The spectral decomposition of near-Toeplitz tridiagonal matrices

Nuo Shen, Zhaolin Jiang and Juan Li

***Abstract*—**Some properties of near-Toeplitz tridiagonal matrices with specific perturbations in the first and last main diagonal entries are considered. Applying the relation between the determinant and Chebyshev polynomial of the second kind, we first give the explicit expressions of determinant and characteristic polynomial, then eigenvalues are shown by finding the roots of the characteristic polynomial, which is due to the zeros of Chebyshev polynomial of the first kind, and the eigenvectors are obtained by solving symmetric tridiagonal linear systems in terms of Chebyshev polynomial of the third kind or the fourth kind. By constructing the inverse of the transformation matrices, we give the spectral decomposition of this kind of tridiagonal matrices. Furthermore, the inverse (if the matrix is invertible), powers and a square root are also determined.

***Keywords*—**Tridiagonal matrices, Spectral decomposition, Powers, Inverses, Chebyshev polynomials

I. Introduction

TRIDIAGONAL matrices arise frequently in many areas of mathematics and engineering [1]-[2]. In some problems in numerical analysis one is faced with solving a linear system of equations in which the matrix of the linear system is tridiagonal and Toeplitz, except for elements at the corners. For example, for the homogeneous difference system

$$\boldsymbol{u}(l+1) = \boldsymbol{A}\boldsymbol{u}(l), \qquad l \in \mathbb{Z}, \tag{1}$$

Where $\boldsymbol{A}$ is a nonsingular constant matrix and $\mathbb{Z}$ is the set of all integers including zero, the general solution can be written as $\boldsymbol{u}(l) = \boldsymbol{A}^l\boldsymbol{c}$, $l \in \mathbb{Z}$, where $\boldsymbol{c}$ is an arbitrary constant vector [3]. Thus, to obtain the general solution of the above homogeneous difference system, we need to give the general expression for $\boldsymbol{A}^l$.

J. Rimas computed arbitrary positive integer powers for tridiagonal matrix

$$\boldsymbol{B} = \begin{bmatrix} -1 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 1 \end{bmatrix}$$

in [4]-[5] and presented $\boldsymbol{B}^l = \frac{1}{2n}(q_{ij}(l))$, here

$$q_{ij}(l) = \sum_{k=1}^{n}(4-\lambda_k^2)\lambda_k^l U_{\frac{2i-3}{2}}\left(\frac{\lambda_k}{2}\right) \times U_{\frac{2j-3}{2}}\left(\frac{\lambda_k}{2}\right), i,j = 1,\ldots,n,$$

$\lambda_k (k = 1,\ldots,n)$ is the eigenvalue of the matrix $\boldsymbol{B}$, n is the order of the matrix $\boldsymbol{B}$. Moreover, even order matrix $\boldsymbol{B}$ is nonsingular and the above expression can be applied for computing negative powers of $\boldsymbol{B}$. Taking $l = -1$, he got the following expression for elements of the inverse matrix $\boldsymbol{B}^{-1}$:

$$\{\boldsymbol{B}^{-1}\}_{ij} = \frac{1}{2n}\sum_{k=1}^{n}\frac{4-\lambda_k^2}{\lambda_k}U_{\frac{2i-3}{2}}\left(\frac{\lambda_k}{2}\right) \times U_{\frac{2j-3}{2}}\left(\frac{\lambda_k}{2}\right), i,j = 1,\ldots,n.$$

But odd order matrix $\boldsymbol{B}$ is singular and its inverse and negative powers do not exist.

J. Gutiérrez-Gutiérrez [6] studied the entries of positive integer powers of an $n \times n$ complex tridiagonal Toeplitz (constant diagonals) matrix

$$\boldsymbol{A}_n = \mathrm{tridiag}_n(a_1, a_0, a_{-1}) = \begin{bmatrix} a_0 & a_{-1} & & & & \\ a_1 & a_0 & a_{-1} & & & \\ & a_1 & \ddots & \ddots & & \\ & & \ddots & \ddots & a_{-1} & \\ & & & a_1 & a_0 & a_{-1} \\ & & & & a_1 & a_0 \end{bmatrix},$$

where $a_1a_{-1} \neq 0$. He gave the following result:
Consider $a_1, a_0, a_{-1} \in \mathbb{C}$, $a_1a_{-1} \neq 0$ and $n \in \mathbb{N}$. Let $\boldsymbol{A}_n = \mathrm{tridiag}_n(a_1, a_0, a_{-1})$, $\beta = \sqrt{\frac{a_1}{a_{-1}}}$ and $\lambda_h = -2\cos\frac{h\pi}{n+1}$ for every $1 \leq h \leq n$. Then

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$$\begin{aligned}[\boldsymbol{A}_n^q]_{j,k} =& \frac{\beta^{j-k}}{2n+2}\Bigg[2(1+(-1)^{n+1})a_0^q U_{j-1}(0)U_{k-1}(0) \\ &+ \sum_{h=1}^{\lfloor \frac{n}{2} \rfloor}(4-\lambda_{n-h+1}^2)U_{j-1}\left(\frac{\lambda_{n-h+1}}{2}\right) \\ &\times U_{k-1}\left(\frac{\lambda_{n-h+1}}{2}\right)[(a_0+a_{-1}\beta\lambda_{n-h+1})^q \\ &+ (-1)^{j+k}(a_0-a_{-1}\beta\lambda_{n-h+1})^q]\Bigg]\end{aligned}$$

for all $q \in \mathbb{N}$ and $1 \leq j,k \leq n$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

In this paper, we consider the near-Toeplitz tridiagonal matrices of order $n (n \in \mathbb{N}, n \geq 2)$ with specific perturbations in the first and last main diagonal entries as follows:

$$\boldsymbol{A} = \begin{pmatrix} \alpha+b & c & & & & \\ a & b & c & & & \\ & a & b & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & b & c \\ & & & & a & -\alpha+b \end{pmatrix}_{n\times n}, \quad (2)$$

where $\alpha, a, b, c \in \mathbb{C}$, and $\alpha = \pm\sqrt{ac}$, $ac \neq 0$.

If $a = c$, then $\boldsymbol{A}$ is symmetric. For a general real symmetric matrix is orthogonally equivalent to a symmetric tridiagonal matrix, so solving the spectral decomposition problem of the symmetric tridiagonal matrices makes a contribution to that of the general real symmetric matrices.

The outline of the paper is as follows. In next section, we review some basic definition and facts about the Chebyshev polynomials and an equality on the sum of trigonometric function without proof. In section 3, we first compute trace, determinant, the characteristic polynomial, the eigenvalues and eigenvectors by using root-finding scheme and solving symmetric tridiagonal linear system of equations respectively, which are different from the techniques used in [7]. As we all know, the powers are easily determined if we know the spectral decomposition. Therefore, we present the spectral decomposition by constructing the inverse of the similarity matrix of which column vectors are the eigenvectors. On the grounds of the spectral decomposition, we discuss the conditions under which $\boldsymbol{A}$ can be unitarily diagonalizable. In addition, we give some conclusions when $\boldsymbol{A}$ is a symmetric tridiagonal matrix. In section 4, using the results in section 3, we present the powers, inverse (if invertible) and a square root of $\boldsymbol{A}$. In the end, to make the application of the obtained results clear, we solve a difference system as example and verify the result obtained by J. Rimas is a special case of our conclusion. Moreover, the algorithms of Maple 13 are given.

II. Preliminaries

There are several kinds of Chebyshev polynomials. In particular we shall introduce the first and second kind polynomials $T_n(x)$ and $U_n(x)$, as well as a pair of related (Jacobi) polynomials $V_n(x)$ and $W_n(x)$, which we call the Chebyshev polynomials of the third and fourth kinds [8].

Definition 1 The Chebyshev polynomials $T_n(x)$, $U_n(x)$, $V_n(x)$ and $W_n(x)$ of the first, second, third and fourth kinds are polynomials in x of degree n defined respectively by

$$\begin{aligned} T_n(x) &= \cos n\theta, \\ U_n(x) &= \sin(n+1)\theta / \sin\theta, \\ V_n(x) &= \cos\left(n+\frac{1}{2}\right)\theta / \cos\frac{1}{2}\theta, \\ W_n(x) &= \sin\left(n+\frac{1}{2}\right)\theta / \sin\frac{1}{2}\theta, \end{aligned}$$

when $x = \cos\theta, \ -1 \leq x \leq 1$.

Lemma 1 The four kinds of Chebyshev polynomial satisfy the same recurrence relation

$$X_n(x) = 2xX_{n-1}(x) - X_{n-2}(x),$$

with $X_0(x) = 1$ in each case and $X_1(x) = x$, $2x$, $2x-1$, $2x+1$, respectively. Furthermore, three relationships can be derived from the above relations as follows

$$\begin{aligned} 2T_n(x) &= U_n(x) - U_{n-2}(x) \\ V_n(x) &= U_n(x) - U_{n-1}(x), \\ W_n(x) &= U_n(x) + U_{n-1}(x). \end{aligned}$$

By expanding the following determinant along the last row and using the three-term recurrence for $U_n(x)$ in Lemma 1, we find $U_n(x)$ can be expressed by the determinant, namely,

$$\begin{aligned} U_0(x) &= 1, \\ U_1(x) &= 2x, \\ U_n(x) &= \begin{vmatrix} 2x & z & & & \\ y & 2x & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2x & z \\ & & & y & 2x \end{vmatrix}, \end{aligned}$$

where $yz = 1$.

Lemma 2 The equality

$$\sum_{h=1}^{n}\cos\frac{(2h-1)k\pi}{2n} = 0$$

holds for every $n \in \mathbb{N}$, $k = 1, \ldots, 2n-1$.

III. Spectral Decomposition

Employing Laplace expansion, the expression of $U_n(x)$ in terms of determinant, and the relation between the Chebyshev polynomial of the first kind and second kind, we have the following assertions.

Lemma 3 If $\boldsymbol{A}$ is a tridiagonal matrix of the form (2), then

$$\mathrm{tr}\boldsymbol{A} = nb,$$

$$\det \boldsymbol{A} = 2|\alpha|^n T_n\left(\frac{b}{2|\alpha|}\right)$$

and the characteristic polynomial of $\boldsymbol{A}$ is

$$p_{\boldsymbol{A}}(\lambda) = \det(\lambda\mathbf{I} - \boldsymbol{A}) = 2|\alpha|^n T_n\left(\frac{\lambda - b}{2|\alpha|}\right) \quad (3)$$

where $\mathbf{I}$ is the identity matrix.

Proof : The trace of $\boldsymbol{A}$ is equal to the sum of all the diagonal entries, so we have $\operatorname{tr}\boldsymbol{A} = nb$ from the form of $\boldsymbol{A}$.

By expanding the determinant of $\boldsymbol{A}$ along the first column and the last column, we have

$$\det \boldsymbol{A} = \begin{vmatrix} \alpha+b & c & & & & \\ a & b & c & & & \\ & a & b & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & b & c \\ & & & & a & -\alpha+b \end{vmatrix}_{n\times n}$$

$$= (b^2-\alpha^2)\begin{vmatrix} b & c & & & \\ a & b & c & & \\ & \ddots & \ddots & \ddots & \\ & & a & b & c \\ & & & a & b \end{vmatrix}_{(n-2)\times(n-2)}$$

$$-2bac\begin{vmatrix} b & c & & & \\ a & b & c & & \\ & \ddots & \ddots & \ddots & \\ & & a & b & c \\ & & & a & b \end{vmatrix}_{(n-3)\times(n-3)}$$

$$+a^2c^2\begin{vmatrix} b & c & & & \\ a & b & c & & \\ & \ddots & \ddots & \ddots & \\ & & a & b & c \\ & & & a & b \end{vmatrix}_{(n-4)\times(n-4)}$$

According to the expression of $U_n(x)$ in terms of determinant and Lemma 1, we have

$$\begin{aligned}\det \boldsymbol{A} &= (b^2-\alpha^2)|\alpha|^{n-2}U_{n-2}\left(\frac{b}{2|\alpha|}\right) \\ &\quad - 2b|\alpha|^{n-1}U_{n-3}\left(\frac{b}{2|\alpha|}\right) + |\alpha|^n U_{n-4}\left(\frac{b}{2|\alpha|}\right) \\ &= |\alpha|^n\left[U_n\left(\frac{b}{2|\alpha|}\right) - U_{n-2}\left(\frac{b}{2|\alpha|}\right)\right] \\ &= 2|\alpha|^n T_n\left(\frac{b}{2|\alpha|}\right).\end{aligned}$$

Similar to the determinant, the characteristic polynomial can be calculated. □

Consequently, the eigenvalues of $\boldsymbol{A}$ can be obtained through computing the zeros of the characteristic polynomial (3). In view of the roots of $T_n(x)$ are $x_i = \cos\frac{(2i-1)\pi}{2n}$, $i = 1, 2, \ldots, n$, so the eigenvalues of $\boldsymbol{A}$ are

$$\lambda_i = b + 2|\alpha|\cos\frac{(2i-1)\pi}{2n}, \quad i = 1, 2, \ldots, n.$$

From this, we can obtain the following conclusions:

1) The expression of determinant can be also written as $\det \boldsymbol{A} = \prod_{i=1}^n \left(b + 2|\alpha|\cos\frac{(2i-1)\pi}{2n}\right)$, namely,
$$2|\alpha|^n T_n\left(\frac{b}{2|\alpha|}\right) = \prod_{i=1}^n \left(b + 2|\alpha|\cos\frac{(2i-1)\pi}{2n}\right).$$
2) If n is even, then $\lambda_{n+1-i} = 2b - \lambda_i$, $i = 1, 2, \ldots, \frac{n}{2}$; If n is odd, then $\lambda_{n+1-i} = 2b - \lambda_i$, $i = 1, 2, \ldots, \lfloor\frac{n}{2}\rfloor$, and $\lambda_{\frac{n}{2}+1} = b$. From this, we can again obtain $\operatorname{tr}\boldsymbol{A} = nb$. In addition, the spectral radius of $\boldsymbol{A}$ will converge to $b + 2|\alpha|$ as $n \to \infty$.
3) If $b \neq -2|\alpha|\cos\frac{(2i-1)\pi}{2n}$ $(i = 1, 2, \ldots, n)$, then $\boldsymbol{A}$ is invertible.

The corresponding eigenvectors of $\boldsymbol{A}$ can be attained via solving the following equation system

$$(\lambda\mathbf{I} - \boldsymbol{A})\boldsymbol{v} = 0, \quad \boldsymbol{v} \neq 0, \tag{4}$$

in which the coefficient matrix $\lambda\mathbf{I} - \boldsymbol{A}$ is nonsymmetric. It is more convenient to solve the equation system if we change the coefficient matrix into a symmetric matrix.

Let $\boldsymbol{D}$ = diag$(d_0, d_1, \ldots, d_{n-1})$ and $d_k = (a/c)^{k/2}$.

Suppose $\boldsymbol{u}$ solves equations

$$(\lambda\mathbf{I} - \boldsymbol{A})\boldsymbol{D}\boldsymbol{u} = 0, \tag{5}$$

which can be deduced equivalently to the linear system of equations with the symmetric tridiagonal matrix, then $\boldsymbol{v} = \boldsymbol{D}\boldsymbol{u}$ is a solution of (4).

When $\alpha = -\sqrt{ac}$, the equation (5) can be written as

$$\begin{aligned} \left(\tfrac{\lambda-b}{|\alpha|} + 1\right)u_1 - u_2 &= 0 \\ -u_1 + \tfrac{\lambda-b}{|\alpha|}u_2 - u_3 &= 0 \\ -u_2 + \tfrac{\lambda-b}{|\alpha|}u_3 - u_4 &= 0 \\ &\vdots \\ -u_{n-2} + \tfrac{\lambda-b}{|\alpha|}u_{n-1} - u_n &= 0 \\ u_{n-1} + \left(\tfrac{\lambda-b}{|\alpha|} - 1\right)u_n &= 0. \end{aligned}$$

Solving the above equations, we have some solutions

$$\boldsymbol{u}^{(i)} = \left[W_0(x_i), W_1(x_i), \ldots, W_{n-1}(x_i)\right]^{\mathrm{T}}, i = 1, \ldots, n,$$

where $x_i = \cos\frac{(2i-1)\pi}{2n}$.

Hence, solutions of the characteristic equation (4), the eigenvectors of $\boldsymbol{A}$ with $\alpha = -\sqrt{ac}$, are

$$\boldsymbol{v}^{(i)} = \left[d_0W_0(x_i), d_1W_1(x_i), \ldots, d_{n-1}W_{n-1}(x_i)\right]^{\mathrm{T}}, \quad i = 1, \ldots, n,$$

where $x_i = \cos\frac{(2i-1)\pi}{2n}$.

When $\alpha = \sqrt{ac}$, the equation (5) can be written as

$$\begin{aligned} \left(\tfrac{\lambda-b}{|\alpha|} - 1\right)u_1 - u_2 &= 0 \\ -u_1 + \tfrac{\lambda-b}{|\alpha|}u_2 - u_3 &= 0 \\ -u_2 + \tfrac{\lambda-b}{|\alpha|}u_3 - u_4 &= 0 \\ &\vdots \\ -u_{n-2} + \tfrac{\lambda-b}{|\alpha|}u_{n-1} - u_n &= 0 \\ u_{n-1} + \left(\tfrac{\lambda-b}{|\alpha|} + 1\right)u_n &= 0. \end{aligned}$$

The system has solutions

$$\boldsymbol{u}^{(i)} = \left[V_0(x_i), V_1(x_i), \ldots, V_{n-1}(x_i)\right]^{\mathrm{T}}, i = 1, \ldots, n.$$

Therefore, the solutions of the characteristic equation (4) are

$$\boldsymbol{v}^{(i)} = \left[d_0V_0(x_i), d_1V_1(x_i), \ldots, d_{n-1}V_{n-1}(x_i)\right]^{\mathrm{T}}, \quad i = 1, \ldots, n,$$

which are the eigenvectors of $\boldsymbol{A}$ with $\alpha = \sqrt{ac}$.

Using the above results, we give the spectral decomposition of $\boldsymbol{A}$ and demonstrate it. Note that $\boldsymbol{\Lambda} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\lambda_i (i = 1, 2, \ldots, n)$ are eigenvalues of $\boldsymbol{A}$ in the remainder of the paper. We introduce the fact about the spetral decomposition in [9] as the following lemma.

Lemma 4 If $\boldsymbol{A}$ has n linearly independent eigenvectors $\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(2)}, \ldots, \boldsymbol{v}^{(n)}$, form a nonsingular matrix $\boldsymbol{S}$ with them as columns, then $\boldsymbol{A} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1}$, where

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

and $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $\boldsymbol{A}$.

Theorem 1 If $\boldsymbol{A}$ has the form (2) with $\alpha = -\sqrt{ac}$. Then $\boldsymbol{A} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{T}\boldsymbol{S}^{T}(\boldsymbol{D}^{-1})^2$, where

$$\boldsymbol{S} = \begin{bmatrix} d_0 W_0(x_1) & d_0 W_0(x_2) & \cdots & d_0 W_0(x_n) \\ d_1 W_1(x_1) & d_1 W_1(x_2) & \cdots & d_1 W_1(x_n) \\ \vdots & \vdots & \cdots & \vdots \\ d_{n-2} W_{n-2}(x_1) & d_{n-2} W_{n-2}(x_2) & \cdots & d_{n-2} W_{n-2}(x_n) \\ d_{n-1} W_{n-1}(x_1) & d_{n-1} W_{n-1}(x_2) & \cdots & d_{n-1} W_{n-1}(x_n) \end{bmatrix},$$

$\boldsymbol{T} = \mathrm{diag}(t_1, \ldots, t_n)$ and $t_h = (1 - x_h)/n$, $h = 1, 2, \ldots, n$.

Proof : From Lemma 4, we know that the only thing we need to do is to show that $\boldsymbol{S}\boldsymbol{T}\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2 = \mathbf{I}$, that is, $\boldsymbol{T}\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2$ is the inverse of $\boldsymbol{S}$.

If $i = j$, then

$$\begin{aligned} &[\boldsymbol{S}\boldsymbol{T}\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2]_{ii} \\ &= \sum_{h=1}^{n} d_{i-1} W_{i-1}(x_h)\, t_h d_{i-1} W_{i-1}(x_h)\, (1/d_{i-1})^2 \\ &= \sum_{h=1}^{n} t_h W_{i-1}^2(x_h) \\ &= \frac{1}{n} \sum_{h=1}^{n} \left(1 - \cos\frac{(2h-1)\pi}{2n}\right) \frac{\sin^2 \frac{(i-1/2)(2h-1)\pi}{2n}}{\sin^2 \frac{(2h-1)\pi}{4n}} \\ &= \frac{1}{n} \sum_{h=1}^{n} \left(1 - \cos\frac{(2i-1)(2h-1)\pi}{2n}\right) \\ &= \frac{1}{n} \left(n - \sum_{h=1}^{n} \cos\frac{(2i-1)(2h-1)\pi}{2n}\right). \end{aligned}$$

From Lemma 2, we have

$$\sum_{h=1}^{n} \cos\frac{(2i-1)(2h-1)\pi}{2n} = 0.$$

Then $[\boldsymbol{S}\boldsymbol{T}\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2]_{ii} = 1$.

If $i \neq j$, then

$$\begin{aligned} &[\boldsymbol{S}\boldsymbol{T}\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2]_{ij} \\ &= \sum_{h=1}^{n} d_{i-1} W_{i-1}(x_h)\, t_h d_{j-1} W_{j-1}(x_h)\, (1/d_{j-1})^2 \\ &= d_{i-j} \sum_{h=1}^{n} t_h W_{i-1}(x_h)\, W_{j-1}(x_h) \\ &= \frac{d_{i-j}}{n} \left(\sum_{h=1}^{n} \cos\frac{(i-j)(2h-1)\pi}{2n} \right. \\ &\qquad \left. - \sum_{h=1}^{n} \cos\frac{(i+j-1)(2h-1)\pi}{2n} \right). \end{aligned}$$

According to Lemma 2, we have

$$\sum_{h=1}^{n} \cos\frac{(i-j)(2h-1)\pi}{2n} = \sum_{h=1}^{n} \cos\frac{(i+j-1)(2h-1)\pi}{2n} = 0.$$

Thus $[\boldsymbol{S}\boldsymbol{T}\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2]_{ij} = 0$. Therefore, $\boldsymbol{S}\boldsymbol{T}\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2 = \mathbf{I}$, and $\boldsymbol{T}\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2$ is the inverse of $\boldsymbol{S}$. $\boldsymbol{A} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{T}\boldsymbol{S}^{T}(\boldsymbol{D}^{-1})^2$ is the spectral decomposition of $\boldsymbol{A}$ with $\alpha = -\sqrt{ac}$. □

Corollary 1 Let $\boldsymbol{A}$ be a tridiagonal matrix of the form (2) with $\alpha = -\sqrt{ac}$. If $|a| = |c|$, then $\boldsymbol{A}$ can be unitarily diagonalizable.

Proof : A scalar multiple of an eigenvector of $\boldsymbol{A}$ is still an eigenvector of $\boldsymbol{A}$, So

$$\boldsymbol{v}_1^{(i)} = \sqrt{\tfrac{1-x_i}{n}}\,[d_0 W_0(x_i), d_1 W_1(x_i), \ldots, d_{n-1} W_{n-1}(x_i)]^{\mathrm{T}}, \quad i = 1, \ldots, n,$$

are a set of eigenvectors of $\boldsymbol{A}$. Let $\boldsymbol{U}$ be a matrix with $\boldsymbol{v}_1^{(i)}$ as columns. Namely,

$$\boldsymbol{U} = \begin{bmatrix} \sqrt{\tfrac{1-x_1}{n}} d_0 W_0(x_1) & \sqrt{\tfrac{1-x_2}{n}} d_0 W_0(x_2) & \cdots & \sqrt{\tfrac{1-x_n}{n}} d_0 W_0(x_n) \\ \sqrt{\tfrac{1-x_1}{n}} d_1 W_1(x_1) & \sqrt{\tfrac{1-x_2}{n}} d_1 W_1(x_2) & \cdots & \sqrt{\tfrac{1-x_n}{n}} d_1 W_1(x_n) \\ \vdots & \vdots & \cdots & \vdots \\ \sqrt{\tfrac{1-x_1}{n}} d_{n-2} W_{n-2}(x_1) & \sqrt{\tfrac{1-x_2}{n}} d_{n-2} W_{n-2}(x_2) & \cdots & \sqrt{\tfrac{1-x_n}{n}} d_{n-2} W_{n-2}(x_n) \\ \sqrt{\tfrac{1-x_1}{n}} d_{n-1} W_{n-1}(x_1) & \sqrt{\tfrac{1-x_2}{n}} d_{n-1} W_{n-1}(x_2) & \cdots & \sqrt{\tfrac{1-x_n}{n}} d_{n-1} W_{n-1}(x_n) \end{bmatrix}.$$

If we want to prove that $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^*$, then what we need to do is to verify that $\boldsymbol{U}\boldsymbol{U}^* = \boldsymbol{I}$. Obviously,

$$\begin{aligned} [\boldsymbol{U}\boldsymbol{U}^*]_{ij} &= \sum_{h=1}^{n} \sqrt{\tfrac{1-x_h}{n}}\, d_{i-1} W_{i-1}(x_h) \sqrt{\tfrac{1-x_h}{n}}\, \overline{d_{j-1}} W_{j-1}(x_h) \\ &= d_{i-1}\overline{d_{j-1}} \sum_{h=1}^{n} \frac{1-x_h}{n} W_{i-1}(x_h) W_{j-1}(x_h). \end{aligned}$$

If $i = j$, then

$$[\boldsymbol{U}\boldsymbol{U}^*]_{ii} = d_{i-1}\overline{d_{i-1}}\sum_{h=1}^{n}\frac{1-x_h}{n}W_{i-1}^2(x_h)$$
$$= |d_{i-1}|^2\sum_{h=1}^{n}\frac{1-x_h}{n}W_{i-1}^2(x_h).$$

From the proof of Theorem 1, we have
$$\sum_{h=1}^{n}\frac{1-x_h}{n}W_{i-1}^2(x_h) = 1.$$
Since $|a| = |c|$, $|d_{i-1}|^2 = |\frac{a}{c}|^{i-1} = 1$. Thus, $[\boldsymbol{U}\boldsymbol{U}^*]_{ii} = 1$.
If $i \neq j$, then
$$\sum_{h=1}^{n}\frac{1-x_h}{n}W_{i-1}(x_h)W_{j-1}(x_h) = 0$$
by the proof of Theorem 1.
From the above discussion, we know that the transformation matrix $\boldsymbol{U}$ is unitary and $\boldsymbol{A}$ with $\alpha = -\sqrt{ac}$ can be unitarily diagonalizable when $|a| = |c|$. □

Theorem 2 If $\boldsymbol{A}$ is a tridiagonal matrix of the form (2) with $\alpha = \sqrt{ac}$. Then $\boldsymbol{A} = \boldsymbol{P\Lambda Q P}^T(\boldsymbol{D}^{-1})^2$, where $\boldsymbol{P}$ consists of the eigenvectors of $\boldsymbol{A}$, i.e.,
$$\boldsymbol{P} = \begin{bmatrix} d_0V_0(x_1) & d_0V_0(x_2) & \cdots & d_0V_0(x_n) \\ d_1V_1(x_1) & d_1V_1(x_2) & \cdots & d_1V_1(x_n) \\ \vdots & \vdots & \cdots & \vdots \\ d_{n-2}V_{n-2}(x_1) & d_{n-2}V_{n-2}(x_2) & \cdots & d_{n-2}V_{n-2}(x_n) \\ d_{n-1}V_{n-1}(x_1) & d_{n-1}V_{n-1}(x_2) & \cdots & d_{n-1}V_{n-1}(x_n) \end{bmatrix}.$$
Moreover, $\boldsymbol{Q} = \mathrm{diag}(q_1, \ldots, q_n)$ and $q_h = (1 + x_h)/n$, $h = 1, 2, \ldots, n$.

Proof : The technique used in the proof is the same as Theorem 1. First, we derive that
$$[\boldsymbol{PQP}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2]_{ij}$$
$$= \sum_{h=1}^{n} d_{i-1}V_{i-1}(x_h)\, q_h d_{j-1}V_{j-1}(x_h)\,(1/d_{j-1})^2$$
$$= d_{i-j}\sum_{h=1}^{n} q_h V_{i-1}(x_h)\, V_{j-1}(x_h)$$
$$= \frac{d_{i-j}}{n}\left(\sum_{h=1}^{n}\cos\frac{(i-j)(2h-1)\pi}{2n} + \sum_{h=1}^{n}\cos\frac{(i+j-1)(2h-1)\pi}{2n}\right).$$
According to Lemma 2, we obtain the following conclusions:
If $i = j$, then
$$[\boldsymbol{PQP}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2]_{ii}$$
$$= \frac{1}{n}\left(n + \sum_{h=1}^{n}\cos\frac{(i+j-1)(2h-1)\pi}{2n}\right)$$
$$= 1$$
following from
$$\sum_{h=1}^{n}\cos\frac{(2i-1)(2h-1)\pi}{2n} = 0.$$
If $i \neq j$, then
$$\sum_{h=1}^{n}\cos\frac{(i-j)(2h-1)\pi}{2n} + \sum_{h=1}^{n}\cos\frac{(i+j-1)(2h-1)\pi}{2n} = 0$$
and $[\boldsymbol{PQP}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2]_{ij} = 0$. Thus, $\boldsymbol{PQP}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2 = \mathbf{I}$, and $\boldsymbol{QP}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2$ is the inverse of $\boldsymbol{P}$. Hence
$\boldsymbol{A} = \boldsymbol{P\Lambda QP}^T(\boldsymbol{D}^{-1})^2$ is the spectral decomposition of $\boldsymbol{A}$ with $\alpha = \sqrt{ac}$. □

Corollary 2 Let $\boldsymbol{A}$ be a tridiagonal matrix of the form (2) with $\alpha = \sqrt{ac}$. If $|a| = |c|$, then $\boldsymbol{A}$ can be unitarily diagonalizable.

Proof : First we know that
$$\boldsymbol{v}_2^{(i)} = \sqrt{\tfrac{1+x_i}{n}}\left[d_0V_0(x_i), d_1V_1(x_i), \ldots, d_{n-1}V_{n-1}(x_i)\right]^{\mathrm{T}}, \quad i = 1, \ldots, n,$$
are a set of eigenvectors of $\boldsymbol{A}$. Let $\boldsymbol{V}$ be a matrix with $\boldsymbol{v}_2^{(i)}$ as columns. Namely,
$$\boldsymbol{V} = \begin{bmatrix} \sqrt{\tfrac{1+x_1}{n}}d_0V_0(x_1) & \sqrt{\tfrac{1+x_2}{n}}d_0V_0(x_2) & \cdots & \sqrt{\tfrac{1+x_n}{n}}d_0V_0(x_n) \\ \sqrt{\tfrac{1+x_1}{n}}d_1V_1(x_1) & \sqrt{\tfrac{1+x_2}{n}}d_1V_1(x_2) & \cdots & \sqrt{\tfrac{1+x_n}{n}}d_1V_1(x_n) \\ \vdots & \vdots & \cdots & \vdots \\ \sqrt{\tfrac{1+x_1}{n}}d_{n-2}V_{n-2}(x_1) & \sqrt{\tfrac{1+x_2}{n}}d_{n-2}V_{n-2}(x_2) & \cdots & \sqrt{\tfrac{1+x_n}{n}}d_{n-2}V_{n-2}(x_n) \\ \sqrt{\tfrac{1+x_1}{n}}d_{n-1}V_{n-1}(x_1) & \sqrt{\tfrac{1+x_2}{n}}d_{n-1}V_{n-1}(x_2) & \cdots & \sqrt{\tfrac{1+x_n}{n}}d_{n-1}V_{n-1}(x_n) \end{bmatrix}.$$
In order to prove that $\boldsymbol{A} = \boldsymbol{V\Lambda V}^*$, we need to demonstrate that $\boldsymbol{VV}^* = \boldsymbol{I}$.
$$[\boldsymbol{VV}^*]_{ij} = \sum_{h=1}^{n}\sqrt{\tfrac{1+x_h}{n}}d_{i-1}V_{i-1}(x_h)\sqrt{\tfrac{1+x_h}{n}}\overline{d_{j-1}}V_{j-1}(x_h)$$
$$= d_{i-1}\overline{d_{j-1}}\sum_{h=1}^{n}\frac{1+x_h}{n}V_{i-1}(x_h)V_{j-1}(x_h).$$
According to the proof of Theorem 2, we have the following arguments.
If $i = j$, then
$$\sum_{h=1}^{n}\frac{1+x_h}{n}V_{i-1}(x_h)V_{j-1}(x_h) = 1.$$
If $i \neq j$, then
$$\sum_{h=1}^{n}\frac{1+x_h}{n}V_{i-1}(x_h)V_{j-1}(x_h) = 0.$$
Furthermore, $d_{i-1}\overline{d_{i-1}} = 1$ for $|a| = |c|$. Therefore, $\boldsymbol{VV}^* = \boldsymbol{I}$, that is $\boldsymbol{V}$ is unitary. Then $\boldsymbol{A}$ with $\alpha = \sqrt{ac}$ can be unitarily diagonalizable when $|a| = |c|$. □

Corollary 3 Let $\boldsymbol{A}$ be a tridiagonal matrix of the form (2) with $\alpha = -\sqrt{ac}$ or $\alpha = \sqrt{ac}$. If $a = c$, then two arbitary tridiagonal matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ with this kind form are simultaneously diagonalizable, that is, there is a single similarity matrix $\boldsymbol{S}$ such that $\boldsymbol{S}^{-1}\boldsymbol{AS}$ and $\boldsymbol{S}^{-1}\boldsymbol{BS}$ are both diagonal.

Proof : If $a = c$, then $\boldsymbol{D}$ is the identity matrix in Theorem 1 and Theorem 2. The conclusion can be obtained directly from Theorem 1 and Theorem 2.

Corollary 4 Let $\mathfrak{F}$ be a family of the matrices of the form (2) with $a = c$, $\alpha = |a|$ or $\alpha = -|a|$. Then $\mathfrak{F}$ is a simultaneously diagonalizable family and a commuting family.

Proof : From Corollary 3, we know that $\mathfrak{F}$ is a simultaneously diagonalizable family, that is, for any $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{F}$, there exists a single similarity matrix $\boldsymbol{S}$ such that $\boldsymbol{S}^{-1}\boldsymbol{AS} = \boldsymbol{\Lambda}_1$ and $\boldsymbol{S}^{-1}\boldsymbol{BS} = \boldsymbol{\Lambda}_2$, where $\boldsymbol{\Lambda}_1$, $\boldsymbol{\Lambda}_2$ are diagonal matrices. Then

$$\begin{aligned}\boldsymbol{AB} &= \boldsymbol{S\Lambda}_1\boldsymbol{S}^{-1}\boldsymbol{S\Lambda}_2\boldsymbol{S}^{-1} = \boldsymbol{S\Lambda}_1\boldsymbol{\Lambda}_2\boldsymbol{S}^{-1} \\ &= \boldsymbol{S\Lambda}_2\boldsymbol{\Lambda}_1\boldsymbol{S}^{-1} = \boldsymbol{S\Lambda}_2\boldsymbol{S}^{-1}\boldsymbol{S\Lambda}_1\boldsymbol{S}^{-1} = \boldsymbol{BA}.\end{aligned}$$

Therefore, $\mathfrak{F}$ is not only a simultaneously diagonalizable family but also a commuting family. □

IV. Powers and Inverse

As we all know, if the matrix $\boldsymbol{A}$ has spectral decomposition $\boldsymbol{A} = \boldsymbol{S\Lambda S}^{-1}$, then the lth $(l \in \mathbb{N})$ power of $\boldsymbol{A}$ can be obtained by $\boldsymbol{A}^l = \boldsymbol{S\Lambda}^l\boldsymbol{S}^{-1}$, where $\boldsymbol{\Lambda}$ is diagonal matrix, the diagonal entries of which are eigenvalues of $\boldsymbol{A}$. $\boldsymbol{S}$ is the transforming matrix formed by eigenvectors of $\boldsymbol{A}$ with them as columns [9]. In the previous section, we have stated the spectral decomposition of $\boldsymbol{A}$. In this section, we calculate the powers, inverse and a square root of $\boldsymbol{A}$.

Theorem 3 If $\boldsymbol{A}$ has the form (2) with $\alpha = -\sqrt{ac}$ and $x_h = \cos\frac{(2h-1)\pi}{2n}$, $h = 1, 2, \ldots, n$. Then the i, j entry of $\boldsymbol{A}^l$ $(l \in \mathbb{N})$ is

$$[\boldsymbol{A}^l]_{ij} = \frac{d_{i-j}}{n}\sum_{h=1}^{n}(b + 2|\alpha|x_h)^l(1 - x_h)W_{i-1}(x_h)W_{j-1}(x_h).$$

Proof : According to Theorem 1, we have

$$\begin{aligned}[\boldsymbol{A}^l]_{ij} &= \sum_{h=1}^{n} d_{i-1}W_{i-1}(x_h)\lambda_h^l t_h d_{j-1}^{-1}W_{j-1}(x_h) \\ &= \frac{d_{i-j}}{n}\sum_{h=1}^{n}(b + 2|\alpha|x_h)^l(1 - x_h)W_{i-1}(x_h)W_{j-1}(x_h).\end{aligned}$$

The proof is completed. □

Theorem 4 If $\boldsymbol{A}$ has the form (2) with $\alpha = \sqrt{ac}$ and $x_h = \cos\frac{(2h-1)\pi}{2n}$, $h = 1, 2, \ldots, n$. Then the i, j entry of $\boldsymbol{A}^l$ $(l \in \mathbb{N})$ is

$$[\boldsymbol{A}^l]_{ij} = \frac{d_{i-j}}{n}\sum_{h=1}^{n}(b + 2\alpha x_h)^l(1 + x_h)V_{i-1}(x_h)V_{j-1}(x_h).$$

Proof : According to Theorem 2, we have

$$\begin{aligned}[\boldsymbol{A}^l]_{ij} &= \sum_{h=1}^{n} d_{i-1}V_{i-1}(x_h)\lambda_h^l q_h d_{j-1}^{-1}V_{j-1}(x_h) \\ &= \frac{d_{i-j}}{n}\sum_{h=1}^{n}(b + 2\alpha x_h)^l(1 + x_h)V_{i-1}(x_h)V_{j-1}(x_h).\end{aligned}$$

The proof is completed. □

Corollary 5 Let $\boldsymbol{A}$ be a tridiagonal matrix of the form (2) with $\alpha = -\sqrt{ac}$ and $x_h = \cos\frac{(2h-1)\pi}{2n}$, $h = 1, 2, \ldots, n$. If $b \neq -2|\alpha|\cos\frac{(2i-1)\pi}{2n}$, $i = 1, 2, \ldots, n$, then l can be taken negative integer in Theorem 3 and

$$[\boldsymbol{A}^{-1}]_{ij} = \frac{d_{i-j}}{n}\sum_{h=1}^{n}\frac{1 - x_h}{b + 2|\alpha|x_h}W_{i-1}(x_h)W_{j-1}(x_h).$$

Moreover, the matrix $\boldsymbol{C}$

$$c_{ij} = \frac{d_{i-j}}{n}\sum_{h=1}^{n}\sqrt{b + 2|\alpha|x_h}(1 - x_h)W_{i-1}(x_h)W_{j-1}(x_h)$$

is a square root of $\boldsymbol{A}$ with $\alpha = -\sqrt{ac}$.

Corollary 6 Let $\boldsymbol{A}$ be a tridiagonal matrix of the form (2) with $\alpha = \sqrt{ac}$ and $x_h = \cos\frac{(2h-1)\pi}{2n}$, $h = 1, 2, \ldots, n$. If $b \neq -2\alpha\cos\frac{(2i-1)\pi}{2n}$, $i = 1, 2, \ldots, n$, then l can be taken negative integer in Theorem 4 and

$$[\boldsymbol{A}^{-1}]_{ij} = \frac{d_{i-j}}{n}\sum_{h=1}^{n}\frac{1 + x_h}{b + 2\alpha x_h}V_{i-1}(x_h)V_{j-1}(x_h).$$

In addition, the matrix $\boldsymbol{D}$

$$d_{ij} = \frac{d_{i-j}}{n}\sum_{h=1}^{n}\sqrt{b + 2\alpha x_h}(1 + x_h)V_{i-1}(x_h)V_{j-1}(x_h)$$

is a square root of $\boldsymbol{A}$ with $\alpha = \sqrt{ac}$.

V. Examples

Example 1 Consider the matrix

$$\boldsymbol{B} = \begin{bmatrix} -1 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 1 \end{bmatrix},$$

it is a special case of $\boldsymbol{A}$ we discussed in this paper. On the grounds of the conclusions in preceding part, we derive the following conclusions:

1) The eigenvalues of $\boldsymbol{B}$ are $\lambda_i = 2\cos\frac{(2i-1)\pi}{2n}$, $i = 1, 2, \ldots, n$. The corresponding eigenvectors are $\boldsymbol{v}^{(i)} = [W_0(x_i), W_1(x_i), \ldots, W_{n-1}(x_i)]^{\mathrm{T}}, i = 1, \ldots, n$, where $x_i = \cos\frac{(2i-1)\pi}{2n}$. Moreover, if n is even, then $\lambda_{n+1-i} = -\lambda_i$, $i = 1, 2, \ldots, \frac{n}{2}$; If n is odd, then $\lambda_{n+1-i} = -\lambda_i$, $i = 1, 2, \ldots, \lfloor\frac{n}{2}\rfloor$ and $\lambda_{\frac{n}{2}+1} = 0$. From this, we deduce that if n is even, then $\boldsymbol{B}$ is invertible and if n is odd, then $\boldsymbol{B}$ is singular.
2) The trace of $\boldsymbol{B}$ is $\mathrm{tr}\boldsymbol{B} = 0$. The determinant of $\boldsymbol{B}$ is $\det\boldsymbol{B} = 2T_n(0) = \prod_{i=1}^{n} 2\cos\frac{(2i-1)\pi}{2n}$. In addition, if n is odd, then $\det\boldsymbol{B} = 0$. If $n \equiv 0 \pmod 4$, then $\det\boldsymbol{B} = 2$; If $n \equiv 2 \pmod 4$, then $\det\boldsymbol{B} = -2$.
3) Let $x_h = \cos\frac{(2h-1)\pi}{2n}$, $h = 1, 2, \ldots, n$. The i, j entry of $\boldsymbol{B}^l$ $(l \in \mathbb{N})$ is

$$[\boldsymbol{B}^l]_{ij} = \frac{1}{n}\sum_{h=1}^{n}(2x_h)^l(1 - x_h)W_{i-1}(x_h)W_{j-1}(x_h).$$

If n is even, then the inverse of $\boldsymbol{B}$ is

$$[\boldsymbol{B}^{-1}]_{ij} = \frac{1}{n}\sum_{h=1}^{n}\frac{1 - x_h}{2x_h}W_{i-1}(x_h)W_{j-1}(x_h).$$

The matrix $\boldsymbol{B_1}$

$$b1_{ij} = \frac{1}{n}\sum_{h=1}^{n}\sqrt{2x_h}(1-x_h)W_{i-1}(x_h)W_{j-1}(x_h)$$

is a square root of $\boldsymbol{B}$.

Proof : We demonstrate that the above result 3) we obtained is equivalent to the conclusion presented in [4]-[5].

Let $\theta_h = \frac{(2h-1)\pi}{2n}$, $\lambda_h = -2\cos\frac{(2h-1)\pi}{2n}$, $x_h = \cos\frac{(2h-1)\pi}{2n}$, $h = 1, 2, \ldots, n$, then $\lambda_h = -2x_h$, $x_h = \cos\theta_h$. Since $\lambda_{n+1-i} = -\lambda_i,\ i = 1, 2, \ldots, \lceil \frac{n}{2} \rceil$ (where $\lceil x \rceil$ denotes the smallest integer larger than or equal to x), we have

$$\begin{aligned}
\boldsymbol{B}^l &= \frac{1}{2n}\sum_{h=1}^{n}(4-\lambda_h^2)\lambda_h^l U_{\frac{2i-3}{2}}\left(\frac{\lambda_h}{2}\right)U_{\frac{2j-3}{2}}\left(\frac{\lambda_h}{2}\right)\\
&= \frac{1}{2n}\sum_{h=n}^{1}(4-\lambda_h^2)\lambda_h^l U_{\frac{2i-3}{2}}\left(\frac{\lambda_h}{2}\right)U_{\frac{2j-3}{2}}\left(\frac{\lambda_h}{2}\right)\\
&= \frac{1}{2n}\sum_{h=1}^{n}(4-4x_h^2)(2x_h)^l U_{\frac{2i-3}{2}}(x_h)U_{\frac{2j-3}{2}}(x_h)\\
&= \frac{1}{n}\sum_{h=1}^{n}(2x_h)^l\left(2-2\cos^2\theta_h\right)\frac{\sin\frac{2i-1}{2}\theta_h}{\sin\theta_h}\frac{\sin\frac{2j-1}{2}\theta_h}{\sin\theta_h}\\
&= \frac{1}{n}\sum_{h=1}^{n}(2x_h)^l\left(1-\cos\theta_h\right)\frac{\sin\frac{2i-1}{2}\theta_h}{\sin\frac{\theta_h}{2}}\frac{\sin\frac{2j-1}{2}\theta_h}{\sin\frac{\theta_h}{2}}\\
&= \frac{1}{n}\sum_{h=1}^{n}(2x_h)^l(1-x_h)W_{i-1}(x_h)W_{j-1}(x_h). \quad \square
\end{aligned}$$

In view of the matrix $\boldsymbol{B}$ in [4]-[5], we consider the matrix $\boldsymbol{C}$ of the similar form with $\boldsymbol{B}$ and give the related facts.

Example 2 Consider the matrix

$$\boldsymbol{C} = \begin{bmatrix} 1 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ & & & & 1 & -1 \end{bmatrix},$$

we derive the following results:

1) The eigenvalues of $\boldsymbol{C}$ are $\lambda_i = 2\cos\frac{(2i-1)\pi}{2n}$, $i = 1, 2, \ldots, n$. The corresponding eigenvectors are $\boldsymbol{v}^{(i)} = [V_0(x_i), V_1(x_i), \ldots, V_{n-1}(x_i)]^{\mathrm{T}}, i = 1, \ldots, n,$ where $x_i = \cos\frac{(2i-1)\pi}{2n}$. Moreover, if n is even, then $\lambda_{n+1-i} = -\lambda_i$, $i = 1, 2, \ldots, \frac{n}{2}$; If n is odd, then $\lambda_{n+1-i} = -\lambda_i$, $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ and $\lambda_{\frac{n}{2}+1} = 0$. From this, we deduce that if n is even, then $\boldsymbol{C}$ is invertible and if n is odd, then $\boldsymbol{C}$ is singular.
2) The trace of $\boldsymbol{C}$ is $\mathrm{tr}\boldsymbol{C} = 0$. The determinant of $\boldsymbol{C}$ is $\det\boldsymbol{C} = 2T_n(0) = \prod_{i=1}^{n} 2\cos\frac{(2i-1)\pi}{2n}$. In addition, if n is odd, then $\det\boldsymbol{C} = 0$. If $n \equiv 0 \pmod 4$, then $\det\boldsymbol{C} = 2$; If $n \equiv 2 \pmod 4$, then $\det\boldsymbol{C} = -2$.
3) Let $x_h = \cos\frac{(2h-1)\pi}{2n}$, $h = 1, 2, \ldots, n$. The i, j entry of $\boldsymbol{C}^l$ $(l \in \mathbb{N})$ is
$$[\boldsymbol{C}^l]_{ij} = \frac{1}{n}\sum_{h=1}^{n}(2x_h)^l(1+x_h)V_{i-1}(x_h)V_{j-1}(x_h). \quad (6)$$
If n is even, then the inverse of $\boldsymbol{C}$ is
$$[\boldsymbol{C}^{-1}]_{ij} = \frac{1}{n}\sum_{h=1}^{n}\frac{1+x_h}{2x_h}V_{i-1}(x_h)V_{j-1}(x_h).$$
The matrix $\boldsymbol{C_1}$
$$c1_{ij} = \frac{1}{n}\sum_{h=1}^{n}\sqrt{2x_h}(1+x_h)V_{i-1}(x_h)V_{j-1}(x_h)$$
is a square root of $\boldsymbol{C}$.

Note that $\boldsymbol{C}$ is similar to $\boldsymbol{B}$ in [4]-[5] by the similarity matrix

$$\widehat{\boldsymbol{I}} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

So the eigenvalues, trace, and determinant of $\boldsymbol{C}$ is equal to those of $\boldsymbol{B}$. Furthermore, we have $\boldsymbol{C}^l = \widehat{\boldsymbol{I}}\boldsymbol{B}^l\widehat{\boldsymbol{I}}$. Another expression of $\boldsymbol{C}^l$ is obtained as follows:

$$[\boldsymbol{C}^l]_{ij} = \frac{1}{n}\sum_{h=1}^{n}(2x_h)^l(1-x_h)W_{n-i}(x_h)W_{n-j}(x_h). \quad (7)$$

Next, we prove that the expressions (6) and (7) are equivalent.

$$\begin{aligned}
[\boldsymbol{C}^l]_{ij} &= \frac{1}{n}\sum_{h=1}^{n}(2x_h)^l(1-x_h)W_{n-i}(x_h)W_{n-j}(x_h)\\
&= \frac{1}{n}\sum_{h=1}^{n}(2x_h)^l(1-\cos\theta_h)\frac{\sin(n-i+1/2)\theta_h}{\sin(\theta_h/2)}\\
&\quad\times\frac{\sin(n-j+1/2)\theta_h}{\sin(\theta_h/2)}\\
&= \frac{1}{n}\sum_{h=1}^{n}(2x_h)^l(1+\cos\theta_h)\frac{\cos(i-1/2)\theta_h}{\cos(\theta_h/2)}\frac{\cos(j-1/2)\theta_h}{\cos(\theta_h/2)}\\
&= \frac{1}{n}\sum_{h=1}^{n}(2x_h)^l(1+x_h)V_{i-1}(x_h)V_{j-1}(x_h).
\end{aligned}$$

Example 3 Consider the homogeneous difference system [3], where the matrix $\boldsymbol{A}$ is given by

$$\begin{bmatrix} 5 & 8 & 0 & 0 & 0 \\ 2 & 1 & 8 & 0 & 0 \\ 0 & 2 & 1 & 8 & 0 \\ 0 & 0 & 2 & 1 & 8 \\ 0 & 0 & 0 & 2 & -3 \end{bmatrix},$$

the general solution is $\boldsymbol{u}(l) = \boldsymbol{A}^l\boldsymbol{c}$, where $\boldsymbol{c}$ is an arbitrary constant vector. In particular, according to Theorem 4, we get

$$\boldsymbol{u}(5) = \begin{bmatrix} 20181.00 & 34920.00 & 44160.00 & 56320.00 & 20480.00 \\ 8730.00 & 13761.00 & 26920.00 & 21120.00 & 35840.00 \\ 2760.00 & 6730.00 & 8001.00 & 21800.00 & -1920.00 \\ 880.00 & 1320.00 & 5450.00 & 2241.00 & 13800.00 \\ 80.00 & 560.00 & -120.00 & 3450.00 & -4179.00 \end{bmatrix}\boldsymbol{c}$$

by using Maple 13 programme.

VI. Conclusion

Being inspired by J. Rimas and J. Gutiérrez-Gutiérrez, we not only generalize their work concerning the positive integer powers of tridiagonal matrices, but also other basic properties including trace, determinant, eigenvalues, eigenvectors and so on. Unfortunately, In this paper, we consider only two kinds of tridiagonal matrices. If possible, we can consider more general tridiagonal matrices.

Appendix

Theorem 3 and Theorem 4 can be executed by Maple 13 programme.

The algorithm of Theorem 3:

```
>restart:
>n:=n:l:=l:a:=a:b:=b:c:=c:
 Al:=array(1..n,1..n):
 x:=cos((2*h-1)*pi/(2*n)):
>for i from 1 by 1 to n do
   for j from 1 by 1 to n do
     Al[i,j]:= evalf(sqrt(a/c)^(i-j)/n
              *(sum((b+2*sqrt(a*c)*x)^l*(1-x)
             *(ChebyshevU(i-1,x)+ChebyshevU(i-2,x))
           *(ChebyshevU(j-1,x)+ChebyshevU(j-2,x)),h=1..n)))
   end do
  end do;
 >print(Al);
```

The algorithm of Theorem 4:

```
>restart:
>n:=n:l:=l:a:=a:b:=b:c:=c:
 Al:=array(1..n,1..n):
 x:=cos((2*h-1)*pi/(2*n)):
>for i from 1 by 1 to n do
   for j from 1 by 1 to n do
     Al[i,j]:= evalf(sqrt(a/c)^(i-j)/n
               *(sum((b+2*sqrt(a*c)*x)^l*(1+x)
              *(ChebyshevU(i-1,x)-ChebyshevU(i-2,x))
           *(ChebyshevU(j-1,x)-ChebyshevU(j-2,x)),h=1..n)))
   end do
  end do;
 >print(Al);
```

where a, b, c are the entries of $\boldsymbol{A}$, n is the order of $\boldsymbol{A}$, l is the power index. The lth powers of $\boldsymbol{A}$ is obtained if we input a, b, c, n and l.

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