# Techniques to create multitime extensions of single-time ODEs and PDEs

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***Abstract*—** The present paper describes some methods to extend classical single-time ODEs and PDEs, creating some multitime versions of these equations. Our approach is justified by examples of possible applications in different fields of science. The first section of paper motivates the interest in this subject. Three original ways to pass from the single-time formulation to several evolution variables are proposed: using some geometrical objects that extend some classical single-time PDEs and then finding families of multitime exact soliton solutions for the obtained multitime geometrical prolongations; creating new versions with more evolution variables for significant ODEs and PDEs, by accepting that the variable "time" may be a function of certain parameters; introducing a multitime $(t^{\alpha})$ instead the single-time variable $t$ and a directional derivative instead the partial derivative with respect to $t$, in first order ODEs. All these methods are applied to some classical modeling equations from physics, biology, economy, ecology. Our original techniques may be useful in modeling, in order to obtain efficient representations of some phenomena including more temporal scales evolving from slow to faster.

***Keywords*—** Keller-Segel model, multitime modeling, multitime sine-cosine-Gordon solitons, telegraph equation, Ueda attractor.

## I. History of Multitime Modeling

IN classical modeling, the spatial parameter was naturally accepted as multidimensional, since the temporal parameter was unidimensional. But recently appeared a new idea: the mathematical models for certain natural phenomena can be formulated by means of multitime evolution PDEs. In recent years, some interesting multitime developments of classical, single-time theories and principles from different fields of mathematical research, were debated in the research group of Udrişte (see [4], [7], [8], [12], [14]-[22]). Our paper develops further these ideas.

The term "multitime" ("multi-temporal") was introduced in physics by Dirac, Fock and Podolsky, in 1932, considering multi-temporal wave-functions described by evolution PDEs. It was assumed in mathematics by Friedman and Littman (1962, 1963). But the multi-temporal wave equations appeared recently in the context of harmonic analysis on Riemannian symmetric spaces. Multi-temporal parabolic equations, named sometimes pluri-parabolic or ultra-parabolic equations, appeared in theory of the Brownian motion (diffusion processes with inertia, see [17]), transport theory (Fokker-Planck equations), biology (dynamics of age-structured populations, [2]), Maxwell waves and equations (see [1]), meteorology, ecology (diffusion and dispersion of impurities in rivers), computer science and other practical applications of mathematical physics, economics [24] and engineering sciences. Some interesting papers have in attention initial boundary value problems for some multi-time equations ([10], [11], [13]) .

The multi-temporal formalism can be used to describe the long time evolution of the soliton solutions ([7], [8], [19]). Important problems that include harmonic maps, minimal submanifolds, deformations, multi-temporal oscillators etc can be formulated by multitime modeling (see [4], [16]). It is a technique of modeling adequate to the case when a dynamical system contains nonlinearities due to the friction, the deterioration, the flaw or to the presence of the constituents consisting of intelligent materials. It is also useful in engineering since it allows to predict material properties or system behavior based on knowledge of the associated geometry. In meteorology, multitime modeling refers to interaction between weather systems of different spatial and temporal scales that produces the weather that we experience finally.

In usual quantum mechanics the wavefunction (for N-particles) depends on 3N spatial variables and one time variable, considering simultaneity for all particles. The idea of the multi-time formalism is to add a separate time-variable for each particle. One then has a multi-time wavefunction, i.e., a wavefunction that depends on 3N spatial and N time variables. The main reason for doing this is to get a Lorentz-invariant object and a corresponding theory ([9], [11]). Using different time-variables for different particles is also justified when the coherence between the amplitudes vanishes.

Another problem where the multitime approach is useful is the simulation more efficiently of the transitory regime of certain electric circuits. One used a PDE system with two temporal variables, one for the fast periodic variations, second for the slow, transitional evolution of the system ([10], [11], [13]). The method was successfully implemented in a simulation program (as we see in the paper [3]).

Some possibilities for new theories of physics in several temporal dimensions are in present investigated. New directions in the work to understand the physical phenomena, the origin and the evolution of the Universe, are suggested. Weinstein [23] is also one of the initiators of this flow in thinking and research.

## II. Multitime Version of PDEs via Geometrical Elements

There is a close connection between PDEs systems and dif-

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ferential geometry.

We'll make the passing from single-time to multitime using ingredients in the differential geometry of the manifold $\mathbb{R}^m$ (derivations, trace), which extend the initial PDEs, using the method from the papers [4], [7], [8], [12], [14]-[22]. For generating multitime PDEs, we recall geometrical objects from the first order jet bundle (metric, connection, vectorial fields, tensorial fields), creating multitime extensions for significant PDEs from geometry or physics. As examples, let us introduce and study some *multitime geometrical prolongations of the sine-cosine-Gordon PDE.*

### A. Multitime sine-cosine-Gordon PDE

The sine-cosine-Gordon equation is related to many physical systems, such as spin chains, one-dimensional superconducting arrays and nonlinear optics (see [6]).

The single-time sine-cosine-Gordon PDE is

$$u_{xx} - u_{tt} - a\cos u - b\sin 2u = 0. \tag{1}$$

Now let us introduce and study a *multitime version of the sine-cosine-Gordon PDE.*

Suppose a *multitime* $t = (t^1, ..., t^m)$ is a point in $\mathbb{R}^m$. We endow the manifold $\mathbb{R}^m$ with a *symmetric linear connection* $\Gamma^\gamma_{\alpha\beta}$ and with a *fundamental symmetric contravariant tensor field* $g = (g^{\alpha\beta})$ of constant signature $(r, z, s),\ r + z + s = m$.

Using a $C^2$ function $u : \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}$, we build the *Hessian operator*

$$(Hess_\Gamma u)_{\alpha\beta} = \frac{\partial^2 u}{\partial t^\alpha \partial t^\beta} - \Gamma^\gamma_{\alpha\beta}\frac{\partial u}{\partial t^\gamma},\ \ \alpha,\beta,\gamma = \overline{1,m},$$

its trace, called *ultra-parabolic-hyperbolic operator*,

$$\Box_{\Gamma,g} u = g^{\alpha\beta}(Hess_\Gamma u)_{\alpha\beta}$$

and define the *multitime sine-cosine-Gordon PDE* as

$$\Box_{\Gamma,g} u = u_{xx} - a\cos u - b\sin 2u,$$

that is

$$g^{\alpha\beta}(t)\left(\frac{\partial^2 u}{\partial t^\alpha \partial t^\beta} - \Gamma^\gamma_{\alpha\beta}(t)\frac{\partial u}{\partial t^\gamma}\right) = u_{xx} - a\cos u - b\sin 2u. \tag{2}$$

**Theorem 1** There exists an infinity of geometrical structures $\Gamma^\gamma_{\alpha\beta}$ and $g^{\alpha\beta}$ on $\mathbb{R}^m$ such that a solution of the sine-cosine-Gordon PDE (1) is also a solution of the multitime sine-cosine-Gordon PDE (2).

**Proof** Let $t^1 = t$ and $u = u(x, t^1)$. Suppose $u = u(x, t^1)$ is a solution of single-time PDE (1). The function $v(x, t^1, ..., t^m) = u(x, t^1)$ is a solution of the multitime PDE (2) if the family of geometrical structures $\Gamma^\gamma_{\alpha\beta}$, $g^{\alpha\beta}$ is fixed by

$$g^{11} = 1,\ \Gamma^\gamma_{11} = 0,\ \gamma = \overline{1,m}.$$

It is obvious that we have an infinity of geometrical structures that satisfy this algebraic equation.

This Theorem justifies the term *multitime geometrical prolongation of the sine-cosine-Gordon PDE.*

Conversely, if we want to obtain a solution of a single-time sine-cosine-Gordon PDE from a solution of the multitime sine-cosine-Gordon PDE, we can use:

- a suitable curve $\tau \to \phi(\tau)$, $t^\alpha = \phi^\alpha(\tau)$, $\alpha = \overline{1,m}$, which imposes some conditions on the coefficients; particularly, we can look for a solution of type $u(x, (\tau, ..., \tau))$
- solutions $u(x, t)$ depending only one variable $t^\alpha$, $\alpha$ = fixed; for example $u = u(x, t^1)$.

Let $\phi : I \subset R \to R$ be a function of class $C^2$. We seek for solutions of the PDE (2) in the form of *multitime solitons*

$$u(x, t) = \phi(x - c_\alpha t^\alpha) = \phi(z), \tag{3}$$

where $c_\alpha$, $\alpha = \overline{1,m}$, is a constant vector and $z = x - c_\alpha t^\alpha$. Substituting the derivatives in the PDE (2), we obtain a second order ODE,

$$(g^{\alpha\beta}c_\alpha c_\beta - 1)\Phi'' + g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}c_\gamma\Phi' + a\cos\Phi + b\sin 2\Phi = 0.$$

For a fixed $g^{\alpha\beta}c_\alpha c_\beta = v \neq 1$ and $\Gamma^\gamma_{\alpha\beta} = 0$, after transformations, we obtain a new form that allows us to find the expression of $\Phi'$ as

$$\Phi' = \pm\frac{1}{\sqrt{1-v}}\sqrt{2b\sin^2\Phi + 2a\sin\Phi - k - b},$$

where $k$ is an arbitrary real constant. In order to find some exact soliton solutions, we consider the particular situation $b = \frac{a\sqrt{2}}{2}$ and we choose for $k$ a convenient value, $k = -a\sqrt{2}$. The ODE becomes

$$\Phi' = \pm\frac{1}{\sqrt{1-v}}\sqrt{a\sqrt{2}(\sqrt{2}\sin\Phi + 1)^2}.$$

It follows that

$$\int \frac{d\Phi}{\sqrt{2}\sin\Phi + 1} = \pm\sqrt{\frac{a\sqrt{2}}{1-v}}z + C,$$

where $C$ is an arbitrary real constant.

By a change of variable, $\mathrm{tg}\frac{\Phi}{2} = t$, we have to calculate $\int \frac{2dt}{2\sqrt{2}t + 1 + t^2}$. Because the denominator is a square equation in variable $t$ with the roots $-\sqrt{2} - 1$ and $-\sqrt{2} + 1$, using the procedure of splitting in simple fractions, we obtain

$$\int \frac{2dt}{2\sqrt{2}t + 1 + t^2} = \int \frac{1}{t + \sqrt{2} - 1} - \int \frac{1}{t + \sqrt{2} + 1},$$

which allows to find the result

$$\ln\left|\frac{t + \sqrt{2} - 1}{t + \sqrt{2} + 1}\right| = \pm\sqrt{\frac{a\sqrt{2}}{1-v}}z + C,\ \ C \in \mathbb{R}.$$

We distinguish two cases:

- If we take

$$\ln\left|\frac{t + \sqrt{2} - 1}{t + \sqrt{2} + 1}\right| = \sqrt{\frac{a\sqrt{2}}{1-v}}z + C,$$

then it folows

$$t = \frac{-2}{1 - K_1\exp\left(\sqrt{\frac{a\sqrt{2}}{1-v}}z\right)} - \sqrt{2} - 1,$$

where $K_1$ is an arbitrary real constant. The reverse of the substitution $\mathrm{tg}\frac{\Phi}{2} = t$ gives the first family of solutions

$$\Phi = 2\mathrm{arctg}\left(\frac{2}{K_1\exp\left(\sqrt{\frac{a\sqrt{2}}{1-v}}z\right) - 1} - \sqrt{2} - 1\right);$$

- If consider

$$\ln\left|\frac{t+\sqrt{2}+1}{t+\sqrt{2}-1}\right| = \sqrt{\frac{a\sqrt{2}}{1-v}}z + C,$$

then we find

$$t = \frac{2}{1 - K_2 \exp\left(\sqrt{\frac{a\sqrt{2}}{1-v}}z\right)} - \sqrt{2} + 1,$$

where $K_2$ is arbitrary in $\mathbb{R}$, and a new family of solutions of the ODE

$$\Phi = 2\text{arctg}\left(\frac{2}{1 - K_2 \exp\left(\sqrt{\frac{a\sqrt{2}}{1-v}}z\right)} - \sqrt{2} + 1\right).$$

In summary, we can formulate the next result:

**Theorem 2** Under foregoing hypothesis, the multitime sine-cosine Gordon PDE (2) has two families of multitime soliton solutions defined respectively by

$$u(x,t) = 2\text{arctg}\left(\frac{2}{K_1 e^{A(x-c_\alpha t^\alpha)} - 1} - \sqrt{2} - 1\right),$$

$$u(x,t) = 2\text{arctg}\left(\frac{2}{1 - K_2 e^{A(x-c_\alpha t^\alpha)}} - \sqrt{2} + 1\right),$$

where

$$A = \sqrt{\frac{a\sqrt{2}}{1 - g^{\alpha\beta}c_\alpha c_\beta}}$$

and $K_1$, $K_2$ are arbitrary real constants.

### *B. Another multitime version of sine-cosine-Gordon PDE*

Let's take again the classical sine-cosine-Gordon PDE

$$u_{xx} - u_{tt} - a\cos u - b\sin 2u = 0.$$

In order to obtain a PDE of polynomial form (preferable for eventually finding some solutions), we'll make a change of unknown function, $u = 2\text{arctg}v$. The initial PDE becomes

$$v_{tt} - v_{xx} - \frac{2v}{1+v^2}v_t^2 + \frac{2v}{1+v^2}v_x^2 + $$

$$+\frac{a(1-v^2)}{2} + 2bv\frac{1-v^2}{1+v^2} = 0. \quad (4)$$

To this second order PDE, we apply the multitime extension via geometrical approach.

Suppose the *multitime* $t = (t^1, ..., t^m) \in \mathbb{R}^m$ is a parameter of evolution. We endow the manifold (jet bundle of order one) $J^1(\mathbb{R}\times\mathbb{R}^m, \mathbb{R}\times\mathbb{R}^m)$ with a *distinguished symmetric linear connection* $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta}\left(x, t, v, \frac{\partial v}{\partial t}\right)$, and with a *distinguished fundamental symmetric contravariant tensor field* $h = \left(h^{\alpha\beta}(x, t, v, \frac{\partial v}{\partial t})\right)$ of constant signature $(r, z, s)$, $r + z + s = m$. Using a $C^2$ function $v : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$, we build the *Hessian operator*

$$(Hess_\Gamma v)_{\alpha\beta} = \frac{\partial^2 v}{\partial t^\alpha \partial t^\beta} - \Gamma^\gamma_{\alpha\beta}\frac{\partial v}{\partial t^\gamma}, \quad \alpha, \beta, \gamma = \overline{1,m}$$

its trace, called *ultra-parabolic-hyperbolic operator*,

$$\Box_{\Gamma,h}v = h^{\alpha\beta}(Hess_\Gamma v)_{\alpha\beta},$$

and a *multitime PDE*,

$$\Box_{\Gamma,h}v - \frac{\partial^2 v}{\partial x^2} + \frac{2v}{1+v^2}v_x^2 + \frac{a}{2}(1-v^2) + 2b\frac{v(1-v^2)}{1+v^2} = 0, \quad (5)$$

where $t = (t^1, ..., t^m) \in \mathbb{R}^m$ and $x \in \mathbb{R}$.

Let $C^\gamma(x, t, \eta, \xi)$, $\gamma = \overline{1,m}$ be a *distinguished vector field*. If we adopt the hypothesis

$$h^{\alpha\beta}(x,t,\eta,\xi)\Gamma^\gamma_{\alpha\beta}(x,t,\eta,\xi)\xi_\gamma = C^\gamma(x,t,\eta,\xi)\frac{\eta\xi_\gamma^2}{1+\eta^2}, \quad (6)$$

then we obtain the a multitime extension of PDE (4):

$$h^{\alpha\beta}(x,t,v,\frac{\partial v}{\partial t^\gamma})\frac{\partial^2 v}{\partial t^\alpha \partial t^\beta} - \frac{C^\gamma(x,t,v,\frac{\partial v}{\partial t^\gamma})v}{1+v^2}\left(\frac{\partial v}{\partial t^\gamma}\right)^2 -$$

$$-\frac{\partial^2 v}{\partial x^2} + \frac{2v}{1+v^2}v_x^2 + \frac{a}{2}(1-v^2) + 2b\frac{v(1-v^2)}{1+v^2} = 0. \quad (7)$$

**Remark** Results similar to Theorem 1 can be formulated for this second extension of the sine-cosine-Gordon equation; we can understand this multitime extension of the sine-cosine-Gordon PDE as a multitime geometrical prolongation of the single-time one.

## III. Multitime Extension of Evolution Laws via "Time-Functions"

The evolution laws that appear in theories from different fields of science have a *single-time formulation*. To give a *multitime version*, we accept that the time $t$ can be a $\mathcal{C}^\infty$ function of certain parameters, i.e. $t = t(s^1, s^2, ..., s^m)$. Thus the simple uni-dimensional variable $t$ from the single-time law becomes in the multitime law a function that we can call *time-function*. The substitution of the derivatives with new expressions found by differentiating composed functions leads to multitime new PDEs or systems of PDEs. If it is necessary, some logistics such as the technique of duality can be used for these equations to become symmetric.

We sall see this technique applied for two examples: a second order ODE known as *Ueda attractor* (see [24]) and a second order PDE associated to a system of two PDEs (second order, but first order in $t$) from biology, named *Keller-Segel equations* (see [2]).

### *A. Multitime extensions of Ueda attractor*

Ueda found a strange attractor, a dynamical system modeled as

$$\ddot{x} + 2\gamma\dot{x} + x^3 = F\cos t,$$

where $\gamma$ and $F$ are constants that determine the chaotic behavior, and the unknown is $x(t)$. We propose two ways to extend this second order ODE at the multi-temporal case. The difference consists in the form of expressions that substitute the partial derivatives. Thus we obtain a multitime PDE or a system of multitime PDEs.

**a)** *The first multitime extension*

Let's consider the variable time $t$ becoming a function of a vector parameter of evolution $s = (s^1, ..., s^m)$, that is $t = t(s^1, s^2, ..., s^m)$, $x = x(t((s^1, s^2, ..., s^m))$

Because

$$\frac{\partial x}{\partial s^\alpha} = \dot{x}\frac{\partial t}{\partial s^\alpha}, \forall\alpha = \overline{1,m},$$

we have by addiction

$$\dot{x} = \frac{\sum_{\alpha=1}^{m} \frac{\partial x}{\partial s^\alpha}}{\sum_{\alpha=1}^{m} \frac{\partial t}{\partial s^\alpha}}.$$

The second order derivatives with respect to the components of the vector $s$ are

$$\frac{\partial^2 x}{\partial s^\alpha \partial s^\beta} = \ddot{x}\frac{\partial t}{\partial s^\alpha}\frac{\partial t}{\partial s^\beta} + \dot{x}\frac{\partial^2 t}{\partial s^\alpha \partial s^\beta} =$$

$$= \ddot{x}\frac{\partial t}{\partial s^\alpha}\frac{\partial t}{\partial s^\beta} + \frac{\sum_{\alpha=1}^{m} \frac{\partial x}{\partial s^\alpha}}{\sum_{\alpha=1}^{m} \frac{\partial t}{\partial s^\alpha}}\frac{\partial^2 t}{\partial s^\alpha \partial s^\beta},\ \forall \alpha, \beta = \overline{1, m},$$

so that, by addiction, it results

$$\ddot{x} = \frac{\sum_{\alpha,\beta=1}^{m} \frac{\partial^2 x}{\partial s^\alpha \partial s^\beta} - \frac{\sum_{\alpha=1}^{m} \frac{\partial x}{\partial s^\alpha}}{\sum_{\alpha=1}^{m} \frac{\partial t}{\partial s^\alpha}} \sum_{\alpha,\beta=1}^{m} \frac{\partial^2 t}{\partial s^\alpha \partial s^\beta}}{\sum_{\alpha,\beta=1}^{m} \frac{\partial t}{\partial s^\alpha}\frac{\partial t}{\partial s^\beta}}.$$

The multitime equation is

$$\sum_{\alpha,\beta=1}^{m}\frac{\partial^2 x}{\partial s^\alpha \partial s^\beta} + \frac{\sum_{\alpha=1}^{m} \frac{\partial x}{\partial s^\alpha}}{\sum_{\alpha=1}^{m} \frac{\partial t}{\partial s^\alpha}}\left(2\gamma\sum_{\alpha,\beta=1}^{m}\frac{\partial t}{\partial s^\alpha}\frac{\partial t}{\partial s^\beta} - \sum_{\alpha,\beta=1}^{m}\frac{\partial^2 t}{\partial s^\alpha \partial s^\beta}\right) =$$

$$= \left(F\cos t(s^1, s^2, ...s^m) - x^3\right)\sum_{\alpha,\beta=1}^{m}\frac{\partial t}{\partial s^\alpha}\frac{\partial t}{\partial s^\beta}. \quad (8)$$

We obtain a multitime second order PDE, quite complicated.

**Remark** To keep the periodicity, we can choose the time-function $t = t(s^1, ..., s^m)$ with a special property: we want to exist certain constants $T^1, ..., T^m > 0$ and $k \in \mathbf{Z}$ so that

$$t(s^1 + T^1, ..., s^m + T^m) = t(s^1, ...s^m) + 2k\pi, \quad (9)$$

for all $(s^1, ..., s^m) \in \mathbb{R}^m$. Thus we have

$$\cos t(s^1 + T^1, ..., s^m + T^m) = \cos t(s^1, ..., s^m),$$

$\forall (s^1, ..., s^m) \in \mathbb{R}^m$.

**b)** *The second multitime extension*

Let's $\alpha, \beta$ be arbitrary, fixed in $\{1, ..., m\}$. Then we have

$$\dot{x} = \frac{\frac{\partial x}{\partial s^\alpha}}{\frac{\partial t}{\partial s^\alpha}} = \frac{\frac{\partial x}{\partial s^\beta}}{\frac{\partial t}{\partial s^\beta}} = \frac{\frac{\partial x}{\partial s^\alpha} + \frac{\partial x}{\partial s^\beta}}{\frac{\partial t}{\partial s^\alpha} + \frac{\partial t}{\partial s^\beta}}.$$

The second derivative will be substituted by

$$\ddot{x} = \left[\frac{\partial^2 x}{\partial s^\alpha \partial s^\beta} - \frac{\frac{\partial x}{\partial s^\alpha} + \frac{\partial x}{\partial s^\beta}}{\frac{\partial t}{\partial s^\alpha} + \frac{\partial t}{\partial s^\beta}}\frac{\partial^2 t}{\partial s^\alpha \partial s^\beta}\right]\frac{\partial t}{\partial s^\alpha}\frac{\partial t}{\partial s^\beta},$$

for all $\alpha, \beta \in \{1, ...m\}$ fixed.

This way to transform the single-time equation doesn't use the addiction of the relations that give the partial derivatives. The initial equation becomes

$$\frac{\partial^2 x}{\partial s^\alpha \partial s^\beta} + \frac{\frac{\partial x}{\partial s^\alpha} + \frac{\partial x}{\partial s^\beta}}{\frac{\partial t}{\partial s^\alpha} + \frac{\partial t}{\partial s^\beta}}\left(2\gamma\frac{\partial t}{\partial s^\alpha}\frac{\partial t}{\partial s^\beta} - \frac{\partial^2 t}{\partial s^\alpha \partial s^\beta}\right) =$$

$$= \left(F\cos t(s^1, ..., s^m) - x^3\right)\frac{\partial t}{\partial s^\alpha}\frac{\partial t}{\partial s^\beta},\ \forall \alpha, \beta = \overline{1, m}. \quad (10)$$

This is a system of multitime second order PDEs, with $m^2$ equations. In order to keep the periodicity, we can make the same choice as in the case of the first extension.

*B. Multitime extensions of Keller-Segel model*

For illustrate how to make the extension for a second order PDE, we take a system that models a process from biology.

In biology and biochemistry, chemotaxis is the phenomenon in which cells move according to the concentration of certain molecules in the environment. They could be attracted by food, or repelled by poisons. If the cells themselves secrete the chemotactic molecules, we can describe the movement of the cells by the Keller-Segel model (see [2]). The simplest case is

$$\begin{cases} u_t = du_{xx} - u_x v_x - uv_{xx} \\ v_t = ev_{xx} + u - av. \end{cases}$$

By replacing the expression of $u = v_t + av - ev_{xx}$ from the second equation of the system into the first one, we transform it into a second order PDE (including second order in $t$),

$$v_{tt} + v_t(a + v_{xx}) + v_{tx}v_x - dv_{txx} - ev_{xxt} =$$
$$= F(v, v_x, ..., v_{xxxx}), \quad (11)$$

where

$$F(v, v_x, ..., v_{xxxx}) =$$
$$= dav_{xx} - dev_{xxxx} - a(v_x)^2 + ev_{xxx}v_x - avv_{xx} + e(v_{xx})^2.$$

We propose to extend the variable $t$ as a function, $t = t(s^1, ..., s^m)$. Thus the function $v$ becomes a multitime function $v(t(s^1, ..., s^m), x)$.

Because

$$v_{s^\alpha} = v_t\frac{\partial t}{\partial s^\alpha}, \forall \alpha = \overline{1, m},$$

we add these equalities and it follows that

$$v_t = \frac{\sum_{\alpha=1}^{m} v_{s^\alpha}}{\sum_{\alpha=1}^{m} \frac{\partial t}{\partial s^\alpha}}, \quad v_{tx} = \frac{\sum_{\alpha=1}^{m} v_{s^\alpha x}}{\sum_{\alpha=1}^{m} \frac{\partial t}{\partial s^\alpha}},$$

$$v_{txx} = \frac{\sum_{\alpha=1}^{m} v_{s^\alpha xx}}{\sum_{\alpha=1}^{m} \frac{\partial t}{\partial s^\alpha}}, \quad v_{xxt} = \frac{\sum_{\alpha=1}^{m} v_{xxs^\alpha}}{\sum_{\alpha=1}^{m} \frac{\partial t}{\partial s^\alpha}},$$

$$v_{tt} = \frac{\sum_{\alpha,\beta=1}^{m} v_{s^\alpha s^\beta} - \frac{\sum_{\alpha=1}^{m} v_{s^\alpha} \sum_{\alpha,\beta=1}^{m} \frac{\partial^2 t}{\partial s^\alpha \partial s^\beta}}{\sum_{\alpha=1}^{m} \frac{\partial t}{\partial s^\alpha}}}{\sum_{\alpha,\beta=1}^{m} \frac{\partial t}{\partial s^\alpha}\frac{\partial t}{\partial s^\beta}}.$$

The equation (11) becomes the multitime PDE

$$\frac{\sum_{\alpha,\beta=1}^{m} v_{s^\alpha s^\beta} - \frac{\sum_{\alpha=1}^{m} v_{s^\alpha}}{\sum_{\alpha=1}^{m} \frac{\partial t}{\partial s^\alpha}} \sum_{\alpha,\beta=1}^{m} \frac{\partial^2 t}{\partial s^\alpha \partial s^\beta}}{\sum_{\alpha,\beta=1}^{m} \frac{\partial t}{\partial s^\alpha}\frac{\partial t}{\partial s^\beta}} +$$

$$+(a+v_{xx})\frac{\sum_{\alpha=1}^{m} v_{s^\alpha}}{\sum_{\alpha=1}^{m} \frac{\partial t}{\partial s^\alpha}} + \frac{\sum_{\alpha=1}^{m} v_{s^\alpha x}}{\sum_{\alpha=1}^{m} \frac{\partial t}{\partial s^\alpha}} v_x -$$

$$-d\frac{\sum_{\alpha=1}^{m} v_{s^\alpha xx}}{\sum_{\alpha=1}^{m} \frac{\partial t}{\partial s^\alpha}} - e\frac{\sum_{\alpha=1}^{m} v_{xxs^\alpha}}{\sum_{\alpha=1}^{m} \frac{\partial t}{\partial s^\alpha}} = F(v, v_x, ..., v_{xxxx}),$$

or, by multiplying with $\sum_{\alpha=1}^{m} \frac{\partial t}{\partial s^\alpha}$,

$$\frac{\sum_{\alpha,\beta=1}^{m} v_{s^\alpha s^\beta} \sum_{\alpha=1}^{m} \frac{\partial t}{\partial s^\alpha} - \sum_{\alpha=1}^{m} v_{s^\alpha} \sum_{\alpha,\beta=1}^{m} \frac{\partial^2 t}{\partial s^\alpha \partial s^\beta}}{\sum_{\alpha,\beta=1}^{m} \frac{\partial t}{\partial s^\alpha}\frac{\partial t}{\partial s^\beta}} +$$

$$+(a+v_{xx})\sum_{\alpha=1}^{m} v_{s^\alpha} + v_x \sum_{\alpha=1}^{m} v_{s^\alpha x} - d\sum_{\alpha=1}^{m} v_{s^\alpha xx} -$$

$$-e\sum_{\alpha=1}^{m} v_{xxs^\alpha} = F(v, v_x, ..., v_{xxxx}) \sum_{\alpha=1}^{m} \frac{\partial t}{\partial s^\alpha}. \tag{12}$$

This new multitime PDE is complicated; to find a solution is not so easy, but our technique applied for many types of PDEs can leads to certain equations with different properties, more or less complicated.

## IV. Multitime PDEs Systems Using Directional Derivative

Consider a system of ODEs in the form

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1(x,y,t) \\ f_2(x,y,t) \end{pmatrix}.$$

If we take $t = t^1$ and introduce some new temporal variables, $t^2, t^3, ..., t^m$, we need a substitute for the derivative with respect to $t$. Let $h = (h_1(t^1, \ldots, t^m), ..., h_m(t^1, \ldots, t^m))$ be a direction in $\mathbb{R}^m$. By substituting $\frac{d}{dt}$ with the directional derivative along $h$, we obtain a multitime PDEs system that extends the initial single-time system. For the paricular direction $h = (1, 0, 0, ..., 0)$, this system coincides with the single-time system.

The choice of $t^i$, $i = \overline{1,m}$ instead of $t$ is arbitrar and it may generally depend on the meaning and the function of each $t^i$ in the application and in the model that one study.

**Example** We consider again the Ueda attractor from [24], a dynamical system modeled as

$$\ddot{x} + 2\gamma\dot{x} + x^3 = F\cos t,$$

where $\gamma$ and $F$ are constants that determine the chaotic behaviour, and the unknown is $x(t)$. In order to transform this second order ODE into a system of first order ODEs, we denote $\dot{x}(t)$ by $y(t)$ and obtain

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -2\gamma y - x^3 + F\cos t \end{pmatrix}.$$

Let an evolution parameter $t = (t^1, \ldots, t^m) \in \mathbb{R}^m$ be, called *multitime*. Then $x$ and $y$ will be functions of $(t^1, \ldots, t^m)$. Consider a direction $h = (h_1(t^1, \ldots, t^m), ..., h_m(t^1, \ldots, t^m))$. By substituting the derivative with respect to $t$ with the directional derivative along this direction, the initial ODE system becomes a multitime PDE system in $(x, y)$,

$$h_i \frac{\partial}{\partial t^i}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -2\gamma y - x^3 + F\cos t^1 \end{pmatrix},$$

or, equivalent,

$$\begin{cases} h_i \dfrac{\partial x}{\partial t^i} = y \\ h_i \dfrac{\partial y}{\partial t^i} = -2\gamma y - x^3 + F\cos t^1, \end{cases}$$

where $i$ is a summation index.

In fact, here we consider $t = t^1$ and introduce some new temporal variables, $t^2, t^3, ..., t^m$.

## V. Multitime Extensions of Telegraph PDE

Partial differential equations are used to model some ecological phenomena and processes related to the movement of multiple species of animals: dispersal, ecological invasions, critical patch size, dispersal-mediated coexistence, diffusion-driven spatial patterning ([2], [5])

Animals do not zig-zag back and forth wildly like molecules. They tend to continue forward in the direction of their existing path. This process is modeled by a PDE named *telegraph equation*:

$$u_t = \frac{V^2}{2\lambda}(u_{xx} + u_{yy}) - \frac{1}{2\lambda}u_{tt}. \tag{13}$$

where $u(x, y, t)$ is the density of organisms with respect to spatial coordinates $x, y$ and time $t$, $\frac{1}{2\lambda}$ is a measure of the correlation between directions of travel from one step to the next and $V$ is he velocity of the organisms.

We propose to create multitime extensions of this second order PDE by all the methods above.

### A. Multitime geometrical prolongations of telegraph PDE

The telegraph PDE can take the form

$$u_{tt} + 2\lambda u_t = V^2(u_{xx} + u_{yy}).$$

Consider a point in $\mathbb{R}^m$ (*multitime*) $t = (t^1, ..., t^m)$. On the manifold $\mathbb{R}^m$, we define a *symmetric linear connection* $\Gamma^\gamma_{\alpha\beta}$ and a *fundamental symmetric contravariant tensor field* $g = (g^{\alpha\beta})$ of constant signature $(r, z, s)$, $r + z + s = m$. For a $C^2$ function $u : \mathbb{R}^2 \times \mathbb{R}^m \to \mathbb{R}$, we construct the *Hessian operator*

$$(Hess_\Gamma u)_{\alpha\beta} = \frac{\partial^2 u}{\partial t^\alpha \partial t^\beta} - \Gamma^\gamma_{\alpha\beta}\frac{\partial u}{\partial t^\gamma}, \quad \alpha, \beta, \gamma = \overline{1,m},$$

its trace, named *ultra-parabolic-hyperbolic operator*,

$$\square_{\Gamma,g} u = g^{\alpha\beta}(Hess_\Gamma u)_{\alpha\beta}$$

and define the *multitime telegraph PDE* as

$$\Box_{\Gamma,g} u = V^2(u_{xx} + u_{yy}),$$

or, equivalent,

$$g^{\alpha\beta}(t)\left(\frac{\partial^2 u}{\partial t^\alpha \partial t^\beta} - \Gamma^\gamma_{\alpha\beta}(t)\frac{\partial u}{\partial t^\gamma}\right) = V^2(u_{xx} + u_{yy}). \quad (14)$$

We can understand this multitime extension of the telegraph PDE as a multitime geometrical prolongation of the single-time one, because Theorem 1 can be also formulated for this equation.

The proof is similar: let $t^1 = t$ and $u = u(x, y, t^1)$ and suppose $u = u(x, y, t^1)$ is a solution of single-time PDE . The function $v(x, y, t^1, ..., t^m) = u(x, y, t^1)$ is a solution of the multitime PDE (14) if the family of geometrical structures $\Gamma^\gamma_{\alpha\beta},\ g^{\alpha\beta}$ is fixed by

$$g^{11} = 1,\ \Gamma^1_{11} = -2\lambda,\ \Gamma^\gamma_{11} = 0,\ \gamma = \overline{2, m}.$$

We have an infinity of geometrical structures that satisfy this algebraic equation.

Conversely, to obtain a solution of the single-time telegraph PDE from a solution of the multitime one, we can use:

- a suitable curve $\tau \to \phi(\tau),\ t^\alpha = \phi^\alpha(\tau),\ \alpha = \overline{1, m}$, which imposes some conditions on the coefficients; particularly, we can look for a solution of type $u(x, y, (\tau, ..., \tau))$
- solutions $u(x, y, t)$ depending only one variable $t^\alpha$, $\alpha$ = fixed; for example $u = u(x, y, t^1)$.

**Remark:** The multitime telegraph PDE has stationary solutions (i.e. independent on temporal variables). Such a solution must satisfy the condition $u_{xx} + u_{yy} = 0$, or, equivalent, $\Delta u = 0$; it means that $u$ is an harmonic function with respect to the spatial variables.

We want to find special solutions of this multitime PDE, in the form of *multitime solitons*

$$u(x, y, t) = \phi(x + ay - c_\alpha t^\alpha) = \phi(z),$$

where $\phi : I \subset R \to R$ is a function of class $C^2$, $a$ is a real constant, $c_\alpha$, $\alpha = \overline{1, m}$, is a constant vector and $z = x + ay - c_\alpha t^\alpha$.

Substituting the derivatives of $u$ in the equation, we obtain a second order ODE,

$$\phi''(z)(g^{\alpha\beta}(t)c_\alpha c_\beta - V^2 - a^2V^2) + g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}(t)c_\gamma\phi'(z) = 0.$$

We choose the vector $c_\alpha$, $\alpha = \overline{1, m}$ such as to fix $g$ and $\Gamma$ in the equalities

$$g^{\alpha\beta}(t)c_\alpha c_\beta = A = const.,\quad g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}(t)c_\gamma = B = const.$$

and the ODE becomes

$$\phi''(z)(A - V^2 - a^2V^2) + B\phi'(z) = 0,$$

that is, after an integration,

$$\phi'(z)(A - V^2 - a^2V^2) + B\phi(z) = 0.$$

This is a linear ODE and its solutions are

$$\phi(z) = C\ \exp\left(-\frac{Bz}{A - V^2 - a^2V^2}\right) + \frac{K}{B},\quad C, K \in \mathbb{R}.$$

**Theorem 3** Under foregoing hypothesis, the multitime telegraph PDE (14) has a family of multitime soliton solutions defined by

$$u(x, y, t) = C\ \exp\left(-\frac{B(x + ay - c_\alpha t^\alpha)}{A - V^2 - a^2V^2}\right) + \frac{K}{B},$$

where

$$A = g^{\alpha\beta}(t)c_\alpha c_\beta, B = g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}(t)c_\gamma$$

and $C$, $K$ are arbitrary real constants.

*B. Multitime telegraph PDEs via "time-functions"*

Using the technique presented in Section III, the temporal variable $t$ becomes a function, $t = t(s^1, s^2, ...s^m)$. Thus the single-time function $u$ becomes a multitime function $u(x, y, t(s^1, s^2, ...s^m))$.

**a)***First multitime extension:*

We add the equalities

$$u_{s^\alpha} = u_t\frac{\partial t}{\partial s^\alpha},\quad \alpha \in \{1, ..., m\}$$

and it follows that

$$u_t = \frac{\sum_{\alpha=1}^m u_{s^\alpha}}{\sum_{\alpha=1}^m \frac{\partial t}{\partial s^\alpha}}.$$

Taking the second derivative and adding again, it follows

$$u_{tt} = \frac{\sum_{\alpha,\beta=1}^m u_{s^\alpha s^\beta} - \frac{\sum_{\alpha=1}^m u_{s^\alpha}}{\sum_{\alpha=1}^m \frac{\partial t}{\partial s^\alpha}}\sum_{\alpha,\beta=1}^m \frac{\partial^2 t}{\partial s^\alpha \partial s^\beta}}{\sum_{\alpha,\beta=1}^m \frac{\partial t}{\partial s^\alpha}\frac{\partial t}{\partial s^\beta}}.$$

The single-time telegraph equation (13) becomes a multitime PDE,

$$\sum_{\alpha=1}^m u_{s^\alpha}\sum_{\alpha,\beta=1}^m \frac{\partial t}{\partial s^\alpha}\frac{\partial t}{\partial s^\beta} = \frac{V^2}{2\lambda}(u_{xx} + u_{yy})\sum_{\alpha,\beta=1}^m \frac{\partial t}{\partial s^\alpha}\frac{\partial t}{\partial s^\beta}\times$$

$$\sum_{\alpha=1}^m \frac{\partial t}{\partial s^\alpha} - \frac{1}{2\lambda}\Big(\sum_{\alpha,\beta=1}^m u_{s^\alpha s^\beta}\sum_{\alpha=1}^m \frac{\partial t}{\partial s^\alpha} - \sum_{\alpha=1}^m u_{s^\alpha}\sum_{\alpha,\beta=1}^m \frac{\partial^2 t}{\partial s^\alpha \partial s^\beta}\Big),$$

that is

$$\sum_{\alpha=1}^m u_{s^\alpha}\Big(\sum_{\alpha,\beta=1}^m \frac{\partial t}{\partial s^\alpha}\frac{\partial t}{\partial s^\beta} - \frac{1}{2\lambda}\sum_{\alpha,\beta=1}^m \frac{\partial^2 t}{\partial s^\alpha \partial s^\beta}\Big) =$$

$$= \frac{1}{2\lambda}\sum_{\alpha=1}^m \frac{\partial t}{\partial s^\alpha}\Big[V^2(u_{xx} + u_{yy})\sum_{\alpha,\beta=1}^m \frac{\partial t}{\partial s^\alpha}\frac{\partial t}{\partial s^\beta} - \sum_{\alpha,\beta=1}^m u_{s^\alpha s^\beta}\Big].$$

**b)***Second multitime extension:*

For each $\alpha, \beta$ arbitrary, fixed in $\{1, ..., m\}$, we have

$$u_t = \frac{\frac{\partial u}{\partial s^\alpha}}{\frac{\partial t}{\partial s^\alpha}} = \frac{\frac{\partial u}{\partial s^\beta}}{\frac{\partial t}{\partial s^\beta}} = \frac{\frac{\partial u}{\partial s^\alpha} + \frac{\partial u}{\partial s^\beta}}{\frac{\partial t}{\partial s^\alpha} + \frac{\partial t}{\partial s^\beta}}.$$

The second derivative becomes

$$u_{tt} = \left[ \frac{\partial^2 u}{\partial s^\alpha \partial s^\beta} - \frac{\dfrac{\partial u}{\partial s^\alpha} + \dfrac{\partial u}{\partial s^\beta}}{\dfrac{\partial t}{\partial s^\alpha} + \dfrac{\partial t}{\partial s^\beta}} \frac{\partial^2 t}{\partial s^\alpha \partial s^\beta} \right] \frac{\partial t}{\partial s^\alpha} \frac{\partial t}{\partial s^\beta},$$

for each $\alpha, \beta \in \{1, ..., m\}$.

Thus the single-time telegraph PDE (13) produces a multitime second order PDEs system with $m^2$ equations

$$\frac{\dfrac{\partial u}{\partial s^\alpha} + \dfrac{\partial u}{\partial s^\beta}}{\dfrac{\partial t}{\partial s^\alpha} + \dfrac{\partial t}{\partial s^\beta}} = \frac{V^2}{2\lambda}(u_{xx} + u_{yy}) - \frac{1}{2\lambda} \frac{\partial t}{\partial s^\alpha} \frac{\partial t}{\partial s^\beta} \times$$

$$\left[ \frac{\partial^2 u}{\partial s^\alpha \partial s^\beta} - \frac{\dfrac{\partial u}{\partial s^\alpha} + \dfrac{\partial u}{\partial s^\beta}}{\dfrac{\partial t}{\partial s^\alpha} + \dfrac{\partial t}{\partial s^\beta}} \frac{\partial^2 t}{\partial s^\alpha \partial s^\beta} \right], \quad \alpha, \beta = \overline{1, m},$$

that is equivalent to

$$\frac{\dfrac{\partial u}{\partial s^\alpha} + \dfrac{\partial u}{\partial s^\beta}}{\dfrac{\partial t}{\partial s^\alpha} + \dfrac{\partial t}{\partial s^\beta}} \Big( 1 - \frac{1}{2\lambda} \frac{\partial t}{\partial s^\alpha} \frac{\partial t}{\partial s^\beta} \frac{\partial^2 t}{\partial s^\alpha \partial s^\beta} \Big) =$$

$$= \frac{1}{2\lambda} \Big[ V^2(u_{xx} + u_{yy}) - \frac{\partial t}{\partial s^\alpha} \frac{\partial t}{\partial s^\beta} \frac{\partial^2 u}{\partial s^\alpha \partial s^\beta} \Big],$$

with $\alpha, \beta = \overline{1, m}$.

*C. Multitime telegraph PDEs system using directional derivative*

In order to transform the second order PDE (13) into a system of first order PDEs, we denote $u_t(x, y, t)$ by $v(x, y, t)$ and it results

$$v_t = u_{tt} = V^2(u_{xx} + u_{yy}) - 2\lambda v(x, y, t).$$

We obtain the PDEs system

$$\frac{\partial}{\partial t} \begin{pmatrix} u(x, y, t) \\ v(x, y, t) \end{pmatrix} = \begin{pmatrix} v(x, y, t) \\ V^2(u_{xx} + u_{yy}) - 2\lambda v(x, y, t) \end{pmatrix}. \tag{15}$$

Consider a multitime $t = (t^1, \dots, t^m) \in \mathbb{R}^m$ and a direction $h = (h^1(t^1, \dots, t^m), ..., h^m(t^1, \dots, t^m))$. The unknowns $u$ and $v$ will be functions of $(x, y, t^1, \dots, t^m)$.

We refer to the derivative with respect to $t$. By substituting this derivative with the directional derivative along the direction $h$, the single-time PDEs system (15) becomes a multitime PDE system in $(u, v)$,

$$\begin{cases} h^i \dfrac{\partial u}{\partial t^i} = v \\ h^i \dfrac{\partial v}{\partial t^i} = V^2(u_{xx} + u_{yy}) - 2\lambda v, \end{cases} \tag{16}$$

where $i$ is a summation index.

For $h = (1, 0, \dots, 0)$, the multitime system coincides with the single-time system. Thus we understand the new multitime system (16) as a more general form of (15).

## VI. Conclusions

The specialists [1]-[24] consider that in certain cases it is more efficient to represent the evolution by a function in two evolution variables $(t^1, t^2)$, like a surface, than a curve (trajectory), i.e., by a function of one variable $t$.

The multitime formalism has the main advantage that the mathematical information is reach and the thinking is easier. That is why, the techniques for creating multitime PDEs is useful in mathematical modeling and graphical representations of the evolution of certain processes, and also in signal theory.

The single-time ODEs and PDEs arise in the mathematical modeling of many physical, chemical and biological phenomena and many diverse subject areas such as fluid dynamics, electromagnetism, material science, astrophysics, economy, financial modeling etc. To reinforce some ideas we can change them into multi-temporal models which are sometimes more convenient in understanding some evolutions.

A new understanding of multi-dimensional consistency has been a major breakthrough of the papers [1]-[4], [7], [8], [10]-[22]. Our aim is to continue these ideas, extending the known results by extensive applications of multitime prolongations.

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