Research of processes in continuous systems with multiple eigenvalues of state matrix

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***Abstract*—**The steady continuous system with state matrix has real spectrum of the multiple eigenvalues which multiplicity is equal to dimension of its state vector is considered. There is shown that if the eigenvalues modulus is less than unit, in free transient motion of system on norm of state vector the oscillativity which become apparent by initial overshoot, being replaced by monotonous movement to quiescent state is found. It is established that the size of overshoot is more, than it is less the modulus of eigenvalue and more its multiplicity. Later in this article the steady continuous system which state matrix has spectrum of the multiple complex conjugated eigenvalues which multiplicity is equal to a half dimension of its state vector is considered. The special attention is paid on the situation when the modulus of real part of eigenvalue is less than unit. Is established that in this case at a small oscillativity of eigenvalues there is a noticeable overshoot in processes on norm of free transient motion on state vector and the size of overshoot more, than more its multiplicity and imaginary part and then the real component of eigenvalue and is less on the modulus.

***Keywords*—** real eigenvalue, complex conjugate eigenvalue, multiplicity, free transient motion, norm, overshoot.

I. Introduction

The task is to research the influence of the multiplicity and the absolute value of the eigenvalues of the state matrix on the free motion steady continuous multidimensional linear dynamic system in the norm of the state vector. It is assumed that the multiplicity of the eigenvalue equal to the dimension of the state vector. As will be shown, we have to state systemic phenomenon, which consists in the fact that in the aperiodic system at a multiplicity of eigenvalues greater than one, and absolute value of the eigenvalues less than one there is the possibility of appreciable overshoot of the norm of the state vector in free motion. Found that the value of overshoot increases with the decrease in the absolute value of the eigenvalues and with increase their multiplicity. Moreover, there is an opportunity to "exchange" absolute value of the eigenvalues for their multiplicity in the class of systems with fixed value of overshoot. At first the problem is solved for the case of a state matrix representation in the Jordan canonical form [1], and then research activities are carried on an arbitrary case.

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II. Analytical Researches. Case of Multiple Real Eigenvalues

Consider the linear continuous Hurwitz multidimensional dynamic system given by [2,3] in the vector - matrix form

$$\dot{x}(t) = Fx(t), x(t)\big|_{t=0} = x(0), \tag{1}$$

where $x(0), x(t)$ is vector of initial and current states of the system respectively; F is its state matrix; $x(0), x(t) \in R^n; F \in R^{n\times n}$. Matrix F of system (1) given in a random basis, such that it has the following characteristic polynomial $D(\lambda)$ representation

$$D(\lambda) = det(\lambda I - F) = \\ = \left\{ (\lambda - \alpha)^n = \lambda^n + \sum_{i=1}^{n} (-1)^i C_n^i \alpha^i \lambda^{n-i}; \alpha : Jm(\alpha) = 0 \right\} \tag{2}$$

Eigenvalues spectrum of matrix F:

$$\sigma\{F\} = \left\{\lambda_i = \arg[det(\lambda I - F) = 0] : \lambda_i = \alpha; i = \overline{1,n}\right\} \tag{3}$$

Defect of characteristic matrix of matrix [1] F:

$$def(\lambda I - F) = 1 \tag{4}$$

From (3), (4) it follows [1] that canonical form of matrix is $(n \times n)$-Jordan block $J(\alpha)$. It's represented in the following form

$$J(\alpha) = \begin{bmatrix} \alpha & 1 & 0 & \dots & 0 \\ 0 & \alpha & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \alpha \end{bmatrix}. \tag{5}$$

Matrix in Jordan form $J(\alpha)$ generates an autonomous dynamical system of the type (1), defined in the Jordan canonical basis

$$\dot{\tilde{x}}(t) = J(\alpha)\tilde{x}(t), \tilde{x}(t)\big|_{t=0} = \tilde{x}(0). \tag{6}$$

Vector $\tilde{x}$ and matrix $J(\alpha)$ are related with vector x and matrix F by the following vector–matrix ratios

$$x = S\tilde{x},\ SJ(\alpha) = FS \tag{7}$$

where $S - (n \times n)$ is the non-singular similarity transformation matrix , allowing representation of the matrix in the form of

$$F = SJ(\alpha)S^{-1}. \tag{8}$$

In turn, Jordan matrix $J(\alpha)$ (5) can be decomposed in a following additive form

$$J(\alpha) = diag\{\lambda_i = \alpha; i = \overline{1,n}\} + J(0) = \alpha I + J(0) \tag{9}$$

where $J(0)$ is nilpotent matrix [1] with the index $\nu = n$.

Now, the task is to the research of the free motion of the system (6) in its state vector in a scalar form. The solution of the system (6) is [1,2,3]

$$\tilde{x}(t) = \tilde{x}(t, \tilde{x}(0)) = exp\{J(\alpha)t\}\tilde{x}(0). \tag{10}$$

We will do scalarization of vector process (10), based on the use of consistent [1] vector and matrix norms. As a result, on the basis of (9), we obtain the sequence of ratios

$$\begin{aligned} \|\tilde{x}(t)\| &= \|exp\{J(\alpha)t\}\tilde{x}(0)\| \le \|exp\{J(\alpha)t\}\| \cdot \|\tilde{x}(0)\| = \\ &= e^{\alpha t}\|exp\{J(0)t\}\| \cdot \|\tilde{x}(0)\|, \end{aligned} \tag{11}$$

where

$$\begin{aligned} exp\{J(0)t\} &= \exp\left\{\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} t\right\} = \\ &= \begin{bmatrix} 1 & t & (2)^{-1}t^2 & \dots & [(\mu-1)!]^{-1}t^{\mu-1} \\ 0 & 1 & t & \dots & [(\mu-2)!]^{-1}t^{\mu-2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & t \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \end{aligned} \tag{12}$$

From (12) it follows that the column norm $\|exp\{J(0)t\}\|_1$, the row norm $\|exp\{J(0)t\}\|_\infty$ and the spectral norm $\|exp\{J(0)t\}\|_2$ are the same. And they are defined by the following majorizing inequality

$$\begin{aligned} \|exp\{J(0)t\}\|_p &= 1 + t + (1/2)t^2 + \dots + (1/(\mu\text{ -}1)!)t^{\mu-1} = \\ &= \sum_{k=0}^{\mu-1}(1/k!)t^k, (p = 1,2,\infty). \end{aligned} \tag{13}$$

Thus, norm of the matrix exponential satisfy the following relation:

$$\|exp\{J(\alpha)t\}\| = e^{\alpha t}\sum_{k=0}^{\mu-1}(1/(k!))t^k. \tag{14}$$

From (11) and (14) it follows:

$$\|\tilde{x}(t)\| = \|exp\{J(\alpha)t\}\| \cdot \|\tilde{x}(0)\|_{\|\tilde{x}(0)\|=1} = e^{\alpha t}\sum_{k=0}^{\mu-1}(1/(k!))t^k. \tag{15}$$

Now, the task is to evaluation sign of the velocity change of norm $\|\tilde{x}(t)\|$ at time $t = 0$. We are fixing multiplicity $\mu = n$ eigenvalue $\lambda = \alpha$. We differentiate equation (15) on time:

$$\begin{aligned} \frac{d}{dt}\|\tilde{x}(t)\| &= \frac{d}{dt}\left\{e^{\alpha t}\sum_{k=0}^{\mu-1}(1/(k!))t^k\right\}\Bigg|_{t=0} = \\ &= \left\{\alpha e^{\alpha t}\sum_{k=0}^{\mu-1}(1/(k!))t^k + e^{\alpha t}\sum_{k=0}^{\mu-2}(1/(k!))t^k\right\}\Bigg|_{t=0} = \alpha + 1. \end{aligned} \tag{16}$$

Relation (16) allows us to separate the processes by their quality in the system (6) as a function of multiplicity eigenvalue $\lambda = \alpha$. Clear is that processes in the system (6) are convergent for any negative value $\lambda = \alpha$ and any multiplicity, because the multiplicative term $e^{\alpha t}$ in (16) for $\|\tilde{x}(t)\|$ has an infinite number of elements of the expansion in powers of t, and the term $\sum_{k=0}^{\mu-1}(1/(k!))t^k$ has a finite number elements.

Consequently, there is always a moment of time $t = t^*$, at which the dominance of the exponential multiplier $e^{\alpha t}$ begins to emerge. Now, we consider the following situations.

Situation 1: $\alpha < 0, |\alpha| > 1$, $\left\{\frac{d}{dt}\|\tilde{x}(t)\|\right\}\Big|_{t=0} < 0$, process $\|\tilde{x}(t)\|$ converges to zero, and is majorized by an exponent in the form $\|\tilde{x}(t)\| \le e^{(\alpha+1)t}\|\tilde{x}(0)\|$.

Situation 2: $\alpha = -1, \left\{\frac{d}{dt}\|\tilde{x}(t)\|\right\}\Big|_{t=0} = 0$, the initial velocity is zero, but at $t > 0$ by (16) is set to a negative velocity. It is defined by the following relations

$$\begin{aligned} \frac{d}{dt}\|\tilde{x}(t)\| &= \left\{\alpha e^{\alpha t}\sum_{k=0}^{\mu-1}(1/(k!))t^k + e^{\alpha t}\sum_{k=0}^{\mu-2}(1/(k!))t^k\right\}\Bigg|_{\alpha=-1} =. \\ &= -(1/(\mu-1)!)e^{-t}t^{(\mu-1)} \end{aligned} \tag{17}$$

The velocity of change norm $\|\tilde{x}(t)\|$ on the system trajectories is characterized by the extremum, which is observed at time t_m. It is defined by (15) the following relations

$$\begin{aligned} t_m &= arg\left\{\frac{d^2}{dt^2}\|\tilde{x}(t)\| = 0\right\} = \\ &= arg\left\{\frac{d}{dt}\left(e^{-t}t^{(\mu-1)}\right) = 0\right\} = \mu - 1 \end{aligned} \tag{18}$$

And the Velocity of change norm $\|\tilde{x}(t)\|$ is determined by the relation

$$\max_t\left(\frac{d}{dt}\|\tilde{x}(t)\|\right) = -\frac{(\mu-1)^{(\mu-1)}}{(\mu-1)!}e^{-(\mu-1)}. \tag{19}$$

The process $\|\tilde{x}(t)\|$ converges to zero by (15). The process is majorized by an exponential function so that the following inequality

$$\|\tilde{x}(t)\| \le \rho e^{\gamma t}\|\tilde{x}(0)\| \tag{20}$$

where

$$(\rho,\gamma) = = arg\left\{\begin{aligned} &\min_{\rho,\gamma}\left\|\|\tilde{x}(t)\| - \rho e^{\gamma t}\|\tilde{x}(0)\|\right\| \& \left(\frac{d}{dt}\left(\rho e^{\gamma t}\|\tilde{x}(0)\|\right)\right)\Bigg|_{t=(\mu-1)} = \\ &= -\frac{(\mu\text{ -}1)^{(\mu-1)}}{(\mu\text{ -}1)!}e^{-(\mu-1)} \,\&\, \rho \ge 1 \end{aligned}\right\}.$$

Situation 3 (the subject of the paper): $\alpha < 0, |\alpha| < 1$,

$\left\{\frac{d}{dt}\|\tilde{x}(t)\|\right\}\Big|_{t=0} > 0$. The process $\|\tilde{x}(t)\|$ at the initial interval time diverges, reaching a maximum at the time t_M. It is defined by the following relations

$$t_M = arg\left\{\frac{d}{dt}\|\tilde{x}(t)\| = 0\right\} = \\ arg\left\{(1+\alpha)\sum_{k=0}^{\mu-2}(1/k!)t^k + \alpha(1/(\mu-1)!)t^{(\mu-1)} = 0\right\} \quad (21)$$

And further process $\|\tilde{x}(t)\|$ converges to zero. Thus, the process $\|\tilde{x}(t)\|$ on the trajectories of free motion of aperiodic system detects overshoot. It is numerically determined by value $\alpha : (\alpha < 0, |\alpha| < 1)$ of the multiple eigenvalue and value μ of its multiplicity. The obvious property of process $\|\tilde{x}(t)\|$: the smaller the value $|\alpha| < 1$ and the more its multiplicity μ, the greater the value of its overshoot over the level $\|\tilde{x}(0)\|$. To illustrate this result, we perform the calculation of the time t_M by (21) and overshoot to the curve $\|\tilde{x}(t)\|$ aperiodic systems for time $t = t_M$ by (15) for different values of $\alpha : (\alpha < 0, |\alpha| < 1)$ and multiplicity μ. The calculation results are shown in Tables 1 and 2.

μ	2	3	4	5	10
α	t_M				
-0.2	4	8.9	13.9	18.8	43.8
-0.02	49	99	149	199	449

Table1.Values of moments overshoot to curve $\|\tilde{x}(t)\|$

μ	2	3	4	5	10
α	$\max_t(\|\tilde{x}(t)\|) = \|\tilde{x}(t_M)\|$				
-0.2	2.25	8.3	34.7	151.6	$3.32*10^5$
-0.02	18.8	690	$2.86*10^4$	$1.25*10^6$	$2.72*10^{14}$

Table 2. Values of overshoot $\max_t(\|\tilde{x}(t)\|) = \|\tilde{x}(t_M)\|$ to curve $\|\tilde{x}(t)\|$

Now, we return to the original system (1) with the state matrix F, defined in an arbitrary basis. Then, by analogy with (10), using (8) we can write the following

$$x(t) = x(t, x(0)) = \exp\{Ft\}x(0) = \\ = S\exp\{J(\alpha)t\}S^{-1}x(0). \quad (22)$$

If in (22) we will proceed to scalarized vector processes in the state vector norm of system (1), we obtain using (14), the following sequence of relations

$$\|x(t)\| = \left\|S\exp\{J(\alpha)t\}S^{-1}x(0)\right\| \le \\ \le \|S\|\|\exp\{J(\alpha)t\}\|\left\|S^{-1}\right\|\|x(0)\| = \quad , \quad (23) \\ = c\{S\}e^{\alpha t}\sum_{k=0}^{\mu-1}(1/(k!))t^k\|x(0)\|$$

where $c\{S\} = \|S\| \cdot \|S^{-1}\|$ is condition number of the matrix S, $1 \le c\{S\} < \infty$ [4].

The values of $\|x(t)\|$ will be $c\{S\}$ times greater than value of $\|\tilde{x}(t)\|$, keeping the same dependence on absolute value $|\alpha|$ of the eigenvalue $\lambda = \alpha$ and its multiplicity μ.

III. Computer Simulations. Case of Multiple Real Eigenvalues

Large Computer simulation the processes in the norm $\|\tilde{x}(t)\|$ as a function of eigenvalue $\lambda = \alpha$ and its multiplicity $\mu = n$ was conducted in accordance with the ratio of $\|\tilde{x}(t)\| = \|exp\{J(\alpha)t\}\tilde{x}(0)\| \le \|exp\{J(\alpha)t\}\| \cdot \|\tilde{x}(0)\|$ on his majorizing part. This research was carried out in Matlab. The results of the simulation of processes in the form of $\|\tilde{x}(t)\|$ for a single set of multiplicity $\mu = n = 2;3;5;10$ and values $\lambda = \alpha = -2; -0.2; -0.02$ are presented in the figures below.

Figure 1 shows the curves for the case $\lambda = \alpha = -2$. Processes $\|\tilde{x}(t)\|$ converge monotonically with no overshoots (look situation 1). The left curve corresponds to the case $\mu = n = 2$, and the right curve corresponds to the case $\mu = n = 10$.

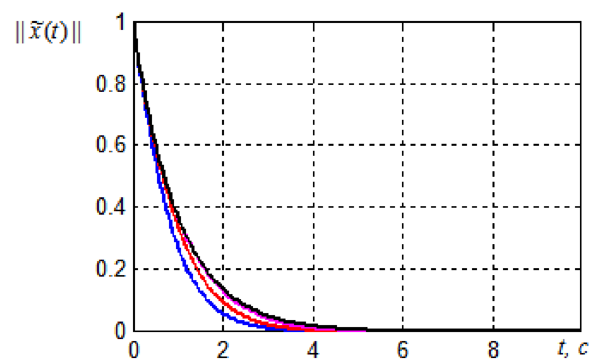

Fig.1. Curves of processes $\|\tilde{x}(t)\|$ for $\lambda = \alpha = -2;$ and $\mu = n = 2;3;5;10$

Figure 2 shows the curves for the case $\lambda = \alpha = -0.2$. Processes $\|\tilde{x}(t)\|$ detect overshoots that increase with increasing $\mu = n$ (look situation 3).

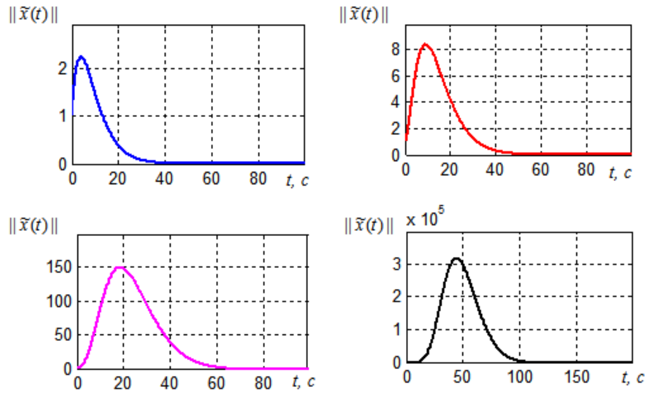

Fig.2. Curves of processes $\|\tilde{x}(t)\|$ for $\lambda = \alpha = -0.2;$ and $\mu = n = 2;3;5;10$

Figure 3 shows the curves for the case $\lambda = \alpha = -0.02$. Processes $\|\tilde{x}(t)\|$ detect the notable overshoots that increase with increasing $\mu = n$ (look situation 3).

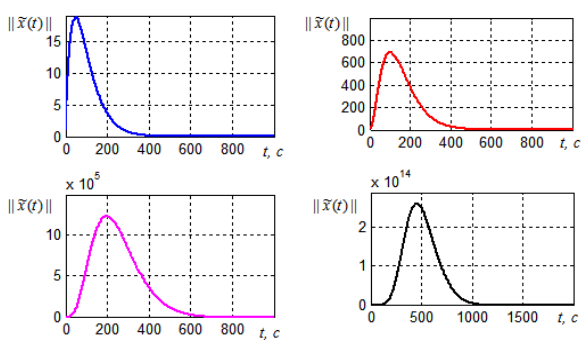

Fig.3. Curves of processes $\|\tilde{x}(t)\|$ for $\lambda = \alpha = -0.02;$ and $\mu = n = 2;3;5;10$

Fig.4 shows the curves of constant values of $\max\limits_t \left(\|\tilde{x}(t)\|\right) = \|\tilde{x}(t_M)\| = \text{const}$ in the plane «$\mu - \lambda$». They illustrate the possibility of «exchange» multiplicity to the value of a multiple eigenvalue of the task at hand.

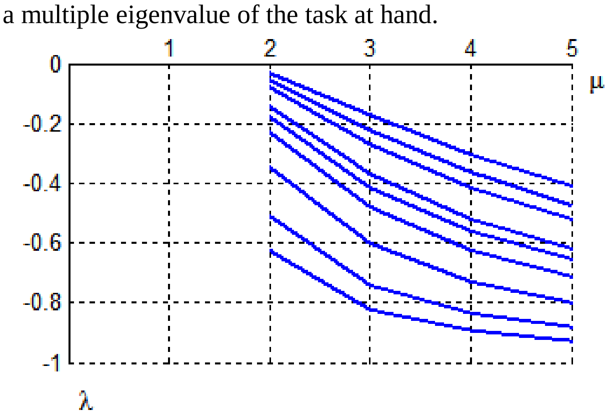

Fig.4. Curves of constant values of $\max\limits_t \left(\|\tilde{x}(t)\|\right) = \|\tilde{x}(t_M)\| = \text{const}$

Figure 5 shows the curves for $\lambda = \alpha = -0.2$ for the case of system (1) with the matrix F is specified in the accompanying row form (frobenius form), and for the case of system (6). Processes $\|x(t)\|$ correspond with the curves $\|\tilde{x}(t)\|$, but each time $\|x(t)\|$ exceed $\|\tilde{x}(t)\|$ in $c\{S\}$ times.

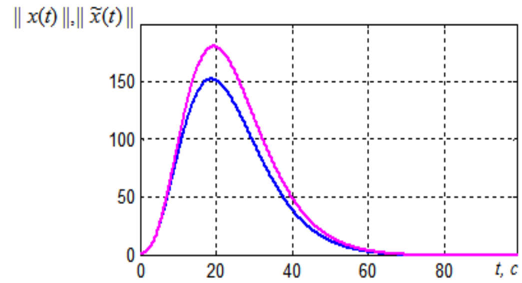

Fig.5. Curves of processes $\|x(t)\|$ (top) and $\|\tilde{x}(t)\|$ for $\lambda = \alpha = -0.2;$ and $\mu = n = 5$

Finally, it should be noted that if the spectrum of eigenvalues of the matrix F has several multiples eigenvalues $\sigma\{F\} = \left\{\lambda_i = \alpha_j : i = \overline{1,\mu_j};\ j = \overline{1,q};\ \sum_{j=1}^{q} \mu_j = n\right\}$, the canonical representation F in the Jordan form will contain q Jordan blocks of $(\mu_j \times \mu_j)$-dimension each. Then for this case, relation (11) takes the following form

$$\|\tilde{x}(t)\| = \left\|diag\left\{exp\{J(\alpha_j)t\};\ j = \overline{1,q}\right\}\tilde{x}(0)\right\| \leq$$
$$\leq e^{\overline{\alpha}t}\left\|exp\{J_{(\overline{\mu}\times\overline{\mu})}(0)t\}\right\| \cdot \|\tilde{x}(0)\|,$$

where

$$\overline{\alpha} = \max_j\left\{\alpha_j : \alpha_j < 0\ \&\ |\alpha_j| < 1;\ j = \overline{1,q}\right\},\ \overline{\mu} =$$
$$= \max_j\left\{\mu_j;\ j = \overline{1,q}\right\}$$

IV. Analytical Researches. Case of Multiple Complex Conjugate Eigenvalues

The Consider the linear continuous Hurwitz multidimensional dynamic system given by [2,3] in the vector - matrix form

$$\dot{x}(t) = Fx(t), x(t)\big|_{t=0} = x(0), \tag{24}$$

where $x(0), x(t)$ is vector of initial and current states of the system respectively; F is its state matrix; $x(0), x(t) \in R^n; F \in R^{n\times n}$. Matrix F of system (24) given in a random basis, such that it has the following eigenvalues spectrum $\sigma\{F\}$ representation

$$\sigma\{F\} = \{\lambda_{2i-1;2i} = \arg[det(\lambda I - F) = 0] : \lambda_{2i-1;2i} = = \alpha \pm j\beta; i = \overline{1, n/2}\}. \tag{25}$$

From (25) implies that the matrix F has a single pair of complex conjugate eigenvalues of multiplicity $\mu = n/2$, where $n = \dim(x)$.

Defect of characteristic matrix of matrix [1] *F:*

$$def\,(\lambda I - F) = 2 \tag{26}$$

From (25), (26) it follows [1] that canonical form of matrix is $(n \times n)$- "pseudo Jordan" block $\tilde{J}(\alpha, \beta)$.

When designing "pseudo Jordan" block $\tilde{J}(\alpha,\beta)$ require that the following conditions

$$\lim_{\beta \to 0}\left\{\tilde{J}(\alpha,\beta)\right\} = J(\alpha). \tag{27}$$

To construct "pseudo Jordan" block $\tilde{J}(\alpha,\beta)$ use structural view of the system

$$\dot{\tilde{x}}(t) = J(\alpha)\tilde{x}(t), \ \tilde{x}(t)\big|_{t=0} = \tilde{x}(0), \tag{28}$$

which is shown in figure 6.

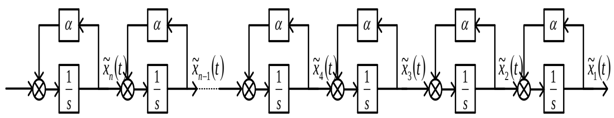

Fig.6. Block diagram of system (28)

Clear that conditions (25) and (27) satisfies the following system

$$\dot{\tilde{x}}(t) = \tilde{J}(\alpha,\beta)\tilde{x}(t), \ \tilde{x}(t)\big|_{t=0} = \tilde{x}(0). \tag{29}$$

Structural representation of the system (28) is obtained from the block diagram in fig. 6. There covered a pair of integrators feedback transmission factor of "$-\beta^2$", so that it turns the diagram shown in fig. 7.

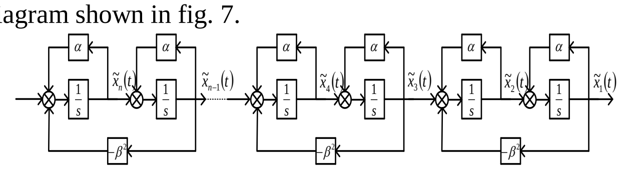

Fig.7. Block diagram of system (29)

With structural realization (fig. 7) of the system (29) can be "written off" the matrix $\tilde{J}(\alpha,\beta)$, which gets the following representation

$$\tilde{J}(\alpha,\beta) = \begin{bmatrix} \alpha & 1 & 0 & 0 & \dots & 0 & 0 \\ -\beta^2 & \alpha & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \alpha & 1 & \dots & 0 & 0 \\ 0 & 0 & -\beta^2 & \alpha & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \alpha & 1 \\ 0 & 0 & 0 & 0 & \dots & -\beta^2 & \alpha \end{bmatrix}. \tag{30}$$

In turn, "pseudo Jordan" matrix $\tilde{J}(\alpha,\beta)$ (30) can be decomposed in a following additive form

$$\tilde{J}(\alpha,\beta) = diag\left\{\lambda_i = \alpha; i = \overline{1,n}\right\} + \\ + \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -\beta^2 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & -\beta^2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & -\beta^2 & 0 \end{bmatrix} = \tag{31}$$

$$\alpha I + \tilde{J}(0,\beta).$$

Now, the task is to the research of the free motion of the system (29) in its state vector in a scalar form. The solution of the system (29) is

$$\tilde{x}(t) = \tilde{x}(t, \tilde{x}(0)) = \\ = exp\left\{\tilde{J}(\alpha,\beta)\ t\right\}\tilde{x}(0) = e^{\alpha t} exp\left\{\tilde{J}(0,\beta)\ t\right\}\tilde{x}(0). \tag{32}$$

We will do scalarization of vector process (32), based on the use of consistent vector and matrix norms. As a result, on the basis of (31), we obtain the sequence of ratios

$$\left\|\tilde{x}(t)\right\| = \left\|e^{\alpha t} exp\left\{\tilde{J}(0,\beta)t\right\}\tilde{x}(0)\right\| = \\ = e^{\alpha t}\left\|exp\left\{\tilde{J}(0,\beta)t\right\}\tilde{x}(0)\right\| \le \\ \le e^{\alpha t}\left\|exp\left\{\tilde{J}(0,\beta)t\right\}\right\| \cdot \left\|\tilde{x}(0)\right\|. \tag{33}$$

Note that in the case of multiple real eigenvalues matrix exponential $exp\{J(0)t\}$ has a clear algorithmic basis for the formation of its representation. And in the case of multiple complex conjugate eigenvalues matrix exponential $exp\left\{\tilde{J}(0,\beta)t\right\}$ does not have this property. Therefore further compute the matrix exponential $exp\left\{\tilde{J}(0,\beta)t\right\}$ for a representative system situation characterized by $n = 6$. $\mu = n/2 = 3$. As a result, we obtain a chain of equations based on the calculation of the inverse Laplace transform of resolvent

$$exp\left\{\tilde{J}(0,\beta)t\right\} = L^{-1}\left\{\left(sI - \tilde{J}(0,\beta)\right)^{-1}\right\} =$$

$$= row\left\{\begin{matrix} [\cos\beta t, -\beta\sin\beta t, 0,0,0,0]^T, [\beta^{-1}\sin\beta t, \cos\beta t, 0,0,0,0]^T, [(2\beta)^{-1} t\sin\beta t, (2\beta)^{-1}(\sin\beta t + \beta t\cos\beta t), \\ \cos\beta t, -\beta\sin\beta t, 0,0]^T, [(2\beta^3)^{-1}(\sin\beta t - \beta t\cos\beta t), (2\beta)^{-1} t\sin\beta t, \beta^{-1}\sin\beta t, \cos\beta t, 0, 0]^T, \\ [(8\beta^3)^{-1} t\ (\sin\beta t - \beta t\cos\beta t), (8\beta^3)^{-1}((1+(\beta t)^2)\sin\beta t - \beta t\cos\beta t), (2\beta)^{-1} t\sin\beta t, \\ (2\beta)^{-1}(\sin\beta t + \beta t\cos\beta t), \cos\beta t, -\beta\sin\beta t]^T, [(8\beta^5)^{-1}((3-(\beta t)^2)\sin\beta t - 3\beta t\cos\beta t), \\ (8\beta^3)^{-1} t(\sin\beta t - \beta t\cos\beta t), (2\beta^3)^{-1}(\sin\beta t - \beta t\cos\beta t), (2\beta)^{-1} t\sin\beta t, \beta^{-1}\sin\beta t, \cos\beta t]^T \end{matrix}\right\}. \tag{34}$$

From (34) it follows that the column norm $\left\|exp\left\{\tilde{J}(0,\beta)t\right\}\right\|_1$, the row norm $\left\|exp\left\{\tilde{J}(0,\beta)t\right\}\right\|_\infty$ and the spectral norm $\left\|exp\{J(0)t\}\right\|_2$ are the same. And they are defined by the following majorizing inequality

$$v(\alpha,\beta,t) = \\ = \left[\frac{\left(3-(\beta t)^2\right)\sin\beta t - 3\beta t\cos\beta t}{8\beta^5}, \frac{t\left(\sin\beta t - \beta t\cos\beta t\right)}{8\beta^3}, \right. \\ \left. \frac{\sin\beta t - \beta t\cos\beta t}{2\beta^3}, \frac{t\sin\beta t}{2\beta}, \frac{\sin\beta t}{\beta}, \cos\beta t\right]^T. \tag{35}$$

It should be noted that condition (27) holds for the norm of the matrix exponential in the form of norm of the vector (35). Indeed, under the $\beta \to 0$ using the limit $\lim_{\beta \to 0}(\sin\beta/\beta) = 1$ and l'Hopital rule we prove the following limits of convergence:

1. $\lim_{\beta \to 0}(\cos(\beta t)) = 1$; 2. $\lim_{\beta \to 0}(\frac{1}{\beta}\sin(\beta t)) = t\lim_{\beta \to 0}\left(\frac{\sin(\beta t)}{\beta t}\right) = t$;

3. $\lim\limits_{\beta\to 0}\left(\dfrac{1}{2\beta}t\sin(\beta t)\right)=\dfrac{t^2}{2}\lim\limits_{\beta\to 0}\left(\dfrac{\sin(\beta t)}{\beta t}\right)=\dfrac{t^2}{2}$;

4. $\lim\limits_{\beta\to 0}\left(\dfrac{1}{2\beta^3}\left(\sin(\beta t)-\beta t\cos(\beta t)\right)\right)=$

$$=\frac{1}{2}\lim_{\beta\to 0}\frac{t\cos(\beta t)-t\cos(\beta t)+\beta t^2\sin(\beta t)}{3\beta^2}=$$

$$=\frac{t^3}{6}\lim_{\beta\to 0}\left(\frac{\sin(\beta t)}{\beta t}\right)=\frac{t^3}{3!};$$

5. $\lim\limits_{\beta\to 0}\left(\dfrac{t}{8\beta^3}\left(\sin(\beta t)-\beta t\cos(\beta t)\right)\right)=$

$$=\frac{t}{8}\lim_{\beta\to 0}\frac{t\cos(\beta t)-t\cos(\beta t)+\beta t^2\sin(\beta t)}{3\beta^2}=$$

$$=\frac{t^4}{24}\lim_{\beta\to 0}\left(\frac{\sin(\beta t)}{\beta t}\right)=\frac{t^4}{4!};$$

6. $\lim\limits_{\beta\to 0}\left(\dfrac{t}{8\beta^5}((3-\beta^2t^2)\sin(\beta t)-3\beta t\cos(\beta t))\right)=$

$$=\frac{t^2}{40}\lim_{\beta\to 0}\frac{\beta\sin(\beta t)-\beta^2 t\cos(\beta t)}{\beta^4}=$$

$$=\frac{t^2}{160}\lim_{\beta\to 0}\frac{\sin(\beta t)-\beta t\cos(\beta t)+\beta^2t^2\sin(\beta t)}{\beta^3}=$$

$$=\frac{t^4}{480}\lim_{\beta\to 0}\frac{3\sin(\beta t)+\beta t\cos(\beta t)}{\beta}=\frac{t^5}{5!}.$$

Thus prove to be fair the following limit transitions

$$\lim_{\beta\to 0}\left\|\exp\left(\tilde{J}(0,\beta)\right)t\right\|_p=\left\|\exp(J(0)t)\right\|_p,$$

$$\lim_{\beta\to 0}\left\|\exp\left(\tilde{J}(\alpha,\beta)\right)t\right\|_p=\left\|\exp(J(\alpha)t)\right\|_p.$$

Next are formulated and solved two tasks. The first task is to evaluate influence of β under the condition $\alpha=\arg\{\alpha<0\vee|\alpha|<1\}$ on the value of overshoot. The second task is to evaluate possibility of the appearance of overshoot under the condition $\alpha=\arg\{\alpha<0\vee|\alpha|\geq 1\}$ and the influence on β value of the overshoot.

The results of solve the first task in the form of $\left\|\tilde{x}(t,\tilde{x}(0))\right\|_{\|\tilde{x}(0)\|=1}$, calculated by (33), on the example of the system situation characterized by $n=6,\ \mu=\dfrac{n}{2}=3$, $\lambda=\alpha\pm j\beta:\alpha=-0.2\,\&\,\beta\text{ - var}$, are summarized in Table 3.

	$\lambda=\alpha\pm j\beta$; n = 6; $\mu=3$						
α	$\alpha=-0.2$						
β	0.01	0.1	0.25	0.5	1	1.25	1.375
$\max\limits_t\left\|\tilde{x}(t,\tilde{x}(0))\right\|_{\|\tilde{x}(0)\|=1}$	150	110	42	10.6	3.9	9.2	16.8
$t_M=\arg\max\limits_t\left\|\tilde{x}(t,\tilde{x}(0))\right\|$	20	17	10.5	7	7.5	9.2	9.25
β	1.5	1.75	2	3	3.5	5	10
$\max\limits_t\left\|x(t,x(0))\right\|_{\|x(0)\|=1}$	30	86	220	3700	11000	13*10^4	17*10^6
$t_M=\arg\max\limits_t\left\|\tilde{x}(t,\tilde{x}(0))\right\|$	9.3	9.35	9.4	9.56	9.6	9.72	9.93

Table 3. Values of overshoot $\max\limits_t\left(\left\|\tilde{x}(t)\right\|\right)=\left\|\tilde{x}(t_M)\right\|$ to curve $\left\|\tilde{x}(t)\right\|$

The second task's start with graphical illustrations known [5] recommendations. They are to limit sector localization of complex conjugate eigenvalues of the system state (29). This limitation ensures that no overshoot in the transient response. This illustration is shown in Fig. 8.

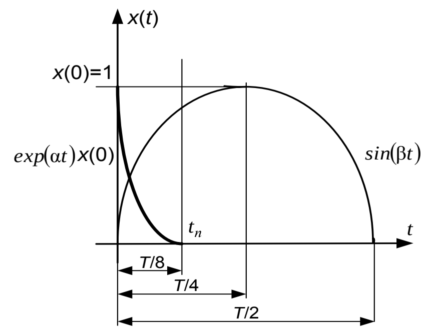

Fig. 8. Graphic illustration of the free motion generated by the real α and imaginary β components of the eigenvalue $\lambda=\alpha\pm j\beta$

We associate trajectory generated by the real part α and the imaginary part β of the ratio $t_n=\gamma T/2=\gamma\pi/\beta=3/\alpha$. From which we obtain that for $\beta\leq\gamma(\pi/3)\alpha$ and $\gamma<0.25$ will not be overshoot otherwise it will be. Allocated by the $i-$th $(i=\overline{1,n/2})$ two-dimensional cell is determined by the following model

$$\tilde{x}_i(t)=\left[\tilde{x}_{2i-1}(t),\tilde{x}_{2i}(t)\right]^T=$$
$$=col\left\{[\alpha,1],\left[-\beta^2,\alpha\right]\right\}\left[\tilde{x}_{2i-1}(t),\tilde{x}_{2i}(t)\right]^T,\tilde{x}_i(0)=$$
$$=\left[\tilde{x}_{2i-1}(0),\tilde{x}_{2i}(0)\right]^T.$$

Motion in cell is described by

$$\tilde{x}_i(t)=\exp(\alpha t)col\left\{[\cos(\beta t),(1/\beta)\sin(\beta t)],[(-\beta)\sin(\beta t),\cos(\beta t)]\right\}\tilde{x}_i(0),$$

for which the norm $\left\|\tilde{x}_i(t)\right\|$ with $\left\|\tilde{x}_i(0)\right\|=1$ is valid covering

$$roof\left\{\left\|\tilde{x}_i(t)\right\|\right\}=\exp(\alpha t)\left\|col\left\{[1,(1/\beta)],[(-\beta),1]\right\}\right\|.$$

For coating processes (35) with $\beta>|\gamma(\pi/3)\alpha|$ justly replace $\cos(\beta t)$ by 1, $\sin(\beta t)$ by 1 so that the coating can be specified as

$$\hat{v}(\alpha,\beta,t)=\left[\left(\frac{\left(3-(\beta t)^2\right)-3\beta t}{8\beta^5}\right),\left(\frac{t(1-\beta t)}{8\beta^3}\right),\left(\frac{1-\beta t}{2\beta^3}\right),\left(\frac{t}{2\beta}\right),\left(\frac{1}{\beta}\right),1\right]^T.$$

In vector $\hat{v}(\alpha,\beta,t)$ dominates the first term. This allows us to construct an analytic representation covering (roof) $\left\|exp\left\{\tilde{J}(0,\beta)t\right\}\right\|$ process in the form of Euclidean vector norm $\tilde{v}(\alpha,\beta,t)=\left[\left(\left(3-(\beta t)^2\right)-3\beta t\right)/\left(8\beta^5\right),0,0,0,0,0\right]$. It takes the form

$roof\left\{\left\|exp\left\{\tilde{J}(0,\beta)t\right\}\right\|\right\}=\left\{\left[\left(\left(3-(\beta t)^2\right)-3\beta t\right)/\left(8\beta^5\right)\right]^2\right\}^{1/2}$. The latter formula makes fair representation coating process $\left\|exp\left\{\tilde{J}(\alpha,\beta)t\right\}\right\|$ in the form

$$roof\left\{\left\|exp\left\{\tilde{J}(\alpha,\beta)t\right\}\right\|\right\}=e^{\alpha t}\left\{\left[\left(\left(3-(\beta t)^2\right)-3\beta t\right)/\left(8\beta^5\right)\right]^2\right\}^{1/2} \qquad (36)$$

Formula (36) we use for research at the extremes of the norm of the matrix exponential calculating $t_M=\arg\max_t\left\{\left\|exp\left\{\tilde{J}(\alpha,\beta)\,t\right\}\right\|\right\}$ of conditions $\frac{d}{dt}\left\{\text{roof}\left\{\left(\left\|exp\left\{\tilde{J}(\alpha,\beta)t\right\}\right\|\right)\right\}\right\}=0$ generates an algebraic equation for computing t_M

$$t_M^4+\frac{6\alpha+2\beta}{\alpha\beta}t_M^3+\frac{3\alpha+9\beta}{\alpha\beta^2}t_M^2+\frac{3\beta-18\alpha}{\alpha\beta^3}t_M+\frac{9(\alpha-\beta)}{\alpha\beta^4}=0 \qquad (37)$$

The results of calculations t_M using (37) are given in Table4.

	β				
α	2	3	5	10	20
-0.2	9.4	9.56	9.72	9.86	9.93
-2		0.913	0.88	0.9	0.938
-8				0.236	0.22
	t_M				

Table 4. Moments t_M overshoot in the curve $\left\|\tilde{x}(t)\right\|$

V. Computer Simulations. Case of multiple complex conjugate eigenvalues

Computer simulations of the processes carried out in the norm $\left\|\tilde{x}(t)\right\|$ for the example of system (29), characterized by the dimension $n=6$ and multiplicity $\lambda=\alpha\pm j\beta$ equal $\mu=n/2=3$ in Matlab. The purpose of research is visualization obtained in the previous section of this paper results. Results were visualized for three system situations.

The first situation is to evaluate the influence of β under the condition $\alpha=\arg\{\alpha<0\vee|\alpha|<1\}$ by the amount of overshoot in the trajectories of the system (29) in the norm of vector $\tilde{x}(t)=\tilde{x}(t,\tilde{x}(0))$. Results of visualization of this situation for

$\alpha=-0.2$ for $\beta=0.01$(fig.a), $\beta=1$(fig.b), $\beta=2$(fig.c), $\beta=5$(fig.d) are shown in Fig. 9.

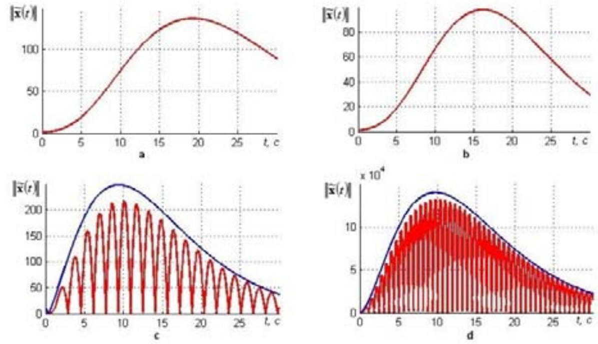

Fig.9. Curves of processes $\left\|\tilde{x}(t,\tilde{x}(0))\right\|_{\|\tilde{x}(0)\|=1}$ and their coverings for $\alpha=\arg\{\alpha<0\vee|\alpha|<1\}$ and $\beta=\text{var}$

The curves to fully correspond to the data in table 4 and are characterized by sharp increase of the value of overshoot with the growth of the value of the imaginary part β in area $\beta>1$.

The second situation is to assess the possibility of occurrence of overshoot in the trajectories of the system (29) in the norm of vector $\tilde{x}(t)=\tilde{x}(t,\tilde{x}(0))$, provided $\alpha=\arg\{\alpha<0\vee|\alpha|\geq 1\}$ and influence β values on the value and character of this overshoot. The investigation of this situation, the authors decided to start with the examination of the same problems for $i-\text{th}\,(i=\overline{1,n/2})$ two-dimensional cell state vector $\tilde{x}_i(t)=\left[\tilde{x}_{2i-1}(t),\tilde{x}_{2i}(t)\right]^T$. Results of visualization of this situation for

$\alpha=-8$ for $\beta=1$(fig.a), $\beta=5$(fig.b), $\beta=20$(fig.c), $\beta=50$(fig.d) are shown in Fig. 10.

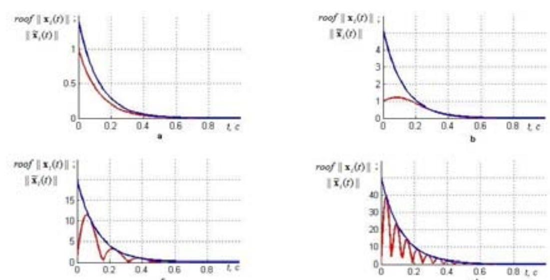

Fig.10. Curves of processes $\left\|\tilde{x}(t,\tilde{x}(0))\right\|_{\|\tilde{x}(0)\|=1}$ and their coverings for $\alpha=\arg\{\alpha<0\vee|\alpha|>1\}$ and $\beta=\text{var}$

Weak damping of complex conjugate eigenvalues is already evident in the emissions trajectories of free motion in the norm state vector of two-dimensional cell. It should be expected that in the case of multiple complex conjugate eigenvalues, this effect will repeatedly increase, despite the condition $\alpha=\arg\{\alpha<0\vee|\alpha|>1\}$. Results of visualization of this situation for

$\alpha=-8$ for $\beta=1$(fig.a), $\beta=3$(fig.b), $\beta=20$(fig.c), $\beta=50$(fig.d) are shown in Fig. 11.

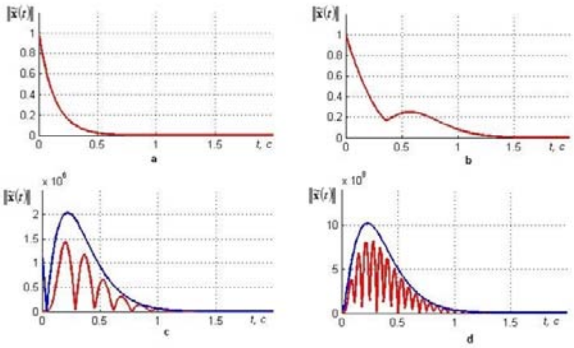

Fig.11. Curves of processes $\left\|\tilde{x}\left(t,\tilde{x}(0)\right)\right\|_{\left\|\tilde{x}(0)\right\|=1}$ and their coverings for $\alpha = \arg\left\{\alpha < 0 \vee \left|\alpha\right| > 1\right\}$ and $\beta = \text{var}$

The curves detect the presence of significant overshoot, value of which increases with increasing values of the imaginary part β.

VI. Conclusion

There is found that multiplicity of the eigenvalues of state matrix of stable continuous systems and the structure of eigenvectors of state matrix [5] is an important factor in the system, endowing a dynamic system processes of specific properties that may lead to undesirable consequences of a destructive nature. In order to prevent the discovered effect "multiplicity of eigenvalues" in the synthesis of modal control methods [3] state matrix F should be given the spectrum of eigenvalues not containing multiple elements.

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