Hermite functions with discontinuous coefficients for the solution of fractal diffusion retrospective problems

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Abstract—In this article we study the retrospective inverse problem. The retrospective inverse problem consists of in the reconstruction of a priori unknown initial condition of the dynamic system from its known final condition. Existence and uniqueness of the solution is proved.

Keywords—Hermite functions, retrospective problem, integral equation, fractal diffusion.

I. INTRODUCTION

IN this article we study the retrospective inverse problem. The retrospective Inverse problem consists of in the reconstruction of a priori unknown initial condition of the dynamic system from its known final condition. The direct problem of heat conductivity is well-posed; the inverse problem is not well-posed. In mathematics the vast majority of inverse problems set not well-posed - small perturbations of the initial data (observations) can correspond to arbitrarily large perturbations of the solution. The French mathematician Jacques Hadamard in 1939 defined, the problem is called correct or well-posed problem if a solution exists, the solution is unique, the solution's behavior hardly changes when there's a slight change in the initial condition. If at least one of these three conditions is not fulfilled, problems are termed ill-posed or not well-posed. The most often in the case of ill-posed problems of the third condition are violated the condition of the stability of solutions. In this case, there is a paradoxical situation: the problem is mathematically generated, but the solution cannot be obtained by conventional methods. A classic example of ill-posed problem is retrospective problem for heat equation on the real axis. Mathematically retrospective problem leads to a Fredholm integral equation of the first kind:

$$\int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi\tau}} e^{\left(-\frac{(x-\xi)^2}{4\tau}\right)} \hat{f}(\xi)d\xi = \hat{u}(\tau, x), \tag{1}$$

in which $\hat{f}(x)$ - is the initial distribution of the temperature field, $\hat{u}(t, x)$ - is the distribution of the fields in the moment

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of time t. As shown in [1], the solution of equation (1) expressed by the formula:

$$\hat{f}(x) = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{\hat{u}^{(j)}(0)}{(2\sqrt{\tau})^{n+1} j!} H_j \left(\frac{x}{2\sqrt{\tau}} \right). \tag{2}$$

II. PROBLEM STATEMENT

In the inverse problem of heat conductivity the initial distribution of sources is unknown. The initial distribution of sources generates the specified temperature distribution in an infinite piecewise-homogeneous rod In. Mathematical statement of the problem consists in finding a solution separatist system (n+1) equations of parabolic type

$$\left(\frac{\partial}{\partial t} - a_j^2 \frac{\partial^2}{\partial x^2} \right) u_j(t, x) = 0, t > 0, x \in I_n, \tag{3}$$

on the initial conditions

$$u_j(t, x)\big|_{t=0} = f_j(x), x \in I_n, \tag{4}$$

on the boundary conditions

$$u_1\big|_{x=-\infty} = 0, u_{n+1}\big|_{x=\infty} = 0 \tag{5}$$

and the coupling conditions

$$\left[\alpha_{m1}^k \frac{\partial}{\partial x} + \beta_{m1}^k \right] u_k = \left[\alpha_{m2}^k \frac{\partial}{\partial x} + \beta_{m2}^k \right] u_{k+1,} \tag{6}$$

$$x = l_k, k = 1, ..., n; m = 1, 2,$$

here $u(t, x)$ - unknown function, $f(x)$ - set function,

$$u(\tau, x) = \sum_{k=2}^{n} \theta(x - l_{k-1})\theta(l_k - x)u_k(\tau, x) +$$

$$+\theta(l_1 - x)u_1(\tau, x) + \theta(x - l_n)u_{n+1}(\tau, x),$$

$$f(x) = \sum_{k=2}^{n} \theta(x - l_{k-1})\theta(l_k - x)f_k(x) +$$

$$+ \theta(l_1 - x)f_1(x) + \theta(x - l_n)f_{n+1}(x)$$

$\alpha_{mi}^k, \beta_{mi}^k, \gamma_{mi}^k, \delta_{mi}^k,$ - given real number, in which the condition of unlimited solvability of the problem considered fulfilled [2].

The solution to problem (3)-(6) is of the form:

$$u_k(t, x) = \sum_{s=1}^{n+1} \int_{l_s}^{l_{s+1}} H_{ks}(t, x, \xi)f_s(\xi)d\xi, \qquad (7)$$

where $H_{ks}(t, x, \xi)k, s = 1, ..., n+1$- influence function [2] of the mixed boundary value problem.

Retrospective problem for the heat equation in the case of infinite piecewise-homogeneous rod consists in the determination of the unknown initial distribution of sources $f(x)$, which generates the specified temperature distribution $u(\tau, x)$ in the moment of time $t = \tau$.

III. TRANSFORMATION OPERATORS

Method of transformation operators is used to solve the problem [2]. Necessary definitions from [2], [11], [12]. The direct $J : \hat{f} \to f$ f and inverse $J^{-1} : f \to \hat{f}$ transformation operators are set equalities:

$$f(x) = \int_{-\infty}^{\infty} \varphi(x, \lambda)\left(\int_{-\infty}^{\infty} e^{-i\lambda\xi}\hat{f}(\xi)d\xi\right)d\lambda,$$

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{-i\lambda\xi}\left(\int_{-\infty}^{\infty} \varphi^*(\xi, \lambda)f(\xi)d\xi\right)d\lambda.$$

Here $\varphi(x, \lambda), \varphi^*(x, \lambda)$ -are the eigenfunctions [13], [14] of the direct and coupling Sturm–Liouville problems for the Fourier operator in piecewise-homogeneous axis In. Eigenfunction

$$\varphi(x, \lambda) = \sum_{k=2}^{n} \theta(x - l_{k-1})\theta(l_k - x)\varphi_k(x, \lambda) +$$

$$+\theta(l_1 - x)\varphi_1(x, \lambda) + \theta(x - l_n)\varphi_{n+1}(x, \lambda)$$

is a solution of the system of separate differential equations

$$\left(a_m^2 \frac{d^2}{dx^2} + \lambda^2\right)\varphi_m(x, \lambda) = 0, x \in (l_m, l_{m+1}); m = 1, ..., n+1,$$

on the coupling conditions

$$\left[\alpha_{m1}^k \frac{d}{dx} + \beta_{m1}^k\right]\varphi_k = \left[\alpha_{m2}^k \frac{d}{dx} + \beta_{m2}^k\right]\varphi_{k+1},$$

on the boundary conditions

$$\varphi_1\big|_{x=-\infty} = 0, \varphi_{n+1}\big|_{x=\infty} = 0$$

Similarly eigenfunction

$$\varphi^*(\xi, \lambda) = \sum_{k=2}^{n} \theta(\xi - l_{k-1})\theta(l_k - \theta)\varphi^*(\xi, \lambda) +$$

$$\theta(l_1 - \xi)\varphi_1^*(\xi, \lambda) + \theta(\xi - l_n)\varphi_{n+1}^*(\xi, \lambda)$$

is a solution of the system of separate differential equations

$$\left(a_m^2 \frac{d^2}{dx^2} + \lambda^2\right)\varphi_m^*(x, \lambda) = 0, x \in (l_m, l_{m+1}); m = 1, ..., n+1,$$

with the coupling conditions

$$\frac{1}{\Delta_{1,k}}\left[\alpha_{m1}^k \frac{d}{dx} + \beta_{m1}^k\right]\varphi_k^* = \frac{1}{\Delta_{2,k}}\left[\alpha_{m2}^k \frac{d}{dx} + \beta_{m2}^k\right]\varphi_{k+1}^*, x = l_k,$$

where

$$\Delta_{i,k} = \det\begin{pmatrix} \alpha_{1i}^k & \beta_{1i}^k \\ \alpha_{2i}^k & \beta_{2i}^k \end{pmatrix} k = 1, ..., n; i, m = 1, 2,$$

on the boundary conditions

$$\varphi_1\big|_{x=-\infty} = 0, \varphi_{n+1}\big|_{x=\infty} = 0$$

Let for some λ of the considered boundary value problems have nontrivial solutions $\varphi(x, \lambda), \varphi^*(x, \lambda)$, in this case the number λ is called the eigenvalue [13], [14], corresponding solutions $\varphi(x, \lambda), \varphi^*(x, \lambda)$ - is called the eigenfunctions of the direct and coupling Sturm–Liouville problems, respectively. In the further we shall adhere to the following normalization of eigenfunctions:

$$\varphi_{n+1}(x, \lambda) = e^{ia_{n+1}^{-1}x\lambda}.\varphi_{n+1}^*(x, \lambda) = e^{-ia_{n+1}^{-1}x\lambda}.$$

IV. ANALOGUES OF THE SYSTEM THE HERMITE FUNCTIONS ON PIECEWISE-HOMOGENEOUS REAL AXIS

Let define analogues of the system the Hermite functions on piecewise-homogeneous real axis:

$$H_{j,n}(x) = \int_{-\infty}^{\infty} \varphi(x,\lambda) H_j\left(\frac{\lambda}{2\sqrt{\tau}}\right) d\lambda,$$

$$H_{j,n}^*(x) = \int_{-\infty}^{\infty} \varphi^*(x,\lambda) H_j\left(2\sqrt{\tau}\lambda\right) d\lambda.$$

where H_j - the system of the classical orthogonal Hermite functions [1].

Lemma 1. Functions $H_{j,n}(x), H_{j,n}^*(x)$ form biorthogonal system of functions by piecewise-homogeneous real axis.

Proof. We have the equality:

$$\int_{-\infty}^{\infty} H_{j,n}(x) H_{k,n}^*(x) dx =$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \varphi(x,\lambda) H_j(\lambda) d\lambda \right) \left(\int_{-\infty}^{\infty} \varphi^*(x,\beta) H_k(\beta) d\beta \right) dx.$$

We change the integrals of places, we get:

$$\int_{-\infty}^{\infty} H_{j,n}(x) H_{k,n}^*(x) dx =$$

$$= \int_{-\infty}^{\infty} H_j(\lambda) \left(\int_{-\infty}^{\infty} \varphi(x,\lambda) \left(\int_{-\infty}^{\infty} \varphi^*(x,\beta) H_k(\beta) d\beta \right) dx \right) d\lambda.$$

On the decomposition theorem, we have:

$$H_k(\lambda) = \int_{-\infty}^{\infty} \varphi(x,\lambda) \left(\int_{-\infty}^{\infty} \varphi^*(x,\beta) H_k(\beta) d\beta \right) dx.$$

Consequently,

$$\int_{-\infty}^{\infty} H_{j,n}(x) H_{k,n}^*(x) dx = \int_{-\infty}^{\infty} H_j(\lambda) H_k(\lambda) d\lambda = \delta_{j,k}.$$

V. MAIN RESULT

The problem of determining the initial distribution of the temperature field $f(x)$ mathematically leads to the separate system of integral equations:

$$\sum_{s=1}^{n+1} \int_{l_s}^{l_{s+1}} H_{ks}(\tau,x,\xi) f_s(\xi) d\xi = u_k(\tau,x).$$

(8)

$$k = 1,...,n+1.$$

Method of transformation operators applicable to solving separate system of integral equations (8).

Theorem 1. If the function $u(\tau,x) \in S'(R)$ and for her the condition

$$e^{\tau^2\lambda}(1+\lambda^2)^{\alpha/2}\tilde{u}(\tau,\lambda) \in L_2(R),$$

that the separate system of integral equations (8) has a unique solution $f(x) \in H_2^\alpha(I_n)$ (definition $H_2^\alpha(I_n)$ [6]), is according to the formula:

$$f(x) = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{D_n(u)}{2^j \, j!} H_{j,n}(x),$$

where

$$D_n(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\lambda)^j \tilde{u}(\tau,\lambda) d\lambda.$$

Proof. Let's apply the transformation operator J^{-1} to separate system of integral equations (8). As a result come to a model integral equation (1). Let's apply the operator J in both parts of the obtained equality (9); as a result, taking into account the continuity of the operator J, we find the unknown distribution of temperature:

$$f(x) = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{\hat{u}^{(j)}(0)\tau^{\frac{j}{2}}}{\left(2\sqrt{\tau}\right)^{n+1} j!} H_{j,n}(x). \qquad (9)$$

Let's calculate numbers $\hat{u}^{(j)}(0)$.

$$\hat{u}^{(j)}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\lambda)^j \left(\int_{-\infty}^{\infty} e^{-i\lambda\xi} \hat{u}(\xi) d\xi \right) d\lambda,$$

from the definition of the operator conversion of J the equality follows:

$$\int_{-\infty}^{\infty} e^{-i\lambda\xi} \hat{u}(\xi) d\xi = \int_{-\infty}^{\infty} \varphi^*(\xi,\lambda) u(\xi) d\xi.$$

Thus,

$$\hat{u}^{(j)}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\lambda)^j \tilde{u}(\tau,\lambda) d\lambda.$$

VI. RETROSPECTIVE PROBLEM FOR ITERATION HEAT EQUATION

a) Homogeneous case. Let $\hat{u}(\tau,x), x \in R$ - decision iteration heat equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right)^{q+1} \hat{u}(t,x) = 0, (t,x) \in D$$

limited on the set

$$D = (0,\infty) \times I_n = \{(t,x): t \geq 0, x \in R\},$$

with initial conditions

$$\hat{u}_i(0,x) = \hat{f}_i(x), x \in R$$

where the designation is accepted:

$$\hat{u}_i(t,x) = \frac{\partial^i}{\partial t^i} \hat{u}(t,x), i = 0,...,q.$$

and boundary conditions

$$\left. \frac{\partial^i}{\partial x^i} \hat{u}(t,x) \right|_{x=-\infty} = 0, \left. \frac{\partial^i}{\partial x^i} \hat{u}(t,x) \right|_{x=+\infty} = 0, i = 0,...,q.$$

We consider the problem of determining of initial values $\hat{f}_i(x), i = 0,...,q$ of the system of functions $\hat{u}_i(\tau,x), i = 0,...,q,$ that is on known to the solution and its derivatives up to order q at time $t = \tau$.

Let $\hat{v}_i(\tau,x), i = 0,...,q$ - solutions of model heat equations:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \hat{v}_i(t,x) = 0, (t,x) \in D, i = 0,...,q \quad (10)$$

with initial conditions

$$\hat{v}_i(0,x) = \hat{f}_i(x), x \in R \quad (11)$$

boundary conditions

$$\left. \hat{v}_i(t,x) \right|_{x=-\infty} = 0, \left. \hat{v}_i(t,x) \right|_{x=+\infty} = 0, i = 0,...,q. \quad (12)$$

Direct check shows, that the system of functions dependent. $\hat{u}_i(\tau,x), i = 0,...,q,$ и $\hat{v}_i(\tau,x), i = 0,...,q.$

Lemma 2. The system of functions $\hat{u}_i(\tau,x), i = 0,...,q,$ и $\hat{v}_i(\tau,x), i = 0,...,q$ is connected among them selves by equalities:

$$\begin{pmatrix} \hat{v}_0(\tau,x) \\ ... \\ \hat{v}_q(\tau,x) \end{pmatrix} = \Gamma(D^2) \sum_{k=0}^{q} \frac{J_k}{k!} t^k \Gamma(-D^2) \begin{pmatrix} \hat{u}_0(\tau,x) \\ ... \\ \hat{u}_q(\tau,x) \end{pmatrix} \quad (13)$$

$\Gamma(D^2)$ - is operator matrix:

$$\Gamma(D^2) = \begin{pmatrix} D^0 & 0 & 0 & ... & 0 \\ D^2 & D^0 & 0 & ... & 0 \\ D^4 & 2D^2 & D^0 & ... & 0 \\ ... & ... & ... & ... & ... \\ C_q^0 D^{2q} & C_q^1 D^{2q-2} & C_q^2 D^{2q-4} & ... & C_q^q D^0 \end{pmatrix},$$

J_k - is matrix of size $(q+1) \times (q+1)$ where the diagonal connecting elements $\gamma_{1,k+1}, \gamma_{q+1-k,q+1},$ consists of units, and all other elements are equal to zero.

Theorem 2. If the condition
$e^{\tau^2 \lambda} (1+\lambda^2)^{q+\alpha/2} F[\hat{u}_i](\tau,\lambda) \in L_2(R),..., i = 0,...,q,$ is fulfilled for the system of functions $\hat{u}_i(\tau,x) \in S'(R), i = 0,...,q,$
then:

$$\begin{pmatrix} \hat{f}_0(x) \\ ... \\ \hat{f}_q(x) \end{pmatrix} = \sum_{j=0}^{\infty} \begin{pmatrix} D^j \hat{v}_0(\tau,0) \\ ... \\ D^j \hat{v}_q(\tau,0) \end{pmatrix} \frac{\tau^{\frac{j}{2}}}{j!} H_j \left(\frac{x}{2\sqrt{\tau}} \right), \quad (14)$$

and $\hat{f}_i(x) \in H_2^\alpha(R), i = 0,...,q,$ values $\hat{v}_i(\tau,x), i = 0,...,q$ must be taken from the formula (13).

Proof. Let's apply the formula (9) to each of $(q+1)$ the initial conditions $\hat{f}_i(x)$ in the model problems (10)-(12). As a result we come to (14).

b) The case of piecewise-homogeneous axis.
Let the initial distribution is unknown, it generates the specified distribution at time

$$f_i(x) = \sum_{k=2}^{n} \theta(x - l_{k-1})\theta(l_k - x)f_{ik}(x) +$$

$$+ \theta(l_1 - x)f_{i1}(x) + \theta(x - l_n)f_{i,n+1}(x)$$

it generates the specified distribution at time $t = \tau$:

$$u_0(\tau, x) = \sum_{k=2}^{n} \theta(x - l_{k-1})\theta(l_k - x)u_{0k}(\tau, x) +$$

$$+ \theta(l_1 - x)u_{01}(\tau, x) + \theta(x - l_n)u_{0,n+1}(\tau, x),$$

$u_{0m}(t, x)$ - is the decision of the separate system of differential equations

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)^q u_{0m}(t, x) = 0, \qquad (15)$$

$$(t, x) \in D, m = 1, ..., n+1$$

the specified distribution at time $t = \tau$ of the function derivatives $u_0(\tau, x)$ up to order q including:

$$u_i(\tau, x) = \sum_{k=2}^{n} \theta(x - l_{k-1})\theta(l_k - x)u_{ik}(\tau, x) +$$

$$+ \theta(l_1 - x)u_{i1}(\tau, x) + \theta(x - l_n)u_{i,n+1}(\tau, x),$$

$$u_{im} = \frac{\partial^i}{\partial t^i}u_{0m}(\tau, x), i = 1, ..., q,$$

initial conditions

$$u_{im}(0, x) = f_{im}(x), x \in I_n, m = 1, ..., n+1 \qquad (16)$$

boundary conditions

$$\frac{\partial^i}{\partial x^i}u_{01}(t, x)\bigg|_{x = -\infty} = 0,$$

$$\frac{\partial^i}{\partial x^i}u_{0,n+1}(t, x)\bigg|_{x = +\infty} = 0, i = 0, ..., q. \qquad (17)$$

and coupling conditions

$$\sum_{i=0}^{2q-1} \alpha_{m1}^k \frac{\partial^i}{\partial x^i}u_{0k} = \sum_{i=0}^{2q-1} \alpha_{m2}^k \frac{\partial^i}{\partial x^i}u_{0k+1}, x = l_k, \qquad (18)$$

are satisfied,
here $\alpha_{js,i}^k; j = 1, ..., 2q, s = 1, 2, i = 0, ..., 2q-1$ -are set real numbers at which the condition of unlimited solvability of the

problem (26.15)-(26.18) is fulfilled. Search of the decision leads to the separate system of the integrated equations:

$$u_{ik}(t, x) = \sum_{s=1}^{n+1} \int_{l_s}^{l_{s+1}} H_{i,ks}(t, x, \xi)f_{is}(\xi)d\xi$$

Here

$$H_{i,ks}(t, x, \xi) = \int_{-\infty}^{\infty} (-\lambda^2)^i \varphi_k(x, \lambda)\varphi_s^*(\xi, \lambda)e^{-\lambda^2 t}d\lambda,$$

$$i = 0, ..., q; k, s = 1, ..., n+1,$$

$$\varphi(x, \lambda) = \sum_{k=2}^{n} \theta(x - l_{k-1})\theta(l_k - x)\varphi_k(x, \lambda) +$$

$$+ \theta(l_1 - x)\varphi_1(x, \lambda) + \theta(x - l_n)\varphi_{n+1}(x, \lambda),$$

$$\varphi^*(\xi, \lambda) = \sum_{k=2}^{n} \theta(\xi - l_{k-1})\theta(l_k - \xi)\varphi_k^*(\xi, \lambda) +$$

$$+ \theta(l_1 - \xi)\varphi_1^*(\xi, \lambda) + \theta(\xi - l_n)\varphi_{n+1}^*(\xi, \lambda),$$

φ, φ^* - are the eigenfunctions of the corresponding to direct and dual Sturm–Liouville problems [5]. They are connected with this mixed boundary problem (15)-(18). Application of the transformation operator J^{-1} leads the studied task to the uniform case, that is to the problem (10)-(12).
Truth of the result is established by the scheme proposed in the proof of theorem 2:

Theorem 3. If the function $u_i(\tau, x) \in S'(R), i = 0, ..., q$ for each of them the condition is fulfilled

$$e^{\tau^2\lambda}(1 + \lambda^2)^{q+\alpha/2}F[u_i](\lambda) \in L_2(R),$$

that system of the separate integrated equations (26.19) has the

unique solution $\begin{pmatrix} f_0(x) \\ ... \\ f_q(x) \end{pmatrix} \in H_2^\alpha(I_n)$ the solution is found by

the formula:

$$\begin{pmatrix} f_0(x) \\ ... \\ f_q(x) \end{pmatrix} = \sqrt{\frac{\pi}{2}} \sum_{j=0}^{\infty} \frac{1}{2^j j!} \begin{pmatrix} D^j v_0(x) \\ ... \\ D^j v_q(x) \end{pmatrix} H_{j,n}(x), \qquad (20)$$

where

$$D^j v_k(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\lambda)^j F_n[v_k](\lambda)d\lambda, k = 0, ..., q,$$

$$\begin{pmatrix} v_0(\tau,x) \\ ... \\ v_q(\tau,x) \end{pmatrix} = \Gamma(D^2)\sum_{k=0}^{q}\frac{J_k}{k!}t^k\Gamma(-D^2)\begin{pmatrix} u_0(\tau,x) \\ ... \\ u_q(\tau,x) \end{pmatrix},$$

F_n - is the Fourier transform on the piecewise-homogeneous axis with the coupling points [5].

VII. POWER FUNCTION WITH DISCONTINUOUS COEFFICIENTS AND ITS APPLICATION

We consider the Fourier transform of the Delta function [15]

$$F[\delta(x)] = \int_{-\infty}^{\infty}e^{-i\lambda x}\delta(x)dx = 1,$$

consequently

$$F^{-1}[1] = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda x}dx = \delta(\lambda).$$

We find as a consequence

$$F[x^k] = i^k\frac{d^k}{d\lambda^k}\left(\int_{-\infty}^{\infty}e^{-i\lambda x}dx\right) = 2\pi i^k\delta^k(\lambda).$$

Let's define analog of the power function as follows

$$x_n^k = (i)^k\frac{\partial^k\varphi(x,\lambda)}{\partial\lambda^k}\bigg|\lambda = 0.$$

We find equality from the definition of the transformation operators

$$J[x^k] = x_n^k;\ J^{-1}[x_n^k] = x^k.$$

The last equality means that the power function with discontinuous coefficients is obtained by the action of the transformation operator to the power function.

Theorem 4. The ratio connects the generalized power function and differentiation

$$\frac{d^2}{dx^2}x_n^k = k(k-1)x_n^{k-2}.$$

Proof. We have a chain of equalities

$$\frac{d^2}{dx^2}(x_n^k) = \frac{d^2}{dx^2}J(x^k) = J\left(\frac{d^2}{dx^2}x^k\right) =$$

$$J(k(k-1)x^{k-2}) = k(k-1)x_n^{k-2}.$$

VIII. RETROSPECTIVE PROBLEM FOR THE SYSTEM OF THE DIFFUSION EQUATIONS

Let's return to the solution of the separate system of integral equations (8) in space of the generalized functions [16] S'

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\lambda^2\tau}\varphi(x,\lambda)\int_{\infty}^{\infty}\varphi^*(x,\xi)u(\tau,\xi)d\xi d\lambda,$$

(21)
$$k = 1,...,n+1.$$

We will convert the found solution. An analogue of the Taylor series for the function $u(\tau,\xi)$ is of the form

$$u(\tau,\xi) = \sum_{j=0}^{\infty}\frac{u_j(\tau)}{j!}x_n^j.$$

(22)

From decomposition theorem have

$$u(\tau,x) =$$
$$= \frac{1}{2\pi}\sum_{s=1}^{n+1}\int_{-\infty}^{\infty}\varphi(x,\lambda)\int_{l_s}^{l_{s+1}}\varphi_s^*(x,\xi)u_s(\tau,\xi)d\xi d\lambda. \qquad (23)$$

Let's find the decomposition of the generalized Taylor series for eigenfunction from definition of the generalized power function

$$\varphi(x,\lambda) = \sum_{j=0}^{\infty}\frac{(i\lambda)^j}{j!}x_n^j.$$

Let's substitute this decomposition in a formula (22) and let's integrate term by term. We come to the formula (23) in which

$$u_j(\tau) = \frac{1}{2\pi}\sum_{s=1}^{n+1}\int_{-\infty}^{\infty}(i\lambda)^j\int_{l_s}^{l_{s+1}}\varphi_s^*(x,\xi)u_s(\tau,\xi)d\xi d\lambda. \qquad (24)$$

Let's substitute decomposition of eigenfunction $\varphi(x,\lambda)$ in the generalized power series in a formula (21) we will receive

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\lambda^2\tau}\varphi(x,\lambda)\int_{\infty}^{\infty}\varphi^*(x,\xi)\sum_{j=0}^{\infty}\frac{u_j(\tau)}{j!}\xi_n^j d\xi d\lambda =$$

$$= \sum_{j=0}^{\infty}\frac{u_j(\tau)}{j!}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\lambda^2\tau}\varphi(x,\lambda)\int_{\infty}^{\infty}\varphi^*(\xi,\lambda)\xi_n^j d\xi d\lambda =$$

$$= \sum_{j=0}^{\infty} \frac{u_j(\tau)}{j!} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda^2 \tau} \varphi(x,\lambda) 2\pi i^j \delta^j(\lambda) d\lambda =$$

$$= \sum_{j=0}^{\infty} \frac{u_j(\tau)(-1)^j}{j!} \int_{-\infty}^{\infty} \frac{\partial^j}{\partial \lambda^j} \left(e^{\lambda^2 \tau} \varphi(x,\lambda) \right)^j \delta^j(\lambda) d\lambda =$$

$$= \sum_{j=0}^{\infty} \frac{u_j(\tau)}{j!} H_{jn}(x),$$

where designation is accepted

$$H_{jn}(x) = (-i)^j \frac{\partial^j}{\partial \lambda^j} \left(e^{\lambda^2 \tau} \varphi(x,\lambda) \right) \bigg|_{\lambda=0}.$$

Thus, the solution of the retrospective problem (21) is obtained.

Remark 1. The generating function for the $H_{jn}(x)$ is the form

$$e^{-\lambda^2 \tau} \varphi(x,-i\lambda) = \sum_{j=0}^{\infty} \frac{H_{jn}(x)}{j!} \lambda^j.$$

Corollary 1. If to choose $t = \frac{1}{2}, n = 1, \varphi(x,\lambda) = e^{i\lambda x}$,

then we obtain

$$e^{-\frac{1}{2}\lambda^2} e^{\lambda x} = \sum_{j=0}^{\infty} \frac{H_{j1}(x)}{j!} \lambda^j,$$

functions $H_{j1}(x)$-are the classical Hermite polynomials.

IX. RETROSPECTIVE PROBLEM FOR FRACTAL SYSTEMS OF DIFFUSION EQUATIONS

Retrospective problem for fractal system of diffusion equations in the space of generalized functions S' leads to the separatist system of integral equations:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{\alpha,1}(\lambda^2 \tau^\alpha) \varphi(x,\lambda) \int_{\infty}^{\infty} \varphi^*(x,\xi) u(\tau,\xi) d\xi d\lambda,$$

$$k = 1,...,n+1,$$

(25)

where $E_{\alpha,1}(z)$ - the Mittag-Leffler function [9].

We get the solution of the fractal retrospective problem repeating reasoning's from paragraph VII.

$$f(x) = \sum_{j=0}^{\infty} \frac{u_j(\tau)}{j!} H_{jn}(x),$$

where designation is accepted

$$H_{jn}(x) = (-i)^j \frac{\partial^j}{\partial \lambda^j} \left(E_{\alpha,1}\left(\lambda^2 \tau^\alpha\right) \varphi(x,\lambda) \right) \bigg|_{\lambda=0}.$$

You can find the generating functions for $H_{jn}(x)$:

$$E_{\alpha,1}\left(-\lambda^2 \tau^\alpha\right) \varphi(x,-i\lambda) = \sum_{j=0}^{\infty} \frac{H_{jn}(x)}{j!} \lambda^j. \qquad (26)$$

We find the explicit expression for the functions $H_{jn}(x)$. We find decomposition of the left part of the formula (26) in the Taylor series

$$E_{\alpha,1}\left(-\lambda^2 \tau^\alpha\right) \varphi(x,-i\lambda) = \sum_{k=0}^{\infty} (-1)^k \frac{\lambda^{2k} \tau^{\alpha k}}{\Gamma(k\alpha+1)} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} x_n^m =$$

$$= \sum_{j=0}^{\infty} \lambda^j \sum_{2k+m=j} (-1)^k \frac{\tau^{\alpha k}}{\Gamma(k\alpha+1)m!} x_n^m =$$

$$= \sum_{j=0}^{\infty} \lambda^j \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k \tau^{\alpha k} j!}{\Gamma(k\alpha+1)(j-2k)!} x_n^{j-2k}.$$

We get the expression, comparing the two views $H_{jn}(x)$

$$H_{jn}(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k \tau^{\alpha k} j!}{\Gamma(k\alpha+1)(j-2k)!} x_n^{j-2k}.$$

If $n = 1$, the formula takes the form

$$H_{j1}(x) = \sum_{k=0}^{\left[\frac{j}{2}\right]} \frac{(-1)^k \tau^{\alpha k} j!}{\Gamma(k\alpha+1)(j-2k)!} x^{j-2k}.$$

Define a "fractal" generalization of the Hermite polynomials

$$H_j(x) = \sum_{k=0}^{\left[\frac{j}{2}\right]} \frac{(-1)^k j!}{\Gamma(k\alpha+1)(j-2k)!} x^{j-2k},$$

then the solution of the fractal retrospective problem has the form

$$f(x) = \sum_{j=0}^{\infty} \frac{u_j(\tau)\tau^{\beta j}}{j!} H_j\left(\frac{x}{\tau^{\beta}}\right), \beta = \frac{\alpha}{2}.$$

Corollary 2. In the hyperbolic case $\alpha = 2, n = 1$ the solution of the retrospective problem has the form

$$f(x) = \sum_{j=0}^{\infty} \frac{u_j(\tau)\tau^{j}}{j!} H_j\left(\frac{x}{\tau}\right), \beta = \frac{\alpha}{2}.$$

Where

$$H_j(x) = \sum_{k=0}^{\left[\frac{j}{2}\right]} \frac{(-1)^k j!}{((2k)!)(j-2k)!} x^{j-2k} = \sum_{k=0}^{\left[\frac{j}{2}\right]} (-1)^k C_j^{2k} x^{j-2k} =$$
$$= \frac{(1+x)^j + (1-x)^j}{2}. \tag{27}$$

Proof. We replace λ for $i\lambda$ in formula (16) for the solution of the direct problem. As a result, we get the formula

$$u(\tau, x) = \sum_{j=0}^{\infty} \frac{f_j \tau^{\beta j}}{j!} H_j^*\left(\frac{x}{\tau^{\beta}}\right), H_j^*(z) = i^j H_j(-iz).$$

We use the formula (26)

$$H_{j1}(x) = (-i)^j \frac{\partial^j}{\partial \lambda^j}(\cos(\lambda\tau)e^{i\lambda x})\Big|_{\lambda=0} =$$
$$= \frac{1}{2}(-i)^j \frac{\partial^j}{\partial \lambda^j}(e^{i\lambda(\tau+x)} + e^{i\lambda(\tau-x)})\Big|_{\lambda=0} =$$
$$= \frac{(\tau+x)^j + (\tau-x)^j}{2}.$$

In the end we find the solution of the Cauchy problem for the hyperbolic equation

$$f(x) = \sum_{j=0}^{\infty} u^j(\tau,0) \frac{(\tau+x)^j + (\tau-x)^j}{2j!}.$$

Thus,

$$f(x) = \frac{u(\tau, \tau+x) + u(\tau, \tau-x)}{2}.$$

Remark. If, as an example, take

$$u(t, x) = g(x+t-\tau) + g(x-t+\tau),$$
the $u(\tau, x) = 2g(x)$ and, so,

$$f(x) = g(x+\tau) + g(x-\tau) = u(0, x).$$

X. THE INVERSE DIRICHLET PROBLEM FOR A HALF-PLANE

Solution of the inverse Dirichlet problem for the right half-plane has the form:

$$f(y) = \text{Re} \frac{1}{\pi} \int_0^{\infty} e^{\lambda l} e^{i\lambda y} \int_{-\infty}^{\infty} e^{-i\lambda \eta} u(l,\eta) d\eta d\lambda.$$

(28)

Let's repeat the above reasoning. Let's receive expressions for analogues of Hermite polynomials:

$$H_{j1}(x) = \text{Re}(-i)^j \frac{\partial^j}{\partial \lambda^j}(e^{\lambda l} e^{i\lambda y})\Big|_{\lambda=0} =$$
$$= \text{Re}(-i)^j (l + iy)^j = \frac{(y+li)^j + (y-li)^j}{2}.$$

As a result for the solution of the inverse Dirichlet problem we receive the representation in the form of the sum of the Taylor series:

$$f(y) = \sum_{j=0}^{\infty} \frac{u^j(l,0)}{j!} \frac{(y+li)^j + (y-li)^j}{2}.$$

Corollary 3. If the function $f(y)$ admit continued with the real axis of the complex plane as a whole, the

$$f(y) = \text{Re}\, u(l, y + li).$$

Example 2. Let $u(x, y) = (x - l)^2 - y^2$, then
$u(l, y) = -y^2$ therefore, we find

$$f(y) = \text{Re}(-(y+li)^2) = l^2 + y^2 = u(0, y).$$

XI. CONCLUSION

In this article the formal solution of the retrospective problem is provided. The third aspect in determining the well-posed problem is not taken into account. Theorem of existence and uniqueness of solution are given. From the analysis of the formula (9): the solution of the retrospective heat problem with discontinuous coefficients is received by replacement in the final result of the Hermite functions [1] on the Hermite functions with discontinuous coefficients, defined in the article. The derivatives $\hat{u}^j(0)$ are necessary to replace on $D_n(u)$. The noticed analogy allows hoping on the possibility of obtaining the solution of problems of

mathematical physics in which the Hermite functions with discontinuous coefficients.

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