Growth of a Class of Plurisubharmonic Function in a Unit Polydisc I

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***Abstract*—**The Growth of a non- constant analytic function of several complex variables is a very classical concept, but for a finite domain it is a recent concept initiated by Juneja and Kapoor[1],and later on substantiated by Sinha[2]. A Unit Polydisc is the most fundamental example of a Compact Riemann Surface, there is an upsurge in the area as being reflected in[4,5]. This is a very important concept and its applications can also be seen in Complex Analytical Dynamics. However in this present article we shall be concentrating on the growth parameter "type" for such functions and also describe its Geometrical Properties.We have investigated upon some finer results on the growth of slowly varying functions.
The above concept of Growth can also be utilized in Computer aided Tomography[7].

***Keywords*—**Analytic Function, Complex Analytical Dynamics,Growth Parameter, Type.

1. INTRODUCTION

In this section we first define a class $E(\beta)$ where $\beta \in L^0$. The class of such functions L^0 was defined by Juneja Kapoor[1] and Sinha in [2]. Such classes of functions were initiated by Seremeta [6] and then has been used extensively by Sinha[3]. To study the functions having fast or slow growths an important concept of (p,q) order and type was studied by many authors in the past. In the present paper we have introduced the concept of (q,1) order and type for
Functions having fast growth. The above results can also be used in Computer Aided Tomography as, by the famous Riemann- Mapping Theorem any simply connected domain can be conformally mapped onto Unit- Disc,so we can project the three dimensional Tumour in a two dim.plane and study its growth through the above methods and then we can apply inverse transform to study its Growth[7]. Work in this direction is in progress.

Let $D \subset C^n$ be a domain.

Definition 1:

Let $E(\beta)$ be a class of functions $\varphi(\overline{R})$ satisfying the following
Properties,

1. $\varphi(\overline{R})$ is upper semi-continuous on D.
2. $\varphi(\overline{R})$ is monotone, nondecreasing in each of the variables $R_1, R_2, \ldots R_n$.
3. $\varphi(\overline{R})$ is pluriconvex in the variables, $-\beta(\log(1-R_1)), -\beta(\log(1-R_2)), \ldots -\beta(\log(1-R_n))$, that is to say for every $\bar{t} = (t_1, t_2, \ldots t_n)$ *and* $\bar{s} = (s_1, s_2, \ldots s_n)$ in I^n, and for all λ, μ *with* $\lambda + \mu = 1$

$$\varphi\begin{bmatrix} 1-\exp\left\{\beta^{-1}(\lambda\beta(\log(1-t_1)) + \mu(\beta(\log(1-s_1))\right\}, \ldots, \\ 1-\exp\left\{\beta^{-1}(\lambda\beta(\log(1-t_n)) + \mu(\beta(\log(1-s_n))\right\} \end{bmatrix}$$
$$\le \lambda\varphi(t_1, t_2, \ldots t_n) + \mu\varphi(s_1, s_2, \ldots s_n) \qquad (1.1).$$

Here $I^n = \left\{(R_1, R_2 \ldots R_n) \in R^n : 0 \le R_i \le 1, i = 1,2,\ldots n\right\}$.

Upon substituting $\beta(x) = x$, the identity function the class reduces to the class of functions $\varphi(\overline{R})$ defined by Juneja and Kapoor[1].

Definition 2: Let $M_D(t,\varphi) = \max_{\overline{R} \subset D^n} \varphi(\overline{R}),\ 0 < t < 1$ be the maximum modulus of the function $\varphi(\overline{R})$, and D is the Unit Polydisc.

The generalized order for $\varphi(\overline{R})$ is defined as,

$$\rho(\varphi) = \limsup_{t \to 1} \frac{\alpha(M_D(t,\varphi))}{-\beta(\log(1-t))} \qquad (1.2)$$

here $\alpha \in \Delta$ *and* $\beta \in L^0$.

Example:

$\varphi(R_1, R_2) = \alpha^{-1}\left[-\beta(\log(1-R_1)) - \beta(\log(1-R_2))\right]$.It can be easily
seen that the above function has generalized order 2.

Definition 3: Let $\alpha \in \Delta$ and $0 < \rho < \infty$ be the generalized type of
$f(\widetilde{Z})$ is defined as, $E(\beta)$

$$\sigma = \limsup_{R \to 1} \frac{\log M_{f,D}(R)}{\alpha^{-1}\left[-\rho\log(1-R)\right]} \qquad (1.3)$$

where $M_{f,D}(R) = \sup_{\widetilde{Z} \in D^R} \left| f(\widetilde{Z}); \right|$

Here $\widetilde{Z}$ belongs to D^R if the point $\left(\frac{z_1}{R}, \frac{z_2}{R}, \ldots \frac{z_n}{R}\right)$ belongs to D.

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Definition 4: For $\alpha \in \Delta$ and $\beta \in L^0$, the generalized type with respect to one of the variables keeping the others fixed is defined as

$$\sigma_n = \limsup_{R_n \to 1} \frac{\log M_{f,D}(R)}{\alpha^{-1}[-\rho\beta(\log(1-R_n)} \qquad (1.4)$$

Definition 5: Let $\alpha \in \Delta \, and \, \beta \in L^0$ and let $\varphi(\bar{R}) \in E(\beta)$ be a function of finite generalized type. Let

$$B_\sigma = B_\sigma(\varphi) = \begin{Bmatrix} (a_1, a_2, ..a_n) \in R_+^n \ such that \\ \varphi(\bar{R}) < \sum_{i=0}^{n} b_i \left[\alpha^{-1}\{\rho_i \beta(\log(1-R_i)\}\right] \ as \ \|\bar{R}\| \to 1 \end{Bmatrix} \qquad (1.5)$$

Properties of B_σ

1.The set B_σ is octant- like.

2. the boundary points of the set $B_\sigma(\varphi)$ form a certain hypersurface $S_\sigma = S_\sigma(\varphi)$ which divides the hyperoctant R_+^n into

Two parts, one in which the inequality (1.5) is true and the other

In which it is false.Thus we call it the hypersurface of generalized associated typesof the plurisubharmonic function in the class $E(\beta)$, and any system of numbers $(\sigma_1, \sigma_2, ...\sigma_n) \in S_\sigma(\varphi)$

will be called a system of generalized associated types of the function.

Remark: Considering $\alpha(x) = \log x, and \ \beta(x) = x \ and \ \varphi(\bar{R}) = \log M(\bar{R}, f)$ where $M(\bar{R}, f) = \max_{|z_i| = R_i} \left| f(\tilde{Z}) \right|$, for every i, the above definition coincides with that of Juneja and Kapoor[1].

Definition 6: For a function $f(\tilde{Z})$ analytic in a unit polydisc D we define,

$$\rho_D(q) = \limsup_{t \to 1} \frac{\log^{[q]} M_D(t, f)}{-\log(1-t)}$$

$$where \ M_D(t, f) = \max_{\bar{r} \in R^n} M(\bar{r}, f), \ 0 < t < 1. \ (1.2)$$

Definition 7: For a function f, $0 < \rho_D < \infty$;

$$T_D(q) = \limsup_{t \to 1} \frac{\log^{(q-1)} M_D(t, f)}{(1-t)^{-\rho_D(q)}} \quad (1.3).$$

Definition 8: Let $\alpha \in \Delta \ and \ \beta \in L^0$, then the generalized order of $\varphi(R) \in E(\beta)$ with respect to the variable R_i (keeping the others $i \neq j$ fixed) is defined as,

$$\rho_n^* = \limsup_{R_n \to 1_n} \frac{\alpha(\varphi^+(\bar{R}))}{-\beta(\log(1-R_n))} \qquad (1.4).$$

II .MAIN RESULTS

Theorem 2.1: The generalized order can be obtained from the Definition of Generalized Type.

Proof: From the definition of limit superior we have for any $\varepsilon > 0,$

$$\frac{\log M_{f,D}(R)}{\alpha^{-1}[-\rho \log(1-R)]} < \sigma \ for \ (1-R) > R_0(\varepsilon) \qquad (2.1).$$

By the properties of $\alpha \in \Delta$ we can easily obtain,

$$\limsup_{R \to 1} \frac{\log M_{f,D}(R)}{\alpha^{-1}[-\rho \log(1-R)]} \leq \rho \quad (2.2)$$

On the other hand from the Definition 3, we get for $0 < \varepsilon < \sigma,$

$$\frac{\log M_{f,D}(R)}{\alpha^{-1}[-\rho \log(1-R_n)]} > (\sigma - \varepsilon) \quad \text{for some}$$

sequence $R_n \to 1,$ from the above we

obtain, $\limsup_{R_{n_n} \to 1} \frac{\alpha(\log M_{f,D}(R))}{-\log(1-R_n)} \geq \rho \qquad (2.3)$

Combining Eqns. (2.2) and (2.3) we arrive at our result.

Theorem 2.2: Let $f(\tilde{Z})$ be an analytic function having generalized order ρ and type σ then $\delta \leq \sigma,$ where;

$$\delta = \limsup_{\|\bar{\kappa}\| \to \infty} \frac{\left(\frac{\|\bar{\kappa}\|}{\rho}\right)}{\alpha^{-1}\left[\rho \log \frac{1}{1 - \exp\left\{\frac{1}{\rho} \frac{1}{\|\bar{\kappa}\|} \log|c_k| d_k(D)\right\}}\right]} \qquad (2.4)$$

$$\|\bar{\kappa}\| = (k_1 + k_2 + ... + k_n) \to \infty \ and$$

and $d_{\bar{k}}(D) = \sup_{Z \in D} \left|\tilde{Z}\right|^{\bar{k}}$

Furthermore if,

$$\frac{d\alpha(x)}{d\log(x)} = 1 \ for \ l \arg e \ x then$$

$\sigma \leq \delta$, hence $\sigma = \delta,$ provided the growth of the function $\alpha^{-1}[\rho \log R]$ is slower than the growth of

$\alpha^{-1}[-\rho\log(1-R)]$.

Proof: The proof follows from the definitions of limit superior as well as careful use of Cauchy's Inequality.

Remark: The above Theorem holds true for analytic functions having positive order only.

Theorem 2.2: Let $\alpha \in \Delta\, and\, \beta \in L^0$ then σ_n is a Convex Function of $\beta\log(1-R_1),...\beta\log(1-R_{n-1})$.

Proof: Choosing,

$t_i = R_i, s_i = R_i' \; for \; i = 1,2...(n-1), and\, t_n = s_n = R$

and putting in Eqn.(1.1) we get,

$$\varphi\left[1-\exp\left\{\beta^{-1}(\lambda\beta\log(1-R_1)+\mu\beta(\log(1-R_1')\right\},...,R\right]$$
$$\le \lambda\varphi(R_1,R_2,...,R)+\mu\varphi(R_1',R_2',...,R)$$

Therefore,

$$\limsup_{R_n\to 1}\frac{\varphi\left[1-\exp\left\{\beta^{-1}(\lambda\beta\log(1-R_1)+\mu\beta(\log(1-R_1')\right\},...,R\right]}{\alpha^{-1}[-\rho\beta\log(1-R_n)]}$$
$$\le \lambda\varphi(R_1,R_2,...,R)+\mu\varphi(R_1',R_2',...,R) \quad (2.5),$$

Proving the assertion.

Theorem 2.3: Let $\varphi(\overline{R}) \in E(\beta)$ and B_σ^0 be the domain consisting of the interior points of the corresponding set B_σ. Then the image of the domain B_σ. under the map $(1-b_i') = \beta(\log(1-b_i)) \;\; \forall\, i$ is a convex domain provided $b_i < \frac{1}{2}\forall\, i.$

Proof: Exploiting the inequality(1.1) with the following values of t_i *and* s_i as,

$$(1-t_i)=\exp\left[\beta^{-1}\left\{\begin{array}{l}\beta(\log(1-R_i)+\\ \dfrac{\mu}{\rho_i}\left(\beta\log(1-b_i')-\beta\log(1-b_i)\right)\end{array}\right\}\right] \quad (2.6),$$

$$(1-s_i)=\exp\left[\beta^{-1}\left\{\begin{array}{l}\beta(\log(1-R_i)+\\ \dfrac{\lambda}{\rho_i}\left(\beta\log(1-b_i')-\beta\log(1-b_i')\right)\end{array}\right\}\right]$$

(2.7).

But $\lambda\beta\log(1-t_i)+\mu\beta(1-s_i) = \beta\log(1-R_i)$

Therefore the result is obtained upon substituting the above values of t_i *and* s_i *in* (1.1).

Remark: The above Theorem is significantly different from what has been obtained in [1].

Lemma 2.1: Let $\alpha \in \Delta.$ then the necessary condition for a point $(b_1, b_2,...b_n) \in R_+^n$ to lie in the interior of the set $B_\sigma(\log M_f(\overline{R}))$ is

$$\limsup_{\|\overline{\kappa}\|\to\infty}\frac{\log\left|c_{\overline{k}}\right|}{\sum_{i=1}^n b_i k_i - \sum_{i=1}^n b_i k_i \log\left[1-\exp\left\{-\dfrac{1}{\rho_i}\alpha(k_i)\right\}\right]} < 1$$

$$where\; \|\overline{\kappa}\| = \sum_{i=1}^n k_i. \qquad (2.8).$$

Proof: Since,

$$\varphi(\overline{R}) < \sum_{i=1}^n b_i\alpha^{-1}\left[-\rho_i\log(1-R_i)\right]$$

we can get, $M_f(\overline{R}) < C_\varepsilon \exp\left[\sum_{i=1}^n b_i\alpha^{-1}[-\rho_i\log(1-R_i)]\right]$

$$for\; some\; C_\varepsilon > 0. \qquad (2.9)$$

From Cauchy's Inequality,

$$\left|C_{\overline{k}}\right| \le \frac{M_f(\overline{R})}{\prod_{i=1}^n R_i^{k_i}}$$

And choosing R_i to be the root of the equation,

$$(1-R_i) = \exp\left[-\frac{1}{\rho_i}\alpha(k_i)\right]$$

We get,

$$\left|C_{\overline{k}}\right| < \prod_{i=1}^n \frac{C_\varepsilon\exp((b_i-\varepsilon)k_i)}{\left[1-\exp\left\{-\dfrac{1}{\rho_i}\alpha(k_i)\right\}\right]^{k_i}} \;\forall k_i. \qquad (2.10),$$

Where from the result follows upon taking limit superior.

Theorem 2.4: For the analytic function $f(\tilde{Z})$

$$\limsup_{\|\overline{k}\|\to\infty}\frac{\log^{(q)}\left|b_{\overline{k}}\right|}{\log\|\overline{k}\|} = \frac{\rho_D}{\rho_D+1} \qquad (2.11)$$

Proof: (if part)

We consider the function $\varphi(\tilde{Z}, w)$ of (n+1) complex variables and write it as

$\varphi(\tilde{Z}, w) = \sum_{m=0}^{\infty} P_m(\tilde{Z})\, w^m$ where

$P_m(\tilde{Z}) = \sum_{\|\overline{k}\|=m}^{\infty} b_{\overline{k}}\tilde{Z}^k$. Then Cauchy's inequality along with the definition 6 results in

$$M_D(1, P_m) \le A\exp^{[q-1]}\left[(1+\rho_D+\varepsilon)\left(\frac{m}{\rho_D+\varepsilon}\right)^{\frac{\rho_D+\theta\varepsilon}{\rho_D+\varepsilon+1}}\right]$$

Which upon majorising under limit superior results in

$$\lim_{\|\bar{k}\|\to\infty}\sup\frac{\log^{(q)}|b_{\bar{k}}|}{\log\|\bar{k}\|} \le \frac{\rho_D}{\rho_D+1}. \qquad (2.12)$$

(only if part):
For this we first assume,

$\lim_{\|\bar{k}\|\to\infty}\sup\frac{\log^{(q)}|b_{\bar{k}}|}{\log\|\bar{k}\|} = \mu < 1.$Now using the definition of lim sup we arrive at,

$$M_D(t, f) < C_1 t^m(\varepsilon) + C_2 + \sum_{m=0}^{\infty}\frac{t^m(1+m)^m}{\exp^{[q-1]}(m^{\mu+\varepsilon})}.$$

$$where\ C_1\ and\ C_2\ are\ arbitrary\ constants.$$

Now the function $F(\tilde{Z}) = \sum_{m=0}^{\infty}\frac{(1+m)^n}{\exp^{[q-1]} m^{(\mu+\varepsilon)}}\tilde{Z}^m$ is analytic in D. Now upon using Definition 6 we arrive at $\rho_D \le \frac{\mu}{1-\mu}$, which implies,

$$\frac{\rho_D}{1+\rho_D} \le \mu \qquad (2.13).$$

Theorem 2.4: For a function $f(\tilde{Z})$, analytic in D and satisfying $0 < \rho_D(q) < \infty$ and,

$$T_D(q) = \lim_{t\to 1}\sup\frac{\log^{[q-1]} M_D(t, f)}{(1-t)^{-\rho_D(q)}} \qquad (2.14)$$

where q = 2,3,… is such that $\rho_D(q-1) = \infty$ and $\rho_D(q) < \infty$, then,

$$\frac{\rho_D+1}{\rho_D}(T_D)^{1/\rho_D} = \lim_{\|\bar{k}\|\to\infty}\sup\frac{\log^{[q-1]}|b_{\bar{k}}|}{\|\bar{k}\|} \qquad (2.15).$$

Proof: The proof follows the same pattern of the proof of Theorem 2.5 and hence is omitted.

Theorem 2.5: Let $\alpha \in \Delta$ and is of the form $\alpha(x) = \lambda(\log(1-x))$, then for any $\lambda > 0$ and $\beta \in L^0$, $\sigma_n < \gamma\ \forall\gamma > 0.$

Proof:
Choosing

$$t_i = 1 - \exp\left\{\beta^{-1}\left(\frac{l_i}{\lambda l_i + \mu m_i}\beta\log(1-R)\right)\right\} \qquad (2.16)$$

$$s_i = 1 - \exp\left\{\beta^{-1}\left(\frac{m_i}{\lambda l_i + \mu m_i}\beta\log(1-R)\right)\right\} \qquad (2.17)$$

for i= 1,2,…(n-1).

and

$$t_n = 1 - \exp\left\{\beta^{-1}\left(\frac{l_n}{\lambda l_n + \mu m_n}\beta\log(1-R_n)\right)\right\} \qquad (2.18)$$

$$s_n = 1 - \exp\left\{\beta^{-1}\left(\frac{m_n}{\lambda l_n + \mu m_n}\beta\log(1-R_n)\right)\right\} \qquad (2.19)$$

$$l_n = \rho_n^* - \varepsilon_1 \quad 0 < \varepsilon_1 < \rho_n^*,\ m_n = \rho \qquad (2.20)$$

$$\frac{l_i}{m_i} = \beta\log(1-R_n) - C(R) \quad for\ i = 1,2,\ldots n-1; \qquad (2.21)$$

$$C(R) = -1 + \frac{\rho}{\rho_n^* - \varepsilon_1}\beta\log(1-R) \qquad (2.22)$$

$$\lambda = \frac{C(R)}{\beta\log(1-R_n)},\ \mu = 1-\lambda \qquad (2.23)$$

and using the inequality (1.1) with the above values, and upon solving the nonlinear equation $\alpha(x) = \lambda(\log(1-x))$ for x we arrive at our result by using the definition of limit superior.

III. CONCLUSION

(a) The Growth of a Class of plurisubharmonic functions are extensively used in the Value Distribution Theory of functions of Several Complex Variables.
(b) Plurisubharmonic functions are the higher dimensional generalization of sub harmonic functions.
(c) In complex analysis, plurisubharmonic functions are used to describe pseudo convex domains, domains of holomorphy and Stein manifolds.

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REFERENCES:

[1] Juneja,O.P. And Kapoor G.P. Analytic functions- growth aspect, Research Notes in Mathematics, Pitman 104,(1985).

[2] Sinha, Amritasu On the growth of a class of Plurisubharmonic Functions in a Unit disc. J. Indian,Inst,Sci,69, Nov-Dec.1989,441-448.

[3] Sinha, Amritasu Kumar Some studies and applications on certain growth aspects of entire functions of several complex Variables Ph.D. Thesis IIT Kanpur April 1984.

[4] Rudiger W.Braun et al: Local radial Phragman- Lindelof estimates for plurisubharmonic functions on analytic varieties. Proc. Amer. Math.Soc. 131 (2003),2423-2433.

[5] A. Lexander et al. Indicators for plurisubharmonic functions of logarithmic growth.iumj.indiana.edu 50 (2001).

[6] Seremeta,M.N. On the connection between the growth of the Maximum modulus of an entire function And moduli of the coefficients of its Power series expansion. Am. Math. Soc. Transl.,1970 88,291-301.

[7] Lane A.P. and Bolger WE. Endoscopic transmaxillary biopsy of pterygopalatine space masses: A.preliminary report. Am J. Rhinol 16:109-112,2002. Endoscopic transmaxillary biopsy