin{aligned}
&u(x) = \int_0^1 G_u(x,\xi) w'^2(\xi) d\xi, \\
&\psi(x) = \int_0^1 G_\psi(x,\xi) w'(\xi) d\xi,
\end{aligned} \tag{5}
$$

where

$$
G_u(x,\xi) = \begin{cases} \frac{1}{2}(x-1), & x > \xi, \\ \frac{1}{2}x, & x < \xi, \end{cases}
$$

$$
G_\psi(x,\xi) = \begin{cases} -\frac{\sqrt{\sigma}}{sh\sqrt{\sigma}} ch\sqrt{\sigma}(x-1) ch\sqrt{\sigma}\xi, & x > \xi, \\ -\frac{\sqrt{\sigma}}{sh\sqrt{\sigma}} ch\sqrt{\sigma} x ch\sqrt{\sigma}(\xi-1), & x < \xi, \end{cases} \tag{6}
$$

$$
\sigma = \frac{6(1-\nu)k_0^2}{h^2}.
$$

Applying (5), and (6), from the second equation of system (3) we obtain the integro-differential equation with respect to $w(x)$

$$
\begin{aligned}
&\frac{Eh}{2(1+\nu)} \left(k_0^2 + \frac{1}{1-\nu} \int_0^1 \left(w'(\xi)\right)^2 d\xi \right) w''(x) \\
&- \frac{3(1-\nu)Ek_0^4}{(1+\nu)hsh\sqrt{\sigma}} \left(sh\sqrt{\sigma}(x-1) \int_0^x ch\sqrt{\sigma}\xi w'(\xi) d\xi \right. \\
&\left. + sh\sqrt{\sigma} x \int_x^1 ch\sqrt{\sigma}(\xi-1) w'(\xi) d\xi \right) + f(x) = 0,
\end{aligned} \tag{7}
$$

which we complement with the corresponding boundary condition

$$
w(0) = w(1) = 0. \tag{8}
$$

Thus problem (3), (4) reduces to problem (7), (8) for the function $w(x)$. After solving the latter problem, we construct the functions $u(x)$ and $\psi(x)$ by explicit formulas of form (5).

Now let us consider the question of approximate solution of problem (7), (8).

IV. ASSUMPTIONS

Assume that for each $i = 1, 2, \ldots$ there exists an integral

$$
f_i = 4 \int_0^1 f(x) \sin i\pi x \, dx
$$

and the inequality

$$
|f_i| \le \frac{\omega}{i^m}, \quad i = 1, 2, \ldots, \tag{9}
$$

is fulfilled with ω and m being some positive constants.

Suppose there exists a solution of problem (7), (8) representable as a series

$$
w_n(x) = \sum_{i=1}^{\infty} w_i \sin i\pi x, \tag{10}
$$

the coefficients of which satisfy the system of equations

$$
\begin{aligned}
&i\pi w_i \left[\frac{2(1-\nu)}{\frac{1}{k_0^2} + \frac{6(1-\nu)}{h^2 (i\pi)^2}} + \sum_{j=1}^{\infty} \left(j\pi w_j \right)^2 \right] \\
&\quad - \frac{2(1-\nu^2)}{Ehi\pi} f_i = 0, \quad i = 1, 2, \ldots.
\end{aligned} \tag{11}
$$

Note that the i-th equation of system (11) is a result of the substitution of (10) into (7) followed by the multiplication of the obtained equation by $\sin i\pi x$ and its integration over x from 0 to 1 and also using the formulas

$$
\begin{aligned}
&\int_0^1 \sin i\pi x \sin j\pi x dx = \begin{cases} 0, & i \ne j, \\ \frac{1}{2}, & i = j, \end{cases} \\
&\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx), \\
&\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx).
\end{aligned} \tag{12}
$$

V. ALGORITHM

Let us write an approximate solution of problem (7), (8) in the form

$$w_n(x) = \sum_{i=1}^{n} w_{ni} \sin i\pi x, \tag{13}$$

where the coefficients w_{ni} are found by the Galerkin method from the system of nonlinear equations

$$i\pi w_{ni}\left[\frac{2(1-\nu)}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{h^2(i\pi)^2}}+\sum_{j=1}^{n}\left(j\pi w_{nj}\right)^2\right] - \frac{2(1-\nu^2)}{Ehi\pi} f_i = 0, \quad i = 1,2,\dots,n, \tag{14}$$

which is obtained by means of formulas (12) and is a system of nonlinear equations with respect to $i\pi w_{ni}$, $i = 1,2,\dots,n$.

System (14) is solved using the Jacobi nonlinear iteration process [4]

$$i\pi w_{ni,k+1}\left[\frac{2(1-\nu)}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{h^2(i\pi)^2}}+\left(i\pi w_{ni,k+1}\right)^2 + \sum_{\substack{j=1\\ j\neq i}}^{n}\left(j\pi w_{nj,k}\right)^2\right] - \frac{2(1-\nu^2)}{Ehi\pi} f_i = 0, \tag{15}$$

$$k = 0,1,\dots, \quad i = 1,2,\dots,n.$$

Here $w_{ni,k+l}$ is the $(k+l)$-th iteration approximation of w_{ni}, $l = 0,1,\;\; k = 0,1,\dots.$

To realize iteration (15), we have to solve a cubic equation with respect to $i\pi w_{ni,k+1}$ at the $(k+1)$-th iteration step for each i. Using Cardano's formula [3] this problem can be taken off by writing $i\pi w_{ni,k+1}$ in explicit form. It is pertinent to recall that for the incomplete cubic equation

$$y^3 + Ry + S = 0 \tag{16}$$

the a priori real root is equal to

$$y = \left[-\frac{S}{2}+\left(\frac{S^2}{4}+\frac{R^3}{27}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} - \left[\frac{S}{2}+\left(\frac{S^2}{4}+\frac{R^3}{27}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}. \tag{17}$$

Let us write (15) in form (16) as follows

$$\left(i\pi w_{ni,k+1}\right)^3 + r_i\left(i\pi w_{ni,k+1}\right) + s_i = 0, \quad i = 1,2\dots,n, \tag{18}$$

where

$$r_i = \frac{2(1-\nu)}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{h^2(i\pi)^2}} + \sum_{\substack{j=1\\ j\neq i}}^{n}\left(j\pi w_{nj,k}\right)^2, \qquad s_i = -\frac{2(1-\nu^2)}{Ehi\pi} f_i. \tag{19}$$

In view of (17), for the solution of equation (18) we can write

$$i\pi w_{ni,k+1} = \sigma_{i,1} - \sigma_{i,2}, \;\; k = 0,1,\dots, \;\; i = 1,2,\dots,n, \tag{20}$$

where

$$\sigma_{i,p} = \left[(-1)^p \frac{s_i}{2} + \left(\frac{s_i^2}{4}+\frac{r_i^3}{27}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}, \quad p = 1,2. \tag{21}$$

The solution algorithm of problem (7), (8) under consideration should be understood as the counting carried out by formula (20). Having $w_{ni,k}$, $i = 1,2,..,n$, we construct the approximate value of the function $w(x)$

$$w_{n,k}(x) = \sum_{i=1}^{n} w_{ni,k} \sin i\pi x\,. \tag{22}$$

VI. Defining the Approximation Error of the Function w

Let us compare the approximate solution (22) with the n-th truncation of the exact solution (10)

$$p_n w(x) = \sum_{i=1}^{n} w_i \sin i\pi x\;. \tag{23}$$

This means that the approximation error of the function $w(x)$ is defined as a difference

$$p_n w(x) - w_{n,k}(x) \tag{24}$$

which we write as a sum

$$p_n w(x) - w_{n,k}(x) = \Delta w_n(x) + \Delta w_{n,k}(x), \tag{25}$$

where $\Delta w_n(x)$ is the Galerkin method error and $\Delta w_{n,k}(x)$ the Jacobi iteration error which are respectively equal to

$$\begin{aligned} \Delta w_n(x) &= p_n w(x) - w_n(x), \\ \Delta w_{n,k}(x) &= w_n(x) - w_{n,k}(x). \end{aligned} \tag{26}$$

We set the task of estimating the $L_2(0,1)$–norm of $p_n w(x) - w_{n,k}(x)$. For this we have to estimate the errors of the Galerkin method and the Jacobi iteration.

VII. Galerkin Method Error

Let us expand $\Delta w_n(x)$ into a series. Taking (26), (23) and (13) into account we write

$$\Delta w_n(x) = \sum_{i=1}^{n} \Delta w_{ni} \sin i\pi x, \tag{27}$$

where

$$\Delta w_{ni} = w_i - w_{ni}, \quad i = 1,2,\ldots,n. \tag{28}$$

(27) implies

$$\left\| \frac{d^l}{dx^l} \Delta w_n(x) \right\|_{L_2(0,1)} = \left(\frac{1}{2} \sum_{i=1}^{n} (i\pi)^{2l} \Delta w_{ni}^2 \right)^{\frac{1}{2}}, \quad l = 0,1. \tag{29}$$

We will come back to (29) later, while now we denote

$$\gamma_{l\,n} = i\pi \left\{ (2-l)\alpha_i + \frac{1}{2}\left[\sum_{j=1}^{n} (j\pi w_j)^2 + (-1)^{l+1} \sum_{j=1}^{n} (j\pi w_{nj})^2 \right] \right\}, \tag{30}$$

$$\begin{aligned} \varepsilon_n &= i\pi \sum_{j=n+1}^{\infty} (j\pi w_j)^2, \\ \nabla_n &= \frac{1}{2} i\pi \sum_{j=1}^{n} (j\pi)^2 (w_j + w_{nj}) \Delta w_{nj}, \end{aligned} \tag{31}$$

$$\begin{aligned} \alpha_i &= 2(1-\nu)\left[\frac{1}{k_0^2} + \frac{6(1-\nu)}{h^2 (i\pi)^2} \right]^{-1}, \\ \beta_i &= \frac{2(1-\nu^2)}{Ehi\pi}. \end{aligned} \tag{32}$$

Let us rewrite (11) and (14) as

$$(\gamma_{1n} + \gamma_{2n} + \varepsilon_n)\, w_i = \beta_i f_i$$

and

$$(\gamma_{1n} - \gamma_{2n})\, w_{ni} = \beta_i f_i.$$

By virtue of (30) and (31) we have $\gamma_{2n} = \nabla_n$ and therefore

$$(\gamma_{1n} + \nabla_n + \varepsilon_n) w_i = \beta_i f_i$$

and

$$(\gamma_{1n} - \nabla_n)\, w_{ni} = \beta_i f_i.$$

Subtracting the last two equalities from each other and taking (28) into account, we obtain the equation

$$\gamma_{1n} \Delta w_{ni} + \nabla_n (w_i + w_{ni}) + \varepsilon_n w_i = 0$$

which we multiply by $i\pi \Delta w_{ni}$ and sum over $i = 1,2,\ldots,n$. Using (30), (31) and the inequality

$$\sum_{i=1}^{n} \nabla_n (w_i + w_{ni})\, i\pi \Delta w_{ni} \geq 0$$

following from (31), we see that

$$\sum_{i=1}^{n} \alpha_i (i\pi \Delta w_{ni})^2 \leq \sum_{i=1}^{n} (i\pi)^2 \left| w_i \Delta w_{ni} \right| \sum_{i=n+1}^{\infty} (i\pi w_i)^2.$$

By the Cauchy-Schwarz inequality, we therefore have

$$\left(\sum_{i=1}^{n} \alpha_i (i\pi \Delta w_{ni})^2 \right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{n} \frac{1}{\alpha_i} (i\pi w_i)^2 \right)^{\frac{1}{2}} \sum_{i=n+1}^{\infty} (i\pi w_i)^2. \tag{33}$$

Let us estimate the right-hand side of inequality (33). We introduce into consideration the values

$$\begin{aligned} c_1 &= \frac{1}{2(1-\nu)} \left(\frac{\omega(1-\nu^2)}{\pi k_0 Eh} \right)^2, \\ c_2 &= 3\left(\frac{\omega(1-\nu^2)}{\pi^2 Eh^2} \right)^2 \end{aligned} \tag{34}$$

and, keeping in mind (32), rewrite (11) as

$$i\pi w_i\left(\alpha_i+\sum_{j=1}^{\infty}(j\pi w_j)^2\right)-\beta_i f_i=0\,. \tag{35}$$

In the sequel we will frequently use relations (32), (9), (34) and the integral test for convergence of series without declaring each time that we have done so. Let $n>1$.

Lemma 1. The estimate

$$\sum_{i=1}^{n}w_i^2\le\frac{1}{\pi^2}\left\{c_1\left[1+\frac{1}{2m+3}\left(1-\frac{1}{n^{2m+3}}\right)\right]+c_2\left[1+\frac{1}{2m+5}\left(1-\frac{1}{n^{2m+5}}\right)\right]\right\}^{\frac{1}{2}} \tag{36}$$

is valid.

Proof. We multiply equation (35) by $\frac{1}{i\pi}w_i$ and sum the resulting equality over $i=1,2,\ldots,n.$ By virtue of the fact that $\sum_{j=1}^{\infty}(j\pi w_j)^2\ge\pi^2\sum_{j=1}^{n}w_j^2,$ we write

$$\sum_{i=1}^{n}\alpha_i w_i^2+\pi^2\left(\sum_{i=1}^{n}w_i^2\right)^2\le\sum_{i=1}^{n}\frac{1}{i\pi}\beta_i|f_iw_i|.$$

Hence we have the inequalities

$$\left(\sum_{i=1}^{n}w_i^2\right)^2\le\frac{1}{4\pi^2}\sum_{i=1}^{n}\frac{1}{\alpha_i}\left(\beta_i\frac{f_i}{i\pi}\right)^2\le\frac{1}{\pi^4}\left[c_1\left(1+\int_1^n\frac{1}{x^{2m+4}}\,dx\right)+c_2\left(1+\int_1^n\frac{1}{x^{2m+6}}\,dx\right)\right],$$

which bring us to (36). □

Lemma 2. The inequality

$$\sum_{i=1}^{n}(i\pi w_i)^2\le\left\{c_1\left[1+\frac{1}{2m+1}\left(1-\frac{1}{n^{2m+1}}\right)\right]+c_2\left[1+\frac{1}{2m+3}\left(1-\frac{1}{n^{2m+3}}\right)\right]\right\}^{\frac{1}{2}} \tag{37}$$

is fulfilled.

Proof. Multiply equation (35) by $i\pi w_i$ and sum the resulting relation over $i=1,2,\ldots,n.$

Since

$$\sum_{j=1}^{\infty}(j\pi w_j)^2\ge\sum_{j=1}^{n}(j\pi w_j)^2,$$

we first obtain

$$\sum_{i=1}^{n}\left(\alpha_i+\sum_{j=1}^{n}(j\pi w_j)^2\right)(i\pi w_i)^2\le\sum_{i=1}^{n}i\pi\beta_i\,|f_iw_i|$$

and then

$$\left(\sum_{i=1}^{n}(i\pi w_i)^2\right)^2\le\frac{1}{4}\sum_{i=1}^{n}\frac{1}{\alpha_i}(\beta_i f_i)^2\le c_1\left(1+\int_1^n\frac{1}{x^{2m+2}}\,dx\right)+c_2\left(1+\int_1^n\frac{1}{x^{2m+4}}\,dx\right)$$

whence we obtain (37). □

Lemma 3. The relation

$$\sum_{i=n+1}^{\infty}(i\pi w_i)^2\le\left(c_1\frac{1}{(2m+1)n^{2m+1}}+c_2\frac{1}{(2m+3)n^{2m+3}}\right)^{\frac{1}{2}} \tag{38}$$

takes place.

Proof. Multiply equation (35) by $i\pi w_i$ and sum the obtained expression over $i=n+1,n+2,\ldots$. Let us apply

$$\sum_{j=1}^{\infty}(j\pi w_j)^2\ge\sum_{j=n+1}^{\infty}(j\pi w_j)^2\,.$$

The result will be as follows

$$\sum_{i=n+1}^{\infty}\left(\alpha_i+\sum_{j=n+1}^{\infty}(j\pi w_j)^2\right)(i\pi w_i)^2\le\sum_{i=n+1}^{\infty}i\pi\beta_i\,|f_iw_i|.$$

Therefore we get

$$\left(\sum_{i=n+1}^{\infty}(i\pi w_i)^2\right)^2\le\frac{1}{4}\sum_{i=n+1}^{\infty}\frac{1}{\alpha_i}(\beta_i f_i)^2\le\int_n^{\infty}\left(c_1\frac{1}{x^{2m+2}}+c_2\frac{1}{x^{2m+4}}\right)dx,$$

which leads to (38). □

Using the notation from (34) and (32), we introduce into consideration the value

$$\tau = \frac{1}{\alpha_1\sqrt{2}}\left\{\left[c_1\left(1+\frac{1}{2m+1}\left(1-\frac{1}{n^{2m+1}}\right)\right) + c_2\left(1+\frac{1}{2m+3}\left(1-\frac{1}{n^{2m+3}}\right)\right)\right]^{\frac{1}{2}} + \left[c_1\left(1+\frac{1}{2m+3}\left(1-\frac{1}{n^{2m+3}}\right)\right) + c_2\left(1+\frac{1}{2m+5}\left(1-\frac{1}{n^{2m+5}}\right)\right)\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}, \tag{39}$$

which is uniformly bounded both below and above with respect to *n*.

From (29), (33), (32) and (36)–(39) it follows that the estimate

$$\left\| \frac{d^l}{dx^l}\Delta w_n(x)\right\|_{L_2(0,1)} \le \frac{1}{\pi^{1-l}}\,\tau\left(\frac{c_1}{(2m+1)n^{2m+1}} + \frac{c_2}{(2m+3)n^{2m+3}}\right)^{\frac{1}{2}}, \quad l = 0,1, \tag{40}$$

is fulfilled for the error of the Galerkin method.

Note that in [2] and [5] the authors considered the convergence of the Galerkin method, but not its accuracy, for both the static and the dynamic one-dimensional Timoshenko system.

VIII. Jacobi Iteration Error

Taking (26), (13) and (22) into account, we represent $\Delta w_{n,k}(x)$ as a series

$$\Delta w_{n,k}(x) = \sum_{i=1}^{n}\Delta w_{ni,k}\sin i\pi x\,, \tag{41}$$

where

$$\Delta w_{ni,k} = w_{ni} - w_{ni,k}\,, \qquad i = 1,2,\ldots,n. \tag{42}$$

Series (41) implies the formula

$$\left\| \frac{d^l}{dx^l}\Delta w_{n,k}(x)\right\|_{L_2(0,1)} = \left(\frac{1}{2}\sum_{i=1}^{n}(i\pi)^{2l}\,\Delta w_{ni,k}^2\right)^{\frac{1}{2}}, \quad l = 0,1, \tag{43}$$

to be used later.

Let us represent system (20) as follows

$$i\pi w_{ni,k+1} = \varphi_i(1\pi w_{n1,k}, 2\pi w_{n2,k}, \ldots, n\pi w_{nn,k}) \tag{44}$$

and consider the Jacobi matrix

$$J = \left(\frac{\partial\varphi_i}{\partial(j\pi w_{nj,k})}\right)_{i,j=1}^{n}$$

(in this paper this is the second notation connected with the name of C. Jacobi, 1804–1851).

By virtue of (19)–(21) and (44) the diagonal terms of the matrix J are equal to zero, while for the nondiagonal terms we have

$$\frac{\partial\varphi_i}{\partial(j\pi w_{nj,k})} = \frac{1}{27}\, j\pi w_{nj,k} r_i^2\left(\frac{s_i^2}{4}+\frac{r_i^3}{27}\right)^{-\frac{1}{2}}\left(\frac{1}{\sigma_{i,1}^2}-\frac{1}{\sigma_{i,2}^2}\right).$$

If in this equality we use the relations

$$\sigma_{i,1}\sigma_{i,2} = \frac{r_i}{3}, \qquad \sigma_{i,2}^3 - \sigma_{i,1}^3 = s_i, \qquad \left(\frac{s_i^2}{4}+\frac{r_i^3}{27}\right)^{\frac{1}{2}} = \frac{\sigma_{i,1}^3+\sigma_{i,2}^3}{2}, \tag{45}$$

which follow from (21), then we get

$$\frac{\partial\varphi_i}{\partial(j\pi w_{nj,k})} = \frac{2}{3}\, j\pi w_{nj,k} s_i\left(\sigma_{i,1}^4 + \left(\frac{r_i}{3}\right)^2 + \sigma_{i,2}^4\right)^{-1}.$$

Apply to the latter equality the estimate

$$\sigma_{i,1}^4 + \sigma_{i,2}^4 \ge 2\left(\frac{r_i}{3}\right)^2,$$

which is obtained from the first relation in (45) and also use (19). We derive the estimate

$$\left|\frac{\partial \varphi_i}{\partial(j\pi w_{nj,k})}\right| \le \frac{4(1-\nu^2)}{Eh i\pi}\,|f_i|\; j\pi\,|w_{nj,k}|$$
$$\times\left(\frac{2(1-\nu)}{\dfrac{1}{k_0^2}+\dfrac{6(1-\nu)}{h^2\pi^2}}+\left(j\pi w_{nj,k}\right)^2\right)^{-2}. \tag{46}$$

Let us use the vector and matrix norms equal respectively to $\sum_{i=1}^{n}|v_i|$ and $\max_{1\le j\le n}\sum_{i=1}^{n}|m_{ij}|$ for $v=(v_i)_{i=1}^{n}$ and $M=(m_{ij})_{i,j=1}^{n}$.

It is required that for an arbitrary set of values $w_{nj,k}$, $j=1,2,\ldots,n,$ the elements of the matrix J satisfy the condition $\max_{1\le j\le n}\sum_{i=1}^{n}\left|\frac{\partial\varphi_i}{\partial(j\pi w_{nj,k})}\right|\le q<1.$

For this, as follows from (46) and the properties of the function

$$g(\xi)=\frac{\xi}{(a+\xi^2)^2},\quad 0\le\xi<\infty,\quad a=const>0,$$

it suffices that

$$\frac{3}{8Eh\pi}\sqrt{\frac{3}{2}}\frac{1+\nu}{\sqrt{1-\nu}}\left(\frac{1}{k_0^2}+\frac{6(1-\nu)}{h^2\pi^2}\right)^{\frac{3}{2}}$$
$$\times\sum_{\substack{i=1\\ i\ne j}}^{n}\frac{|f_i|}{i}\le q<1,\qquad j=1,2,\ldots,n. \tag{47}$$

Then, according to the map compression principle, system (14) has a unique solution w_{ni}, $i=1,2,\ldots,n,$ the iteration process (20) converges, $\lim_{k\to\infty} w_{ni,k}=w_{ni}$, $i=1,2,\ldots,n,$ with the rate defined by the inequality

$$\sum_{i=1}^{n} i\left|w_{ni}-w_{ni,k}\right|\le\frac{q^k}{1-q}\sum_{i=1}^{n} i\left|w_{ni,1}-w_{ni,0}\right|,\quad k=0,1,\ldots.$$

From this, (42) and (43) we obtain the estimate for the Jacobi iteration error

$$\left\|\frac{d^l}{dx^l}\Delta w_{n,k}(x)\right\|_{L_2(0,1)}\le\frac{\pi^l}{\sqrt{2}}\frac{q^k}{1-q}\sum_{i=1}^{n} i\left|w_{ni,1}-w_{ni,0}\right|,$$
$$l=0,1,\qquad k=0,1,\ldots. \tag{48}$$

At the end of this section, condition (47) is replaced by the condition which is simpler to verify but is a stricter one. For this we apply inequality (9), the integral test for convergence of series and ignore the fact that $i\ne j$ when carrying out summation in (47). Eventually we obtain

$$\frac{3\omega}{8mEh\pi}\sqrt{\frac{3}{2}}\frac{1+\nu}{\sqrt{1-\nu}}\left(\frac{1}{k_0^2}+\frac{6(1-\nu)}{h^2\pi^2}\right)^{\frac{3}{2}}\left(2-\frac{1}{n^m}\right)\le q<1. \tag{49}$$

IX. Estimation of the Approximation Error of the Function w

Let us estimate error (24). By (25) we have

$$\left\|\frac{d^l}{dx^l}\left(p_n w(x)-w_{n,k}(x)\right)\right\|_{L_2(0,1)}$$
$$\le\left\|\frac{d^l}{dx^l}\Delta w_n(x)\right\|_{L_2(0,1)}+\left\|\frac{d^l}{dx^l}\Delta w_{n,k}(x)\right\|_{L_2(0,1)}$$

and therefore the application of (40) and (48) gives the inequality

$$\left\|\frac{d^l}{dx^l}\left(p_n w(x)-w_{n,k}(x)\right)\right\|_{L_2(0,1)}\le\frac{1}{\pi^{1-l}}\tau$$
$$\times\left(\frac{c_1}{(2m+1)n^{2m+1}}+\frac{c_2}{(2m+3)n^{2m+3}}\right)^{\frac{1}{2}}$$
$$+\frac{\pi^l}{\sqrt{2}}\frac{q^k}{1-q}\sum_{i=1}^{n} i\left|w_{ni,1}-w_{ni,0}\right|,$$
$$l=0,1,\qquad k=0,1,\ldots. \tag{50}$$

The obtained result can be summarized as follows.

Theorem 1. Let $n>1$ and q be some numbers from the interval $(0,1)$. Assume that the conditions of Subsection IV and restriction (47) or (49) are fulfilled. Then the approximation error of the function $w(x)$ is estimated by inequality (50), where the constants c_1 and c_2 are calculated by formulas (34), while the coefficient τ defined by (39) is uniformly bounded both below and above.

X. Approximation of the Functions u and ψ

Let us turn to formulas (5). Using $p_n w(x)$ and $w_{n,k}(x)$, we construct n-th truncation of the functions $u(x)$ and $\psi(x)$

$$p_n u(x) = \int_0^1 G_u(x,\xi)\left(p_n w(\xi)\right)'^2 d\xi,$$
$$p_n \psi(x) = \int_0^1 G_\psi(x,\xi)\left(p_n w(\xi)\right)' d\xi \tag{51}$$

and the approximation of the same functions

$$u_{n,k}(x) = \int_0^1 G_u(x,\xi) w'^2_{n,k}(\xi) d\xi,$$
$$\psi_{n,k}(x) = \int_0^1 G_\psi(x,\xi) w'_{n,k}(\xi) d\xi. \tag{52}$$

By analogy with (24), we define the approximation errors of the functions $u(x)$ and $\psi(x)$ through the differences $p_n u(x) - u_{n,k}(x)$ and $p_n\psi(x) - \psi_{n,k}(x)$ and estimate the $L_2(0,1)$–norm of either of them. From (51) and (52) we obtain

$$p_n u(x) - u_{n,k}(x) = \int_0^1 G_u(x,\xi)\left[\left(\left(p_n w(\xi)\right)'\right)^2 - \left(w'_{n,k}(\xi)\right)^2\right] d\xi \tag{53}$$

and

$$p_n\psi(x) - \psi_{n,k}(x) = \int_0^1 G_\psi(x,\xi)\left(p_n w(\xi) - w_{n,k}(\xi)\right)' d\xi. \tag{54}$$

XI. Estimation of the Approximation Errors of the Functions u and ψ

From (53) and (6) we get

$$\left(p_n u(x) - u_{n,k}(x)\right)^2 = \frac{1}{4}\left[(x-1)^2\left(\int_0^x H(\xi)d\xi\right)^2 + 2x(x-1)\int_0^x H(\xi)d\xi \int_x^1 H(\xi)d\xi + x^2\left(\int_x^1 H(\xi)d\xi\right)^2\right],$$

where

$$H(\xi) = \left(\left(p_n w(\xi)\right)'\right)^2 - \left(w'_{n,k}(\xi)\right)^2.$$

Therefore

$$\left(p_n u(x) - u_{n,k}(x)\right)^2 \le \left(x - \frac{1}{2}\right)^2\left(\int_0^1 \left|H(\xi)\right| d\xi\right)^2. \tag{55}$$

Let us estimate $\int_0^1 \left|H(\xi)\right| d\xi$. Since

$$w_{n,k}(\xi) = p_n w(\xi) + \left(w_{n,k}(\xi) - p_n w(\xi)\right)$$

we write

$$\int_0^1 \left|H(\xi)\right| d\xi \le \left\| \left(p_n w(x) - w_{n,k}(x)\right)' \right\|_{L_2(0,1)} \times\left(2\left\| \left(p_n w(x)\right)' \right\|_{L_2(0,1)} + \left\| \left(p_n w(x) - w_{n,k}(x)\right)' \right\|_{L_2(0,1)}\right). \tag{56}$$

Note that by virtue of (23) and (37)

$$\left\| \left(p_n w(x)\right)' \right\|_{L_2(0,1)} \le \frac{1}{\sqrt{2}}\left\{c_1\left[1 + \frac{1}{2m+1}\left(1 - \frac{1}{n^{2m+1}}\right)\right] + c_2\left[1 + \frac{1}{2m+3}\left(1 - \frac{1}{n^{2m+3}}\right)\right]\right\}^{\frac{1}{4}}. \tag{57}$$

Now, in view of (55)–(57) and (50) we have

$$\left\|p_n u(x) - u_{n,k}(x)\right\|_{L_2(0,1)} \le \frac{1}{2\sqrt{3}}\sum_{l=0}^{1}\left\{4\left[c_1\left(1 + \frac{1}{2m+1}\left(1 - \frac{1}{n^{2m+1}}\right)\right)\right)\right) + c_2\left(1 + \frac{1}{2m+3}\left(1 - \frac{1}{n^{2m+3}}\right)\right)\Bigg]\Bigg]\Bigg\}^{\frac{l}{4}}$$
$$\times\left[\tau\left(\frac{c_1}{(2m+1)n^{2m+1}} + \frac{c_2}{(2m+3)n^{2m+3}}\right)^{\frac{1}{2}} + \frac{\pi}{\sqrt{2}}\frac{q^k}{1-q}\sum_{i=1}^{n} i\left| w_{ni,1} - w_{ni,0}\right|\right]^{2-l},$$
$$k = 0,1,\dots. \tag{58}$$

Further, (54) and (6) imply

$$\left(p_n\psi(x)-\psi_{n,k}(x)\right)^2=\frac{\sigma}{sh^2\sqrt{\sigma}}\left[ch\sqrt{\sigma}(x-1)\right.$$
$$\times\int_0^x ch\,\sqrt{\sigma}\,\xi\left(p_n w(\xi)-w_{n,k}(\xi)\right)'d\xi$$
$$\left.+ch\,\sqrt{\sigma}x\int_x^1 ch\,\sqrt{\sigma}(\xi-1)\left(p_n w(\xi)-w_{n,k}(\xi)\right)'d\xi\right]^2.$$

Therefore

$$\left\|p_n\psi(x)-\psi_{n,k}(x)\right\|_{L_2(0,1)}\le c_0\left\|\left(p_n w(x)-w_{n,k}(x)\right)'\right\|_{L_2(0,1)}, \quad (59)$$

where

$$c_0=\left(\int_0^1 F(x)dx\right)^{\frac{1}{2}} \quad (60)$$

and

$$F(x)=\frac{2\sigma}{sh^2\sqrt{\sigma}}\left[ch^2\sqrt{\sigma}(x-1)\int_0^x ch^2\sqrt{\sigma}\xi d\xi\right.$$
$$\left.+ch^2\sqrt{\sigma}x\int_x^1 ch^2\sqrt{\sigma}(\xi-1)d\xi\right].$$

Using the formula

$$\int ch^2 x dx=\frac{sh2x}{4}+\frac{x}{2}$$

and after calculating the integrals we get

$$F(x)=\frac{\sigma}{2sh^2\sqrt{\sigma}}\left[1+\frac{1}{2\sqrt{\sigma}}sh2\sqrt{\sigma}\right.$$
$$+ch2\sqrt{\sigma}x+sh\sqrt{\sigma}\left(\frac{1}{\sqrt{\sigma}}ch\sqrt{\sigma}(2x-1)\right.$$
$$\left.\left.-2xsh\sqrt{\sigma}(2x-1)\right)\right].$$

Substituting this expression into (60), once more performing integration and applying in particular the formula

$$\int xshxdx=xchx-shx,$$

we find

$$c_0=\frac{1}{sh\sqrt{\sigma}}\left(\frac{\sigma}{2}+\frac{\sqrt{\sigma}}{4}sh\,2\sqrt{\sigma}+sh^2\sqrt{\sigma}\right)^{\frac{1}{2}}. \quad (61)$$

By (59) and (50) we have

$$\left\|p_n\psi(x)-\psi_{n,k}(x)\right\|_{L_2(0,1)}\le c_0\left[\tau\left(\frac{c_1}{(2m+1)n^{2m+1}}\right.\right.$$
$$\left.\left.+\frac{c_2}{(2m+3)n^{2m+3}}\right)^{\frac{1}{2}}+\frac{\pi}{\sqrt{2}}\frac{q^k}{1-q}\sum_{i=1}^n i\left|w_{ni,1}-w_{ni,0}\right|\right], \quad (62)$$
$$k=0,1,\ldots.$$

The obtained results is formulated as follows.

Theorem 2. Assume the conditions of Theorem 1 to be fulfilled. Then the approximation errors of the functions $u(x)$ and $\psi(x)$ are estimated by inequalities (58) and (62), respectively, where the coefficients c_0, c_1 and c_2 defined by (61) and (34) do not depend on n, while the coefficient τ, given by (39), is uniformly bounded with respect to n both below and above.

References

[1] J. E. Lagnese and J.-L. Lions, *Modelling analysis and control of thin plates*, Masson: Springer-Verlag, 1988.

[2] V. Odisharia, "A generalized solution of a one-dimensional nonlinear Timoshenko system," in *Int. Conf. on Nonclassical Problems of Mechanics*, October 25-27, Kutaisi, 2007, pp. 137-139.

[3] A. Kurosh, *Higher Algebra*, (Russian) Moscow: Mir, 1988.

[4] J. M. Ortega, W. Rheinboldt, *Iterative solution of nonlinear equations in several variables*, Classics in Applied Mathematics, 30. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000.

[5] J. Peradze, "On the solution of a dynamic system of one-dimensional Timoshenko equations," in *Proc. Conf. on Modern Problems of Cybernetics and Applied Mathematics*, December 21-23, Tbilisi, 1987, Tbilisi Univ. Press, 1991, pp.49-52 .

[6] A. S. Volmir, *Nonlinear dynamics of shells and plates*, (Russian) Moscow: Nauka, 1972.

[7] I. I. Vorovich, *Mathematical problems of the nonlinear theory of shallow shells*, (Russian) Moscow: Nauka, 1989.