Compares of some algorithms by using first and second derivative multistep methods

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***Abstract*— As is known, determination the solution of many problems of natural science can be reduced to determination of the solution of first and second orders ordinary differential equations. Typically solutions for ODEs second order are reduced to solving a system differential equations of first order. Here, has been analyzed known numerical methods for chronological manner, determined their advantages and disadvantages. To construct more accurate methods with extended stability region suggested the method constructed at the junction of second derivative multistep methods and hybrid methods. Proved advantage of the proposed method and constructed one-step method with the order of accuracy $p=10$ to which can be considered as a sense generalization of the trapezoid method. Constructed algorithm for using proposed methods.**

Keywords*— Multistep multiderivative methods, hybrid methods, initial value problem, ODEs first and second order.

I. Introduction

Consider, to solving of the following initial value problem

$$y' = f(x,y), \ \ y(x_0) = y_0 , \ \ x_0 \le x \le X, \qquad (1)$$

by applied some modifications of the multistep and onestep methods and also hybrid methods. Suppose that problem (1) has a unique solution $y(x)$ defined on the interval $[x_0, X]$. To determine the approximate values of the solutions of initial-value problem (1), the segment $[x_0, X]$ with a constant step size $h>0$ is divided into N equal parts, and the mesh points denoted as $x_i = x + ih \quad (i = 0,1,2,..,N)$. We denote the approximate value by y_i and y_i' of the solution and its derivatives at a point $x_i \quad (i=0,1,...,N)$, but the exact values of the solution and its first derivative at the point $x_i \quad (i=0,1,...,N)$ by $y(x_i)$ and $y'(x_i)$.

In the middle of XX century, some scientists taking into consideration the advantage of Adams methods began investigated his method in general:

$$\sum_{i=0}^{k} \hat{\alpha}_i y_{n+i} = h \sum_{i=0}^{k} \hat{\beta}_i \, y'_{n+i}. \qquad (2)$$

In [1] proved that the accuracy of the convergent method obtaining from the formula (2) is limited by the value $k+2$. Therefore, to construct more accurate methods, some scientists have proposed using forward-jumping methods. The forward-jumping method in the simplest case can be written as follows:

$$\sum_{i=0}^{k-m} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i \, y'_{n+i} \quad (m>0). \qquad (3)$$

Here the coefficients $\alpha_j, \beta_i \ (i=0,1,...,k;$ $j=0,1,...,k-m)$-are some real numbers, and $\alpha_{k-m} \ne 0$. Obviously, if we formally set $m=0$, then the method (3) represents k-step method with constant coefficients defined by the formula (2). To determine the benefits of the forward-jumping methods, consider the comparison of multistep methods, derived from the formulae (3) for $m>0$. To this end, we set $k=3$ and $m=1$. Then from (3) can be obtained different methods, one of which is as follows:

$$\begin{aligned} y_{n+2} &= (11y_n + 8y_{n+1})/19 + \\ &+ h(10f_n + 57f_{n+1} + 24f_{n+2} - f_{n+3})/57, \end{aligned} \qquad (4)$$

here $f_l = f(x_l, y_l) \ (l=0,1,2,...)$.

Note that the method (4) is stable and has the degree $p=5$, which is unique in this case (see [2]). However, if we consider the case, $k=3$ and $m=0$ then from (3) may be received the method which is stable and has the degree $p_{\max} \le 4$. Here, the concept of stability and the degree of the method is defined in the following form.

Definition 1. Method (3) is said to be stable if all the roots of the polynomial

$$\rho(\lambda) = \alpha_{k-m}\lambda^{k-m} + \alpha_{k-m-1}\lambda^{k-m-1} + ... + \alpha_1\lambda + \alpha_0$$

lie inside of the unique circle, on the boundary which does not have multiple roots.

Definition 2. For a sufficiently smooth function $y(x)$ method (3) has the degree $p>0$, if the following is holds:

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$$\sum_{i=0}^{k-m}\alpha_i y(x+ih)-h\sum_{i=0}^{k}\beta_i y'(x+ih)=O(h^{p+1}),\ h\to 0,$$

here $x = x_0 + nh$ is a fixed point.

It is easy to make sure that when using the method (4), it is required to determine the quantities y_{n+3} that can be solved by the following modification of the method (4):

$$\begin{aligned} y_{n+2} &= (11y_n + 8y_{n+1})/19 + \\ &+ h(10f_n + 57f_{n+1} + 24f_{n+2})/57 - \\ &- hf(x_{n+3}, y_{n+2} + h(23f_{n+2} - 16f_{n+1} + 5f_n)/12)/57. \end{aligned} \qquad (5)$$

This method differs from the method (3) the fact that he is A-stable (see [2]).

It is known that if the method (3) has the degree p, then there exist stable methods of type (3) with the degree

$$p \le k + m + 1. \qquad (6)$$

Consequently, stable methods of type (3) have a higher accuracy for $m = 1$ and odd values of the quantity k, since stable methods received from (2) has the maximum degree $p_{\max} = k + 1$ for odd k.

I.e. by increasing m for the stable k-step methods the degree P increases, too (see in [6]). It needs to remark, that such increase of m is not unbounded, since it should be $k \ge 3m$ for even or odd k and m at the same time and $k \ge 3m+1$, otherwise. But, by increasing m the use of the forward-jumping formulas is found difficult.

As can be seen from (6), the accuracy of the method (3) depends on the values of quantities k, m and relations between them (see [3]). For example, it is obvious that $k > m$. According to this, scientists for constructed a more accurate methods proposed using the following:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = \sum_{j=1}^{r} h^j \sum_{i=0}^{k}\beta_i^{(j)} y_{n+i}^{(j)}. \qquad (7)$$

It is perfectly clear, that multistep multiderivative method (7) can be used for numerical solution of the initial value problem for differential equations higher order. In particular it is proved, that if the formula (7) has the degree $p,\ \alpha_k \ne 0$ and is stable, then (see [3]):

$$p \le \begin{cases} r(k+1)+1 & \text{in } k = 2j \quad \text{and } r = 2j-1, \\ r(k+1) & \text{in } k = 2j-1 \text{ and } r = 2j. \end{cases}$$

For any k, there exist the stable methods with the degree $p = r(k+1)+1$ at $k = 2m,\ r = 2j-1$ and $p = r(k+1)$ at $k = 2m-1, r = 2j.$

For construction the stable methods, which have higher degrees, than the stable interpolation methods, there were considered the next forward-jumping methods.

$$\sum_{i=0}^{k-m}\alpha_i y_{n+i} = \sum_{j=1}^{r} h^j \sum_{i=0}^{k}\beta_i^{(j)} y_{n+i}^{(j)}.$$

However, in the middle of XX century for the numerical solution of problem (1), researchers have proposed new methods that are called hybrid (see for example [4], [5]). In [2] that by using next hybrid method:

$$y_{n+1} = y_n + h(3f_{n+1/3} + f_{n+1})/4, \qquad (8)$$

constructed the following A-stable method:

$$\begin{aligned} y_{n+1} = y_n &+ (3hf(x_n + h/3, (4y_n + 5y_{n+1})/9 - \\ &- 2hf_{n+1}/9) + h\, f_{n+1})/4 \end{aligned},$$

which has the degree $p = 3$.

There are same ways for the construction of the methods with the higher order of accuracy/ for example by using Richardson extrapolation (see for example [6], or singular points (see[7])), which is remember hybrid methods.

Method (8) can be generalized in different ways. In one variant, it can be written as the following (see [8]):

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i} + h\sum_{i=0}^{k}\gamma_i f_{n+i+\nu_i}, \quad (|\nu_i| < 1; i = 0,1,....,k). \qquad (9)$$

Here $\alpha_i, \beta_i, \gamma_i\ (i = 0,1,...,k)$ are some real numbers. We assume that the coefficients of the method (9) satisfy the following conditions:

A: The coefficients $\alpha_i, \beta_i, \gamma_i, \nu_i\ \ (i = 0,1,2,...,k)$ are some real numbers, moreover, $\alpha_k \ne 0$.

B: Characteristic polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k}\alpha_i \lambda^i, \ \ \sigma(\lambda) \equiv \sum_{i=0}^{k}\beta_i\, \lambda^i;$$

$$\gamma(\lambda) \equiv \sum_{i=0}^{k}\gamma_i\, \lambda^{i+\nu_i}.$$

have no common multipliers different from the constant.

C: $\sigma(1) + \gamma(1) \ne 0$ and $p \ge 1$.

It can be shown that in order for the method (9) has the degree p, its coefficients must satisfy the following conditions:

$$\begin{gathered} \sum_{i=0}^{k}\alpha_i = 0,\ \sum_{i=0}^{k} i\alpha_i = \sum_{i=0}^{k}\beta_i + \sum_{i=0}^{k}\gamma_i, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \\ \sum_{i=0}^{k}\frac{i^p}{p!}\alpha_i = \sum_{i=0}^{k}\frac{i^{p-1}}{(p-1)!}\beta_i + \sum_{i=0}^{k}\frac{l_i^{p-1}}{(p-1)!}\gamma_i. \end{gathered} \qquad (10)$$

Here $l_i = i + \nu_i\ (i = 0,1,2,...k)$.

It is known, that solving of the system (10) can be reduced to solving of the two triangulate systems (see [9]). As is known for solving treeangulare systems in baysikly using the recurrent relation.

Note that in the system (10) the number of unknowns is equal to $4k+4$, and the number of equations $p+1$. Given system (10) is homogeneous, we see that the system (10) always has a trivial solution. Obviously, existents of nontrivial solutions for the system (10) must be satisfy the next conditions.

$$p+1<4k+4.$$

Hence we find that

$$p \le 4k+2.$$

However, among the methods with a maximum degree of $p=4k+2$ there are unstable methods. Hence the need to find the connection between degrees p and order k for the method (9). Considering that the stability of the method (9) imposes some bounders on the coefficients α_i $(i=0,1,...,k)$ of the linear part of it. If in the system (10), considered these coefficients as free variable, then the number of unknowns in system (10) will be equal $3k+3$. Consequently, for the stable methods of the type (9) we can put that

$$p \le 3k+3.$$

If we consider the case $\nu_i = 0$ $(i=0,1,....)$, then the method (9) coincides with the method (2) and there exist stable methods of the type (2) with the degree $p=k+2$.

Remark, that the method with degree $p=6$ obtained for $k=1$ is stable, and one can be construct stable methods with the degree $p=8$ and $p=9$ for $k=2$. Consider investigation of method (9) for $k=1$. In this case under assumption $\alpha_1 = -\alpha_0 = 1$, system (10) has the following form:

$$\begin{aligned} &\beta_0+\beta_1+\gamma_0+\gamma_1=1,\\ &\beta_1+l_0\gamma_0+l_1\gamma_1=1/2,\\ &\beta_1+l_0^2\gamma_0+l_1^2\gamma_1=1/3,\\ &\beta_1+l_0^3\gamma_0+l_1^3\gamma_1=1/4,\\ &\beta_1+l_0^4\gamma_0+l_1^4\gamma_1=1/5,\\ &\beta_1+l_0^5\gamma_0+l_1^5\gamma_1=1/6. \end{aligned} \tag{11}$$

By solving the system (11) of nonlinear algebraic equations, we get the following:

$$\beta_0=\beta_1=1/12,\ \gamma_0=\gamma_1=5/12,$$
$$l_0=1/2-\sqrt{5}/10,\ l_1=1/2+\sqrt{5}/10.$$

The method with degree $p=6$ has the following form:

$$\begin{aligned} y_{n+1} &= y_n + h(f_{n+1}+f_n)/12+\\ &5h(f_{n+1/2-\sqrt{5}/10}+f_{n+1/2+\sqrt{5}/10})/12 \end{aligned}. \tag{12}$$

For applying hybrid method to solving of some problems, we should know values of the quantities $y_{n+1/2\pm\sqrt{5/10}}$ and the accuracy of these values should have at least $O(h^6)$ order. Note that hybrid method (12) is implicit and while applying it to solving of initial value problem (1) by the predictor-corrector scheme (see for example [2], [7]) that contains even one explicit method is used. Therefore, we consider construction of an explicit method that in one variant has the following form:

$$\begin{aligned} y_{n+1} &= y_n + hf_n/9 + h((16+\sqrt{6})f_{n+(6-\sqrt{6})/10}+\\ &+(16-\sqrt{6})f_{n+(6+\sqrt{6})/10})/36. \end{aligned} \tag{13}$$

This method is explicit and has degree $p=5$.

For using method (13) it is required to define the quantities $y_{n+\frac{6-\sqrt{6}}{10}}$ and $y_{n+\frac{6+\sqrt{6}}{10}}$. Depending on the method for calculation of these quantities the properties of block methods are determined.

If put $\alpha_2=1, \alpha_1=0, \alpha_0=-1$ in the system (10), then by solving the obtained system of nonlinear algebraic equations, we have:

$$\beta_2=64/180,\ \beta_1=98/180,\ \beta_0=18/180,$$
$$\gamma_2=18/180,\ \gamma_1=98/180,$$
$$\gamma_0=64/180,\ l_2=1+\sqrt{21}/14,\ l_1=1,$$
$$l_0=1-\sqrt{21}/14.$$

Hence we get the following method:

$$\begin{aligned} y_{n+2} &= y_n + h(64y'_{n+2}+98y'_{n+1}+18y'_n)/180+\\ &+h(18y'_{n+l_2}+98y'_{n+1}+64y'_{n+\gamma_0})/180. \end{aligned}$$

The constructed method has the degree $p=8$ and stable.

If we construct a method with the degree $p=6$ of the type (7), then there must be $k \ge 2$ and the method constructed for $k=2$ as follows (see [8]):

$$\begin{aligned} y_{n+2} &= y_{n+1} + h(101f_{n+2}+128f_{n+1}+11f_n)/240+\\ &+h^2(-13g_{n+2}+40g_{n+1}+3g_n)/240 \end{aligned} \tag{14}$$

If we construct a method of the type (7) with the degree $p=6$ for $k=1$, then we have:

$$\begin{aligned} y_{n+1} &= y_n + h(y'_n+y'_{n+1})/2+h^2(y''_n-y''_{n+1})/10+\\ &+h^3(y'''_n+y'''_{n+1})/120. \end{aligned} \tag{15}$$

For using the method (13), here suggested the next algorithm.

Algorithm. To approximate the solution of the initial-value problem

$$y' = f(x,y),\ x_0 \le x \le X,\quad y(x_0)=y_0$$

at (N+1) equally spaced numbers in the interval $[x_0, X]$:

INPUT endpoint x_0, X; integer N;

Initial values $y_0, y_{1/2.}$

OUTPUT approximating y_i to $y(x_i)$ at the $(N+1)$ values of x.

Step 1. Set $h = (x - x_0)/N$;

Step 2. For $i = 1,2,...,N$ do Step 3-6.

Step 3. $\hat{y}_{i+1} = y_i + hf_{i+1/2}$;

$y_{i+1} = y_i + h(\hat{f}_{i+1} + 4f_{i+1/2} + f_i)/6$;

$y_{i+1} = y_{i+1/2} + h(7\hat{f}_{i+1} - 2f_{i+1/2} + f_i)/6$.

Step 4. For $\alpha = (6-\sqrt{6})/10,\ (6+\sqrt{6})/10\ do$

$$y_{i+\alpha} = y_i + \alpha h y_i' + \alpha^2 h((\alpha^2 - 12\alpha + 6) f_{i+1/2} - \\ - (\alpha^2 - 24\alpha + 33) f_i)/18.$$

Step 5.

$$y_{i+1} = y_i + hf_i/9 + h(16+\sqrt{6)} f_{i+(6-\sqrt{6})/10} + \\ -(16-\sqrt{6}) f_{i+(6+\sqrt{6}/10)})/36.$$

Step. OUTPUT (i, y_i).

Step 7. STOP.

II. Some methods for solving ODE of the second-order.

As is known, the solution of many scientific and practical problems will formulate as the solutions of ODE of second order. A famous example is Newton's law, as the more generally can be written in the following form:

$$y'' = F(x,y,y'),\ y(x_0) = y_0, \\ y'(x_0) = y_0',\ x_0 \le x \le X. \tag{16}$$

We assume that the problem (16) has a unique continuous solution defined on the segment $[x_0, X]$. It is known that with the change of variable, solution of the problem (16) can be replaced to solution of initial value problem for the systems of ordinary differential equations of first order. If in this case for solving the problem (16) using the multistep method, then we have:

$$\sum_{i=0}^{k} \alpha_i z_{n+i} = h\sum_{i=0}^{k} \beta_i F_{n+i}, \tag{17}$$

$$\sum_{i=0}^{k} \alpha_i' y_{n+i} = h\sum_{i=0}^{k} \beta_i' z_{n+i}, \tag{18}$$

here $z(x) = y'(x),\ F_m = F(x_m, y_m, z_m)\ \ (m \ge 0)$.

Given that the degree of stable multistep method is bounded by the quantity $k+2$, then for constructed a more accurate method can offer different ways. For example, using the method of type (2) we can construct stable multistep methods with the degree $p = k+2$ (see for example [10]). Therefore, we can replace the method (17) with the methods of the type (7) for $r = 2$ having a degree $p = 2k+2$. In this case the method (18) must have a high order of accuracy. For this purpose, one can use the forward- jumping methods. Then, for the construction method with the high accuracy, it is necessary to increase the value of k, which is undesirable. In view of this fact in [11] proposed the use of hybrid methods. In contrast to the well-known works, here the method (17) - (18) is replaced with the following:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i y'_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i F_{n+i}, \tag{19}$$

$$\sum_{i=0}^{k} \alpha_i' y'_{n+i} = h\sum_{i=0}^{k} \beta_i' F_{n+i} + h\sum_{i=0}^{k} \gamma_i' F_{n+i+\nu_i} \\ (|\nu_i| < 1;\ i = 0,1,...,k). \tag{20}$$

The coefficients $\alpha_i, \alpha_i', \beta_i, \beta_i', \gamma_i, \gamma_i', \nu_i \quad (i = 0,1,2,...,k)$ are real numbers, $\alpha_k' \ne 0$ and $\alpha_k \ne 0$, but

$$F_{m+\nu_i} = F(x_{m+\nu_i}, y_{m+\nu_i}, y'_{m+\nu_i}) \\ (m = 0,1,2,...;|\nu_i| < 1, i = 0,1,...,k).$$

Note that the method (20) was investigated in [12] and it was proved that the relationship between the degree and the order is as follows:

$$p \le 4k+2.$$

We can prove that if the method (20) is stable, then there are stable methods of type (20) having the degree $p = 3k+3$. Thus we find that the methods of the type (20) are more accurate than the methods of the type (19`), which provides a choice of methods with extended stability region. Stable method with the maximum degree $p_{\max} = 6$ for $k = 1$ is as following:

$$y_{n+1} = y_n + h(f_n + f_{n+1})/12 + \\ +5h(f_{n+1/2-\alpha} + f_{n+1/2+\alpha})/12,\ (\alpha = \sqrt{5}/10). \tag{21}$$

It is easy to see that for the constructed methods of the type (21) having the degree $p = 6$ must fulfill the condition $k \ge 2$. However, use of the method (21) is more difficult than the use method of the type (19). Method (21) can be applied to solving of a particular problem, if the values of the variables $y_{n+1/2\pm\alpha}$ are known. Consequently, it is difficult to give an advantage to some of these methods. According to this, each of them is subject to a separate study. The applied of hybrid methods to the investigation of the problem (16) in the first considered in the work [13], which was developed in the works [14] and [15]. As is known, the hybrid method is based at the junction of the Runge-Kutta and Adams methods (see for example [16], [17]). Here, by using the ideas of [17], a constructed methods at the junction of the second derivative multistep methods and the hybrid method, which is:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}(\beta_i y'_{n+i} + \hat{\beta}_i y'_{n+i+\nu_i}) + \\ + h^2\sum_{i=0}^{k}(\gamma_i y''_{n+i} + \hat{\gamma}_i y''_{n+i+\nu_i}),\ (|\nu_i|<1;\ i=0,1,...,k). \tag{22}$$

Coefficients $\alpha_i, \beta_i, \hat{\beta}_i, \gamma_i, \hat{\gamma}_i, \nu_i\ (i=0,1,2,...,k)$ are some real numbers, moreover $\alpha_k \neq 0$. By choosing coefficients in the method (22) can be received the known methods from the works [1] - [17]. Consequently, the basic property of the methods of type (22) depends from the values of their coefficients. Therefore consider to determination coefficients of the method (22). For this purpose we use the method of undetermined coefficients (see for example [15]). Then we obtain a nonlinear system of algebraic equations in the following form:

$$\sum_{i=0}^{k}\alpha_i = 0;\ \sum_{i=0}^{k} i\alpha_i = \sum_{i=0}^{k}(\beta_i + \hat{\beta}_i),$$

$$\sum_{i=0}^{k}\frac{i^2}{2!}\alpha_i = \sum_{i=0}^{k}(i\beta_i + l_i\hat{\beta}_i) + \sum_{i=0}^{k}(\gamma_i + \hat{\gamma}_i),$$

$$\sum_{i=0}^{k}\left(\frac{i^{\nu-1}}{(\nu-1)!}\beta_i + \frac{l_i^{\nu-1}}{(\nu-1)!}\hat{\beta}_i\right) + \tag{23}$$

$$+\sum_{i=0}^{k}\left(\frac{i^{\nu-2}}{(\nu-2)!}\gamma_i + \frac{l_i^{\nu-2}}{(\nu-2)!}\hat{\gamma}_i\right) = \sum_{i=0}^{k}\frac{i^{\nu}}{\nu!}\alpha_i\ (\nu=3,4,...,p;\ l_i = i+\nu_i).$$

It is easy to observe that in particularly from this system can be obtained the system (10). In this system, the numbers of unknowns are equal $6k+6$ but the numbers of equations are equal $p+1$. Then we see that for $p \le 6k+4$, the system (23) has a nontrivial solution. Therefore methods such as (22) are more accurate than the known. Methods of type (22) can be applied to solving of problems (1) and (16). Consider particularly case and put $k=1$, then from the system (26), receive the next:

$$\beta_1 + \beta_0 + \hat{\beta}_1 + \hat{\beta}_0 = 1,$$

$$\gamma_1 + \gamma_0 + \hat{\gamma}_1 + \hat{\gamma}_0 + \beta_1 + l_1\hat{\beta}_1 + l_0\hat{\beta}_0 = \frac{1}{2};$$

$$2(\gamma_1 + l_1\hat{\gamma}_1 + l_0\hat{\gamma}_0) + \beta_1 + l_1^2\hat{\beta}_1 + l_0^2\hat{\beta}_0 = \frac{1}{3};$$

$$3(\gamma_1 + l_1^2\hat{\gamma}_1 + l_0^2\hat{\gamma}_0) + \beta_1 + l_1^3\hat{\beta}_1 + l_0^3\hat{\beta}_0 = \frac{1}{4};$$

$$4(\gamma_1 + l_1^3\hat{\gamma}_1 + l_0^3\hat{\gamma}_0) + \beta_1 + l_1^4\hat{\beta}_1 + l_0^4\hat{\beta}_0 = \frac{1}{5};$$

$$5(\gamma_1 + l_1^4\hat{\gamma}_1 + l_0^4\hat{\gamma}_0) + \beta_1 + l_1^5\hat{\beta}_1 + l_0^5\hat{\beta}_0 = \frac{1}{6};$$

$$6(\gamma_1 + l_1^5\hat{\gamma}_1 + l_0^5\hat{\gamma}_0) + \beta_1 + l_1^6\hat{\beta}_1 + l_0^6\hat{\beta}_0 = \frac{1}{7};$$

$$7(\gamma_1 + l_1^6\hat{\gamma}_1 + l_0^6\hat{\gamma}_0) + \beta_1 + l_1^7\hat{\beta}_1 + l_0^7\hat{\beta}_0 = \frac{1}{8};$$

$$8(\gamma_1 + l_1^7\hat{\gamma}_1 + l_0^7\hat{\gamma}_0) + \beta_1 + l_1^8\hat{\beta}_1 + l_0^8\hat{\beta}_0 = \frac{1}{9};$$

$$9(\gamma_1 + l_1^8\hat{\gamma}_1 + l_0^8\hat{\gamma}_0) + \beta_1 + l_1^9\hat{\beta}_1 + l_0^9\hat{\beta}_0 = \frac{1}{10}.$$

By using the solution of the received nonlinear systems can be constructed here the next one step method of the type (22) having the degree $p=10$. Remark, that the method with the degree $p=10$ more than one. One of them can be written as follows:

$$y_{n+1} = y_n + h(\beta_1 y'_{n+1} + \beta_0 y'_n + \hat{\beta}_1 y'_{n+\nu_i} + \hat{\beta}_0 y'_{n+\nu_0}) + \\ + h^2(\gamma_1 y''_{n+1} + \gamma_0 y''_n + \hat{\gamma}_1 y''_{n+\nu_i} + \hat{\gamma}_0 y''_{n+\nu_0}). \tag{24}$$

Here the coefficients received the next values:

$$\beta_1 = 0.12240071695435434,$$
$$\beta_0 = 0.15683688,$$
$$\hat{\beta}_1 = 0.34141698828506156,$$
$$\hat{\beta}_0 = 0.3793454392342798,$$
$$\gamma_1 = -0.00436364516276104,$$
$$\gamma_0 = 0.0075051355565532,$$
$$\hat{\gamma}_1 = -0.00351131453831187,$$
$$\hat{\gamma}_0 = 0.0000267518351899,$$
$$l_1 = 0.726730303263997,$$
$$l_0 = 0.342295826267691.$$

III. ON A WAY TO CONSTRUCTION ALGORITHM FOR APPLICATION OF METHODS (12) AND (24).

As is known one of the basic questions in the theory of numerical methods is to construct methods with the higher order of accuracy, extended the field of stability. Among these methods the implicit methods are most popular. Remark that the method (12) refers to the mentioned set of methods. Difficulties in using the method (12) is finding a values of the quantities f_{n+1} and $f_{n+1/2\pm\alpha}$. By using some explicit methods one can be calculate the approximate values for variables f_{n+1} such as the method (13).

The resulting method is stable and has the degree $p=5$. Note that the formula (13) can be formally considered as an explicit method, since the right side is independent from the quantity y_{n+1}. However, in finding the values of the quantity $y_{n+3/5\pm\beta}$ $(\beta = \sqrt{6}/10)$, according to our proposed schemes have to use the quantity y_{n+1}. Thus we see that the process is applied to the calculation the quantity y_{n+1} by the method of (13) is implicit. Thus we see that for the use of the method (13), the main difficulty lies in finding the values of variables $y_{n+3/5\pm\beta}$ $(\beta=\sqrt{6}/10)$. For this aim, one can be use different formulas. For example as the following:

$$\begin{aligned} y_{n+1+\alpha} &= y_{n+1}+\frac{h}{720}((96\alpha^5+120\alpha^4+120\alpha^3- \\ &-60\alpha^2)y'_{n+2}-(384\alpha^5+240\alpha^4-480\alpha^2)y'_{n+3/2}+ \\ &+(576\alpha^5-240\alpha^3+720\alpha)y'_{n+1}-(384\alpha^5- \\ &-240\alpha^4+480\alpha^2)y'_{n+1/2}+(96\alpha^5-120\alpha^4+ \\ &+120\alpha^3+60\alpha^2)y'_n). \end{aligned} \tag{25}$$

Method (25) is stable for the $|\alpha|\le 1$ and has a local error of the order $O(h^6)$, which corresponds to the accuracy of the method (12). The main drawback of method (25) is to use the values y_n, $y_{n+1/2}$, y_{n+1}, $y_{n+3/2}$, y_{n+2} on each step. Note that some of these values can be used in the next step. It is shown that the using method (25) is available.

We show that using only the values of the variables y_n, $y_{n+1/2}$, y_{n+1}, it is possible to construct a method for computing the values of variables $y_{n+1/2\pm\alpha}$ with the order $O(h^6)$. To this end, consider the following expression of Taylor:

$$\begin{aligned} y(x_{n+1/2+\beta}) &= y(x_{n+1/2})+\beta h y'(x_{n+1/2})+ \\ &+\frac{(\beta h)^2}{2!}y''(x_{n+1/2})+\frac{(\beta h)^3}{3!}y'''(x_{n+1/2})+ \\ &+\frac{(\beta h)^4}{4!}y^{IV}(x_{n+1/2})+\frac{(\beta h)^5}{5!}y^{V}(x_{n+1/2})+O(h^6). \end{aligned} \tag{26}$$

Depends from the form of numerical differentiation of the quantities $y''(x)$, $y'''(x)$, $y^{IV}(x)$, $y^{V}(x)$ at the point $x_{n+1/2}$ we get different formulas. Consider the following formulas

$$\begin{aligned} y''(x_{n+1/2}) &= \frac{1}{h}(y'(x_n)-y'(x_{n+1}))+\frac{1}{h^2}(y(x_n)-2y(x_{n+1/2})+ \\ &+y(x_{n+1}))+O(h^4), \\ y'''(x_{n+1/2}) &= -\frac{6}{h^2}(y'(x_n)+8y'(x_{n+1/2})+y'(x_{n+1}))+ \\ &+\frac{60}{h^3}(y(x_{n+1})-y(x_n))+O(h^3), \\ y^{IV}(x_{n+1/2}) &= \frac{48}{h^3}(y'(x_{n+1})-y(x_n))-\frac{192}{h^4}(y(x_{n+1})- \\ &-2y(x_{n+1/2})+y(x_n))+O(h^2), \\ y^{V}(x_{n+1/2}) &= \frac{480}{h^4}(y'(x_{n+1})+4y'(x_{n+1/2})+y'(x_n))- \\ &-\frac{2880}{h^5}(y(x_{n+1})-y(x_n))+O(h). \end{aligned}$$

Taking into account these formulas in (26), after discarding the remaining members receive the following method:

$$\begin{aligned} y_{n+1/2+\beta} &= 4\beta^2(1-2\beta^2)(y_{n+1}+y_n)+(1-4\beta^2)^2 y_{n+1/2}+ \\ &+2\beta^3(5-12\beta^2)(y_{n+1}-y_n)+h\beta^3(4\beta^2-1)(y'_{n+1}-y'_n)+ \\ &+h\beta(1-4\beta^2)^2 y'_{n+1/2}-h\beta^2(\frac{1}{2}-2\beta^2)(y'_{n+1}-y'_n). \end{aligned} \tag{27}$$

Now consider the comparison methods (25) and (27). The subject of applications of these methods are coincide and also are coincide the using quantity of the type $y(x+mh)$ and $y'(x+mh)$ values. These methods differ from each other in that the method (25) was constructed by using 5- mesh points, but the method (27) was constructed by 3 mesh points. As mentioned the method (25) is stable for $|\alpha|\le 1$, but the method (27) may be unstable. However, these methods are using as the predictor formula. Thus we receive the block method for applying a hybrid methods to solve some specific problems.

By using simple comparisons obtain that regardless of the use in the methods (12) formulas (25) and (27) it is necessary to define approximate values of the quantities $y(x_n)$, $y(x_n+h/2)$ and $y(x_n+h)$ quantities. Therefore we consider to define values of variables $y(x_n)$, $y(x_n+h/2)$ and values $y(x_n+h)$. Here for this aim offer step by step method which consist from the following three blocks:

Block I

$$\hat{y}_{n+1/2} = y_n + h\,y'_n/2,$$

$$y_{n+1/2} = y_n + h(y'_n+\hat{y}'_{n+1/2})/4,$$

$$\bar{y}_{n+1} = y_n + hy'_{n+1/2},$$

$$y_{n+1} = y_n + h(y'_n+4y'_{n+1/2}+\bar{y}'_{n+1})/6,$$

$$\hat{y}_{n+1/2} = y_n - h(y'_{n+1}-8y'_{n+1/2}-5y'_n)/12.$$

Block II

$$\hat{y}_{n+3/2} = y_{n+1}+h(23y'_{n+1}-16\hat{y}'_{n+1/2}+5y'_n)/24, \tag{28}$$

$$y_{n+3/2} = y_{n+1}+h(9\hat{y}'_{n+3/2}+19y'_{n+1}-5\hat{y}'_{n+1/2}+y'_n)/48, \tag{29}$$

$$y_{n+1/2} = y_n + h(\hat{y}'_{n+3/2}-5y'_{n+1}+19\hat{y}'_{n+1/2}+9y'_n)/48, \tag{30}$$

$$y_{n+1} = y_n + h(y'_n+4y'_{n+1/2}+y'_{n+1})/6, \tag{31}$$

$$y_{n+1} = (11y_n + 8y_{n+1/2})/19 + \\ + h(10y'_n + 57y'_{n+1/2} + 24y'_{n+1} - y'_{n+3/2})/114. \tag{32}$$

Block III

$$\hat{y}_{n+2} = y_{n+1} + h(8y'_{n+3/2} - 5y'_{n+1} + 4y'_{n+1/2} - y'_n)/6,$$

$$y_{n+2} = y_{n+3/2} +$$

$$+ h(251y'_{n+2} + 646y'_{n+3/2} - 264y'_{n+1} + 106y'_{n+1/2} - 19y'_n)/1440,$$

$$y_{n+3/2} = y_{n+1/2} + h(\beta_4 y'_{n+2} + \beta_3 y'_{n+3/2} + \beta_2 y'_{n+1} + \beta_1 y'_{n+1/2} + \beta_0 y'_n),$$

Coefficients β_l $(l = 0,1,2,3,4)$ as follows

$\beta_4 = 0.011111111111111143$, $\beta_3 = 0.3777777777777794$,
$\beta_2 = 1.2666666666666633$, $\beta_1 = 0.3777777777777809$,
$\beta_0 = 0.011111111111111199$,

here $\hat{y}' = f(x, \hat{y})$; $\overline{y}' = f(x, \overline{y})$.

Note that the methods of the block I used once time for $n = 0$, and the methods of Block II is used for all the values of the variable n. But for the define values of the quantities $y_{n+1/2+\beta}$ using the formula (27), then suppose using methods from the block III.

Remark that in block I method (33) is used in that case, when the method (27) applied to determine values of quantity $y_{n+1/2+\beta}$, but for the calculating value $y_{n+1/2}$ which high accuracy, one can be used method (25) for $\alpha = 1/2$. Thus received can used the next sequences to solving problem (1):

Block I, Block II, Block III, calculating values of quantities $y_{n+1/2\pm\alpha}$ by the formula (25) or formula (27). By using the receiver values in the method (12) to define the quantity y_{n+1}, after them put

$$\hat{y}_{n+1/2} = y_{n+1/2};\ \hat{y}_{n+3/2} = y_{n+3/2};\ \hat{y}_{n+2} = y_{n+2}$$

continue from the block II.

We give here very simple algorithm, which can be corrected by different ways.

Now consider the application of the method (22) to the solution of (1) and (16). Obviously, if we consider the case $\nu_i = 0$ of a method (22) with a multistep method of the second derivative which is included in the class of such methods (7). To construct a more accurate methods assume that $\nu_i \neq 0$ $(0 \le i \le k)$. Above have shown that the methods of the type (22) are accurate than the known. Therefore, we consider the construction of an algorithm for the use of the method (24). Note that since in the method of (24) engage the values $y''(x)$, we construct an algorithm using calculation quantity $y''(x)$. Using the calculations of these values allows us to construct more exact formula in depended of increasing of the number of mesh points x_i. However, to increase the order of accuracy of the method (24) one can modify it as follows:

$$y_{n+1} = y_n + h(\beta_1 y'_{n+1} + \beta_0 y'_n + \hat{\beta}_1 y'_{n+l_1} + \hat{\beta}_0 y'_{n+l_0}) + \\ + h(\gamma_1 y''_{n+1} + \gamma_0 y''_n + \hat{\gamma}_1 y''_{n+\nu_1} + \gamma_0 y''_{n+\nu_0}) \tag{33}$$

The number of unknowns in the method (24) equal to 10 and the method of (33) equal to 12. Therefore it can be assumed that the methods of the type (33) can be more accurate than the methods of the type (24). Note that when using the type (33), additional difficulties arise because using the same formula can be computed value of $y_{n+\alpha}$ $(\alpha = l_1, l_0, \nu_1, \nu_0)$.

Let us consider to application of the next method received from the formula (24) for the following coefficients :

$$\begin{gathered} \beta_1 = 0, \\ \beta_0 = 0.22067064832761202, \\ \hat{\beta}_0 = 0.6162572072176795, \\ \hat{\beta}_1 = 0.1630721444547081, \\ \gamma_1 = 0, \\ \gamma_0 = 4.674947171097215, \\ \hat{\gamma}_1 = 0.05722827972230951, \\ \hat{\gamma}_0 = -4.656220896767483, \\ \nu_1 = 0.6088582789777904, \\ \nu_0 = -0.00013437381414762, \\ l_1 = 0.9359620492093735, \\ l_0 = 0.44042666646102624. \end{gathered} \tag{34}$$

to solving following problems:

1. $y' = \cos x,\ y(0) = 0,\ 0 \le x \le 1,$ where the user decides to move $h = 0{,}1$.
2. $y' = -y,\ y(0) = 1,\ 0 \le x \le 1,$ (the exact solution is $y(x) = \exp(-x)$).
3. $y' = 8(x - y) + 1,\ y(0) = 2,\ 0 \le x \le 1$ (the exact solution is $y(x) = x + 2\exp(-8x)$).

The results can be found in Table.

Using methods	x	Problem(1)	Problem(2)	Problem(3)
Method (34)	0.10	0.73E-12	0.4E-12	0.15E-06
	0.40	0.27E-11	0.12E-11	0.57E-07
	0.70	0.42E-11	0.15E-11	0.46E-08
	1.00	0.52E-11	0.16E-11	0.11E-08

REMARK. We believe that among the proposed method are the most promosing hybrid methods, since we can contruct explicit hybrid methods that are more accurate than the known implicit methods. As can be seen from the above proposed algorithms, the block method is often implicit. Note that we have constructed several hybrid methods, some of which are A-stable. Comparing methods (13), (14) and (15) we find that the method (12) has several advantages. But the numerical exammplies shown that the method (24) is very interesting and has the extended domain of application.

CONCLUSION.

Thus showed that if we use a multistep method obtained from the formula (9) for the $\gamma_i = 0 \;\; (i = 0,1,2,...,k)$, then one can constructed a stable method with the degree $p_{\max} = k + 2$. To increase the order of accuracy of the proposed methods can be used stable forward-jumping methods with the degree $p_{\max} = k + m + 1$. Some authors for construction a more accurate methods proposed to use the formula (7), in so for as these methods can be applied to solving of the problem (1) and (16). Recently, for the construction of more accurate methods used hybrid methods that are successfully applied to the solving of integral and integro-differential equations (see for example [18] - [20]). In [21] – [23] given one ways for application hybrid and forward-jumping methods to solving initial value problem for ODE by using first and second derivative. However, by using results of this work we receive that methods of the type (22) are more promising. It is known that mane problems of biology are reduce to solving delay ordinary differential equation to solving these problem one can be used method from the work [24] or any modification above proposed method.

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