Hermite polynomials and some generalizations on the Heat equations

Clemente Cesarano

***Abstract*—** We propose a method employing the pseudo-hyperbolic functions, Hermite polynomials and the related generalizations to use the connected operational techniques, to find general solutions for extended forms of the d'Alembert and the Fourier heat equations.

***Keywords*—** Pseudo-hyperbolic functions, Hermite Polynomials, Generating Functions, Heat equations.

I. INTRODUCTION

IN a previous paper [1], Ricci has developed a systematic and comprehensive treatment of the pseudo-hyperbolic functions of the type:

$$E_j(x\,|\,r) = \sum_{k=0}^{+\infty} \frac{x^{kr+j}}{(kr+j)!}, \tag{1}$$

where $0 \le j \le r$ and showed that they are r-independent solutions of the differential equation:

$$\left(\frac{d}{dx}\right)^r w(x) - w(x) = 0. \tag{2}$$

In the same paper, it has also been proved that the use of the roots of unity allows to cast the j^{th} order pseudo-hyperbolic function in the form:

$$E_j(x\,|\,r) = \frac{1}{r}\sum_{l=1}^{r} \frac{e^{\rho_l x}}{\rho_l^j}, \tag{3}$$

where:

$$\rho_l = e^{\frac{2i\pi l}{r}}. \tag{4}$$

This fairly general conclusion does not hold for pseudo-hyperbolic functions and for the associated pseudo-trigonometric forms only, but it can be extended to a large body of special polynomials and special functions [2,3,4]. For instance, we can immediately get the following statements for the Hermite Kampé de Feriét polynomials [5] of second order

$$\sum_{k=0}^{+\infty} \frac{t^{kr+j}}{(kr+j)!} H_{kr+j}(x,y) = \frac{1}{r}\sum_{l=1}^{r} \frac{e^{\tau_l x + \tau_l^2 j}}{\rho_l^j}, \tag{5}$$

where:

$$\tau_l = \rho_l t, \; H_n(x,y) = n!\sum_{r=0}^{[n/2]} \frac{y^r x^{n-2r}}{r!(n-2r)!},$$

and for the cylindrical Bessel functions of the first kind [6,7,8]:

$$\sum_{k=-\infty}^{+\infty} t^{kr} J_{kr}(x) = \frac{1}{r}\sum_{l=1}^{r} e^{\frac{x}{2\left(\tau_l - \frac{1}{\tau_l}\right)}}, \tag{6}$$

where:

$$J_n(x) = \sum_{s=-\infty}^{+\infty} \frac{(-1)^s}{(n+s)!\,s!}\left(\frac{x}{2}\right)^{n+2s}. \tag{7}$$

We can use these relations to explore the possibility of extending the above considerations and the results contained in the reference [1] to the study of partial differential equations of the type:

$$\left(\frac{\partial}{\partial x}\right)^{mr} w(x,t) - \frac{1}{h}\left(\frac{\partial}{\partial t}\right)^{r} w(x,t) = 0 \tag{8}$$

where:

$$w(x,0) = f(x);$$

with $r, m \in \mathbb{N}$ and relatively primes, $h \in \mathbb{C}$ and x, t in general complex variables.
The equation (8) reduces to the d'Alembert equation when:

$$m = 1 \text{ and } r = 2,$$

while for:

C. Cesarano is with the Faculty of Engineering, International Telematic University UNINETTUNO, Rome, Italy (phone: +39 0669207675; e-mail: c.cesarano@uninettunouniversity.net).

$m=2$ and $r=1$,

we recover the Fourier heat equation.
In the following we will use the results showed above and the operational methods related to the Hermite polynomials [9,10], to obtain general solutions of the partial differential equation stated in the relation (8).

II. Solutions of Generalized Heat Equations

In the previous section we have presented some results related to the pseudo-hyperbolic functions; we can start to write a general solution of the equation (8) for the case $m=1$ and r a positive integer. Let $r=1$, we can formally rewrite the equation as:

$$\frac{\partial}{\partial t}w(x,t)=h\hat{O}\,w(x,t)\,, \tag{9}$$

after with $\hat{O}$ the derivative respect to *x*; we can therefore treat this operator as a constant and write the solution in analogy with the first order ordinary differential equation, that is:

$$w(x,t)=e^{ht\hat{O}}\left[g(x)\right]=e^{ht\frac{\partial}{\partial x}}\left[g(x)\right]=g(x+th)\,. \tag{10}$$

Now, by using the considerations stated in the previous section and the results showed in equations (1-3), we can write:

$$\begin{aligned}w(x,t)&=E_j\left(h^{\frac{1}{r}}t\frac{\partial}{\partial x}\,|\,r\right)g(x)=\frac{1}{r}\sum_{l=1}^{r}\frac{e^{h^{\frac{1}{r}}\rho_l t\frac{\partial}{\partial x}}}{\rho_l^j}\left[g(x)\right]=\\&=\frac{1}{r}\sum_{l=1}^{r}\frac{1}{\rho_l^j}g\left(x+h^{\frac{1}{r}}\rho_l t\right),\end{aligned} \tag{11}$$

which provides the most general solution of equation (8) for $m=1$ and $r\geq 1$. It is worth noting that for $m=1$ and $r=2$, we obtain the classical d'Alembert solution of the wave propagations.

Before going further, it could be useful to make the discussion more complete related to the two-variable Hermite polynomials, defined by:

$$H_n(x,y)=n!\sum_{r=0}^{\left[n/2\right]}\frac{y^r x^{n-2r}}{r!(n-2r)!} \tag{12}$$

which they are linked to the ordinary case, by:

$$H_n\left(x,-\frac{1}{2}\right)=He_n(x)\,. \tag{13}$$

It is well-known that, the generating function [10,11] reads:

$$e^{t\left(x+2y\frac{\partial}{\partial x}\right)}(1)=\sum_{n=0}^{+\infty}\frac{t^n}{n!}H_n(x,y)\,,\ t\in\mathbb{R}\,,\ |t|<+\infty\,. \tag{14}$$

Moreover, it is also evident that:

$$H_n(x,0)=x^n\,. \tag{15}$$

From the above relations, a fairly straightforward conclusion is the proof that the generalized Hermite polynomials of two variables satisfies the heat equation:

$$\frac{\partial}{\partial y}H_n(x,y)=\frac{\partial^2}{\partial x^2}H_n(x,y)\,. \tag{16}$$

The proof is just a consequence of the structure of the generating function itself. By keeping, indeed the derivatives of both sides of (14) with respect to *t* and then equating the *t*-like powers, we find:

$$\frac{\partial}{\partial y}H_n(x,y)=n(n-1)H_{n-2}(x,y) \tag{17}$$

$$\frac{\partial}{\partial x}H_n(x,y)=nH_{n-1}(x,y) \tag{18}$$

for which the heat equation follows. This statement allows a further important result, indeed by regarding it as an ordinary first order equation in the variable *y* and by treating the differential operators as an ordinary number, we can write the polynomials $H_n(x,y)$ in terms of the following operational definition:

$$H_n(x,y)=e^{y\frac{\partial^2}{\partial x^2}}H_n(x,0)=e^{y\frac{\partial^2}{\partial x^2}}x^n\,. \tag{19}$$

Let to return to the heat equation, states in relation (8) and looking for the general solution when $m=2$ and $r=1$. By using the same formalism, we have:

$$\frac{\partial}{\partial t}w(x,t)=h\hat{O}^{\,2}\,w(x,t)\,. \tag{20}$$

By following the same procedure of the previous case, we can immediately write:

$$w(x,t)=e^{ht\frac{\partial^2}{\partial x^2}}\left[g(x)\right]. \tag{21}$$

We note that the Hermite polynomials are an examples of *quasi-monomial* [10], in fact the polynomials $H_n(x,y)$ have been shown as to be *quasi-monomial* under the action of the operators:

$$\hat{M} = x + 2y\frac{\partial}{\partial x}$$
$$\hat{P} = \frac{\partial}{\partial x} \tag{22}$$

The considerations presented in a previous paper suggest that, by using the concepts and the related formalism of the monomiality principle, we can introduce or *"define"* families of isospectral problems [6,10,12] by exploiting the correspondence:

$$\hat{M} \to x$$
$$\hat{P} \to \frac{\partial}{\partial x}$$
$$p_n(x) \to x^n.$$

We can therefore use the a family of polynomials as a basis to introduce *"new"* functions with eigenvalues corresponding to the ordinary case. The most useful example is provided by a *p-based* Bessel function [10], defined as:

$${}_pJ_n(x) = \sum_{r=0}^{+\infty} \frac{(-1)^r p_{n+2r}}{2^{n+2r} r!(n+r)!}. \tag{23}$$

In the present case, we can write the function in the equation (21) in the Hermite-based, so that:

$${}_Hg(x,ht) = \sum_{n=0}^{+\infty} a_n H_n(x,ht), \tag{24}$$

if the $g(x)$ is specified by the polynomial expansion:

$$g(x) = \sum_{n=0}^{+\infty} a_n x^n. \tag{25}$$

Then, we can finally state the general solution of the heat equation when $m = 2$ and $r = 1$; we have, from equation (21):

$$w(x,t) = e^{ht\frac{\partial^2}{\partial x^2}}\left[g(x)\right] = {}_Hg(x,ht). \tag{26}$$

It is well-known that another form of solution of the ordinary heat equation is provided by the Gauss transform, that is:

$$w(x,t) = \frac{1}{2\sqrt{\pi ht}} \int_{-\infty}^{+\infty} e^{-\frac{(x-\xi)^2}{4ht}} g(\xi)d\xi, \tag{27}$$

which have sense only if the integral converges.
By combining these last remarks (eq.s (26) and (27)) we can easily cast the solution of the equation (8) for $m = 2$ and $r \geq 1$ in the form:

$$w(x,t) = \frac{1}{r}\sum_{l=1}^{r} \frac{1}{\rho_l^j} a_l H_l\left(x, h^{\frac{1}{r}}\rho_l t\right), \tag{28}$$

and, if the integral in equation (27) converges, we can also write:

$$w(x,t) = \frac{1}{r}\sum_{l=1}^{r} \frac{1}{\rho_l^j} \frac{1}{2\sqrt{\pi h^{\frac{1}{r}}\rho_l t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-\xi)^2}{4h^{\frac{1}{r}}\rho_l t}} g(\xi)d\xi. \tag{29}$$

By noting that, if the function $g(x)$ is a Gaussian, then following identity is true:

$$e^{\alpha\frac{\partial^2}{\partial x^2}} e^{-\beta x^2} = \frac{1}{\sqrt{1+4\alpha\beta}} e^{-\beta\frac{x^2}{1+4\alpha\beta}}, \tag{30}$$

known as the Glashier identity [11]. We can use the previous relations to state the solution of the equation (8) when $m = 2$ and $r \geq 1$. In fact, by noting that in the Glashier identity α can also be negative, provided that:

$$|\alpha| < \frac{1}{4\beta},$$

we can cast the identity in equation (29) as:

$$w(x,t) = \frac{1}{r}\sum_{l=1}^{r} \frac{1}{\rho_l^j} \frac{1}{2\sqrt{\pi h^{\frac{1}{r}}\rho_l t}} e^{-\frac{x^2}{1+4h^{\frac{1}{r}}\rho_l t}}. \tag{31}$$

In this section we have seen how the use of the relations stated in reference [1] allow us to provide the general solution of the equations of the type:

$$\left(\frac{\partial}{\partial x}\right)^{mr} w(x,t) - \frac{1}{h}\left(\frac{\partial}{\partial t}\right)^{r} w(x,t) = 0, \tag{32}$$

when $m = 2$ and $r \geq 1$. To explore the case $m > 2$, $r > 1$ it is necessary to make further considerations and to present more results on the theory of Hermite-based functions [10]. In the next sections we will deeply discuss these issues.

III. GENERALIZED HERMITE POLYNOMIALS AND RELATED APPLICATIONS TO THE HEAT EQUATIONS

In the previous sections we have introduced the two-variable Hermite polynomials $H_n(x, y)$. More in general the above

Hermite polynomials can be derive, as a particular case, from the two-variable Hermite polynomials of the type $H_n^{(m)}(x,y)$. We will call Hermite Kampé de Feriét polynomials [5,9] of m^{th}-order, $m \in \mathbb{N}$, the polynomials defined by the formula:

$$H_n^{(m)}(x,y) = n! \sum_{r=0}^{[n/m]} \frac{y^r x^{n-mr}}{r!(n-mr)!} \quad . \tag{33}$$

We can immediately note that the polynomials $H_n^{(m)}(x,y)$ satisfy the following partial differential equation:

$$\frac{\partial}{\partial y} H_n^{(m)}(x,y) = \frac{\partial^m}{\partial x^m} H_n^{(m)}(x,y) . \tag{34}$$

It is evident to observe the similarity with the equation (16) involving the two-variable Hermite polynomials of second order. To prove the above relation is easy to note that the generating function of the Hermite polynomials of the type $H_n^{(m)}(x,y)$ reads:

$$\exp(xt + yt^m) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} H_n^{(m)}(x,y) . \tag{35}$$

By differentiating with respect to *y*, we have:

$$\sum_{n=0}^{+\infty} \frac{t^n}{n!} \frac{\partial}{\partial y} H_n^{(m)}(x,y) = \sum_{n=0}^{+\infty} \frac{t^{n+m}}{n!} H_n^{(m)}(x,y) , \tag{36}$$

and after manipulating the l.h.s. of the above equation and by equating the like *t*-powers, we can immediately write:

$$\frac{n!}{(n-m)!} H_{n-m}^{(m)}(x,y) = \frac{\partial}{\partial y} H_n^{(m)}(x,y) . \tag{37}$$

Otherwise, by deriving *m*-times with respect to *x* in the equation (35), we have:

$$\frac{\partial^m}{\partial x^m} H_n^{(m)}(x,y) = \frac{n!}{(n-m)!} H_{n-m}^{(m)}(x,y) . \tag{38}$$

and then by comparing the relation (38) and the (37) we immediately obtain the partial differential equation (34).
The result stated above, allows us to derive a similar operational definition for the Hermite polynomials $H_n^{(m)}(x,y)$ as in the case of the two-variable Hermite Kampé de Feriét polynomials (see eq. (19)).
We note in fact that for $y = 0$ in the equation (33), we have:

$$H_n^{(m)}(x,0) = x^n \quad . \tag{39}$$

By considering the equation in (34) an ordinary differential equation in the variable *y*, we can immediately conclude that, since is a linear first order, the solution writes:

$$H_n^{(m)}(x,y) = e^{y \frac{\partial^m}{\partial x^m}} x^n \quad . \tag{40}$$

or, in more explicit terms:

$$H_n^{(m)}(x,y) = \left[\sum_{r=0}^{[n/m]} \frac{y^r}{r!} \left(\frac{\partial}{\partial x} \right)^{mr} \right] x^n . \tag{41}$$

By following the same considerations done in previous section, in particular related to the function $g(x)$, it is therefore immediately to specify it in terms of the m^{th}-order Hermite polynomials to set:

$$e^{y\left(\frac{\partial}{\partial x}\right)^m} \left[g(x) \right] = {}_{H^{(m)}} g(x,y) = \sum_{n=0}^{+\infty} a_n H_n^{(m)}(x,y) , \tag{42}$$

where the function ${}_{H^{(m)}} g(x,y)$ denotes the m^{th}-order Hermite-based function of $g(x)$.
Finally, the solution of the equation (8), for the case $m > 2$, $r > 1$, can be written in the following terms:

$$w(x,t) = \frac{1}{r} \sum_{l=1}^{r} \frac{1}{\rho_l^j} \, {}_{H^{(m)}} g\left(x, \rho_l h^{\frac{1}{r}} t \right) , \tag{43}$$

which completes the purpose of the paper.

IV. FURTHER REMARKS

Before to conclude we can extended the obtained results and go further in the analysis by making use of the previous considerations and of the properties of the generalized Hermite polynomials.
We remind the following property of the generalized Hermite polynomials which have played within the present context a non secondary role. We start to note that, the two-variable Hermite Kampé de Feriét polynomials satisfy the following relation:

$$e^{z \frac{\partial^2}{\partial x^2}} H_n(x,y) = H_n(x, y+z) \quad , \tag{44}$$

since for any analytic function $f(x,y)$ the following relations hold:

$$e^{\pm w \frac{\partial}{\partial x}} f(x,y) = f(x \pm w, y) ,$$

$$e^{\pm w\frac{\partial}{\partial y}} f(x,y) = f(x, y \pm w) \ .$$

Then, from equation (19), we can immediately write:

$$e^{-y\frac{\partial^2}{\partial x^2}} H_n(x,y) = x^n \tag{45}$$

and, likewise for the m^{th}-order Hermite polynomials of the type $H_n^{(m)}(x,y)$, we can state:

$$e^{z\frac{\partial^p}{\partial x^p}} H_n^{(m)}(x,y) = H_n^{(m,p)}(x,y,z) \ , \tag{46}$$

where the further generalization of the Hermite polynomials has the following explicit forms [5,9]:

$$H_n^{(m,p)}(x,y,z) = n! \sum_{r=0}^{\left[n/p\right]} \frac{H_{n-pr}^{(m)}(x,y) z^r}{r!(n-pr)!} \ . \tag{47}$$

By remembering that the two-variable Hermite polynomials $H_n(x,y)$ are linked to the ordinary Hermite polynomials by the relations stated in equation (13), and by supposing that the function $g(x)$ can be cast in the form:

$$g(x) = \sum_{n=0}^{+\infty} a_n He_n(x) \ , \tag{48}$$

we can write the solution of the equation (8) in the case $r = 1$ and *m* generic in the following form:

$$w(x,t) = \sum_{n=0}^{+\infty} a_n H_n^{(2,p)}\left(x, -\frac{1}{2}, ht\right) . \tag{49}$$

Before the ending the paper, we consider interesting to make the following concluding comments.
The generalized Hermite polynomials of the type $H_n^{(m,p)}(x,y,z)$, in one of their special case, satisfy the following partial differential equation:

$$\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial y}\right) H_n^{(2,3)}(x,y,z) = \left(\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}\right) H_n^{(2,3)}(x,y,z) \ , \tag{50}$$

where:

$$H_n^{(2,3)}(x,0,0) = x^n \ .$$

Then, the solution of analogous equations of the general equation (8) can be therefore expressed in terms of the Hermite-based functions related to the polynomials of the type $H_n^{(2,3)}(x,y,z)$.

In a forthcoming paper we will discuss deeply these arguments related to generalized Hermite-based functions in the context of the further extended forms of the d'Alembert and of the Fourier heat equations.

REFERENCES

[1] Ricci, P.E., "Le funzioni pseudo-iperboliche e pseudo-trigonometriche", *Ist. Di matematica applicta*, 192, 1978.

[2] Cesarano, C., "Identities and generating functions on Chebyshev polynomials", *Georgian Mathematical Journal*, 19 (3), pp. 427-440, 2012.

[3] Dattoli, G., Cesarano, C., "On a new family of Hermite polynomials associated to parabolic cylinder functions", *applied Mathematics and Computation*, 141 (1), pp. 143-149, 2003.

[4] Dattoli, G., Lorenzutta, Cesarano, C., "Bernestein polynomials and operational methods", *Journal of Computational Analysis and Applications*, 8 (4), pp. 369-377, 2006.

[5] Appell, P., Kampé de Fériet, J., "Fonctions hypergéométriques et hypersphériques. Polinomes d'Hermite", *Gauthier-Villars*, Paris, 1926.

[6] Cesarano, C., Germano, B., Ricci, P.E., "Laguerre-type Bessel functions", Integral Transforms and Special Functions, 16 (4), pp. 315-322, 2005.

[7] Watson, J.H., "A treatise on Bessel functions", *Cambridge University press*, 1958

[8] Assante, D., Verolino, L., "Efficient evaluation of the longitudinal coupling impedance of a plane strip", *Progress In Electromagnetics Research M*, vol. 26 , pp. 251-265, 2012.

[9] Gould, H.W., Hopper, A.T., "Operational formulas connected with two generalizations of Hermite polynomials", *Duke Math. J.*, 29, pp. 51-62, 1962.

[10] Cesarano, C., "Monomiality Principle and related operational techniques for Orthogonal Polynomials and Special Functions", *Int. J. of Math. Models and Methods in Appl. Sci.*, to appear, 2013.

[11] Srivastava, H.M., Manocha, H.L., "A treatise on generating functions", Wiley, New York, 1984.

[12] Dattoli, G., Srivastava, H.M., Cesarano, C., "The Laguerre and Legendre polynomials from an operational point of view", *Applied Mathematics and Computation*, 124 (1), pp. 117-127 (2001).

Clemente Cesarano is assistant professor of Mathematical Analysis at Faculty of Engineering- International Telematic University UNINETTUNO-Rome, ITALY. He is coordinator of didactic planning of the Faculty and he also is coordinator of research activities of the University. Clemente Cesarano is Honorary Research Associates of the Australian Institute of High Energetic Materials His research activity focuses on the area of Special Functions, Numerical Analysis and Differential Equations.
He has work in many international institutions as ENEA (Italy), Ulm University (Germany), Complutense University (Spain) and University of Rome La Sapienza (Italy). He has been visiting researcher at Research Institute for Symbolic Computation (RISC), Johannes Kepler University of Linz (Austrian). He is Editorial Board member of the Research Bulletin of the Australian Institute of High Energetic Materials, the Global Journal of Pure and Applied Mathematics (GJPAM), the Global Journal of Applied Mathematics and Mathematical Sciences (GJ-AMMS), the Pacific-Asian Journal of Mathematics, the International Journal of Mathematical Sciences (IJMS), the Advances in Theoretical and Applied Mathematics (ATAM), the International Journal of Mathematics and Computing Applications (IJMCA).
Clemente Cesarano has published two books and over than fifty papers on international journals in the field of Special Functions, Orthogonal Polynomials and Differential Equations.