The finite volume method in incompressible fluid flow simulation: flow in the gap between cone and cylinder

E. Kornaeva, L. Savin, V. Arkhipov, A. Kornaev

***Abstract*—**The goal of this work is to research the mathematical model of the laminar flow of viscous incompressible fluid in the gap between the static outer cone (stator) and the rotating inner cylinder (rotor). This mathematical model is based on the Navier-Stokes and the continuity equations. Numerical solution is based on the finite volume approach. The results are compared with particular results calculated by other authors and with analytical solutions for asymptotic cases.

***Keywords*—**Cone-cylinder gap, finite volume approach, incompressible fluid, viscosity.

I. Introduction

MATHEMATICAL modeling of enforced and shear flows of viscous fluids in gaps of various geometry is a topical question in hydrodynamics. Among examples of such flows are flows in noncontact seals and journal bearings which are widely used in mechanical engineering, metallurgical and rocket industry. It is widely known that there are many life-time and reliability requirements for seals and bearings [1], [2].

In this article we research mathematical and simulation models of the three – dimensional enforced and shear flow of viscous incompressible fluid in the gap between the steady state cone and the rotating eccentric cylinder. The main equations in this mathematical model are the Navier – Stokes equation and the continuity equation. Its numerical solution has some complications such as the considerable difference in terms of the flow region, curvilinear boundary of the flow region and nonlinear terms of the Navier – Stokes equation because of the inner cylinder high rotational velocity and the tapered flow canal.

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II. Modeling

The flow of the viscous incompressible Newtonian fluid in the confusor is under investigation. The flow region is formed by a stationary truncated cone (stator) and a rotating cylinder (rotor), which are shown in Figure 1. The Cone has radii R_1 and R_2 respectively. The Cylinder with radius r is off-centered in cone and is rotating at a constant angular velocity ω. Under pressure P_1 the fluid flows from one end towards the channel shrinkage and escapes from the other end under pressure P_0.

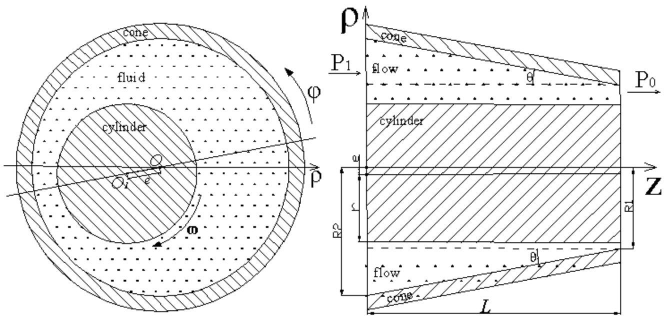

Fig. 1 geometry of the channel

The fluid is assumed to fill up the whole channel, the flow is laminar. The temperature is assumed as constant.

A. Mathematical model

The Navier-Stokes equation and the continuity equation are the fundamental equations which describe the flow process [4], [6]:

$$\begin{cases} \rho_f \nabla \cdot \left(\vec{V} \otimes \vec{V}\right) = -\nabla P + \nabla \cdot D_\sigma, \\ \nabla \cdot \vec{V} = 0, \end{cases} \tag{1}$$

where ρ_f is a density of fluid, $\vec{V}$ is a velocity vector, ∇P is a gradient of the pressure, D_σ is a stress deviator, $\nabla \cdot \vec{V}$ is a velocity divergence.

The stress deviator is determined by the Newton's generalized hypothesis:

$$D_\sigma = 2\mu D_\xi, \tag{2}$$

where μ is the coefficient of the dynamic viscosity.

For the incompressible medium:

$$T_\xi = D_\xi = \frac{1}{2}\left(\nabla \otimes \vec{V} + \vec{V} \otimes \nabla\right). \quad (3)$$

Due to the nondimensionalization by means of characteristic quantities, system (1) can be rearranged as follows:

$$\begin{cases} \delta^2 \hat{V}_\rho \frac{\partial \hat{V}_\rho}{\partial \hat{\rho}} + \delta \frac{\hat{V}_\varphi}{\hat{\rho}+\psi}\frac{\partial \hat{V}_\rho}{\partial \varphi} + \delta^2 \hat{V}_z \frac{\partial \hat{V}_\rho}{\partial \hat{z}} - \frac{\hat{V}_\varphi^2}{\hat{\rho}+\psi} = -Eu \frac{\partial \hat{P}}{\partial \hat{\rho}} + \\ + \frac{\delta}{\mathrm{Re}}\left(\frac{\partial^2 \hat{V}_\rho}{\partial \hat{\rho}^2} + \frac{1}{\hat{\rho}+\psi}\frac{\partial \hat{V}_\rho}{\partial \hat{\rho}} + \frac{1}{(\hat{\rho}+\psi)^2}\frac{\partial^2 \hat{V}_\rho}{\partial \varphi^2} + \delta^2 \frac{\partial^2 \hat{V}_\rho}{\partial \hat{z}^2} - \right. \\ \left. - \frac{\hat{V}_\rho}{(\hat{\rho}+\psi)^2} - \frac{2}{\delta(\hat{\rho}+\psi)^2}\frac{\partial \hat{V}_\varphi}{\partial \varphi} \right), \\ \delta \hat{V}_\rho \frac{\partial \hat{V}_\varphi}{\partial \hat{\rho}} + \frac{\hat{V}_\varphi}{\hat{\rho}+\psi}\frac{\partial \hat{V}_\varphi}{\partial \varphi} + \delta \hat{V}_z \frac{\partial \hat{V}_\varphi}{\partial \hat{z}} + \delta \frac{\hat{V}_\rho \hat{V}_\varphi}{\hat{\rho}+\psi} = \\ = -Eu \frac{1}{\hat{\rho}+\psi}\frac{\partial \hat{P}}{\partial \varphi} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \hat{V}_\varphi}{\partial \hat{\rho}^2} + \frac{1}{\hat{\rho}+\psi}\frac{\partial \hat{V}_\varphi}{\partial \hat{\rho}} + \right. \\ \left. \frac{1}{(\hat{\rho}+\psi)^2}\frac{\partial^2 \hat{V}_\varphi}{\partial \varphi^2} + \delta^2 \frac{\partial^2 \hat{V}_\varphi}{\partial \hat{z}^2} - \frac{\hat{V}_\varphi}{(\hat{\rho}+\psi)^2} + \frac{2\delta}{(\hat{\rho}+\psi)^2}\frac{\partial \hat{V}_\rho}{\partial \varphi} \right), \\ \delta \hat{V}_\rho \frac{\partial \hat{V}_z}{\partial \hat{\rho}} + \frac{\hat{V}_\varphi}{\hat{\rho}+\psi}\frac{\partial \hat{V}_z}{\partial \varphi} + \delta \hat{V}_z \frac{\partial \hat{V}_z}{\partial \hat{z}} = -\delta Eu \frac{\partial \hat{P}}{\partial \hat{z}} + \\ + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \hat{V}_z}{\partial \hat{\rho}^2} + \frac{1}{\hat{\rho}+\psi}\frac{\partial \hat{V}_z}{\partial \hat{\rho}} + \frac{1}{(\hat{\rho}+\psi)^2}\frac{\partial^2 V_z}{\partial \varphi^2} + \delta^2 \frac{\partial^2 V_z}{\partial z^2} \right), \\ \frac{\partial \hat{V}_\rho}{\partial \hat{\rho}} + \frac{\hat{V}_\rho}{\hat{\rho}+\psi} + \frac{1}{\delta}\frac{1}{\hat{\rho}+\psi}\frac{\partial \hat{V}_\varphi}{\partial \varphi} + \frac{\partial \hat{V}_z}{\partial \hat{z}} = 0, \end{cases} \quad (4)$$

where $\hat{\rho} = \frac{\rho - (r-e)}{h_{02}+e}$, $\hat{z} = \frac{z}{L}$ are dimensionless radial and axes coordinates, $\hat{V}_\rho = \frac{V_\rho}{\delta V^*}$, $\hat{V}_\varphi = \frac{V_\varphi}{V^*}$, $\hat{V}_z = \frac{V_z}{V^*}$ are dimensionless radial, tangential and axes components of the velocity vector, $\hat{P} = \frac{P - P_0}{\Delta P}$ are dimensionless pressure function, $\delta = (\xi+1)(\beta+\eta\gamma)$, $\psi = \left(\gamma - \xi(\beta+\eta\gamma)\right)\delta^{-1}$, $\beta = \frac{R_2 - R_1}{L}$, $\gamma = \frac{r}{L}$, $\eta = \frac{h_{01}}{r}$, $\xi = \frac{e}{h_{02}}$ are dimensionless coefficients and geometric parameters, $Re = \frac{V^*(h_{02}+e)}{\nu}$, $Eu = \frac{\Delta P}{\rho_{ж}(V^*)^2}$ are Reynolds and Euler numbers respectively.

The equations of the channel boundaries may be shown as follows:

$$\hat{R}(\hat{z}) = -\frac{\beta}{\delta}\hat{z} + 1,$$

$$\hat{r}(\varphi) = \frac{1}{\delta}(\xi(\beta+\eta\gamma)(1+\cos(\varphi-\alpha)) - \gamma + \sqrt{1-\xi^2(\beta+\eta\gamma)^2 \sin^2(\varphi-\alpha)}), \quad \varphi \in [0;2\pi].$$

System (4) can be solved with boundary conditions simultaneously. The boundary conditions for the velocity components can be presented as follows:

$$\begin{cases} \hat{V}_\rho(\hat{r}(\varphi),\varphi,\hat{z}) = 0, \\ \hat{V}_\varphi(\hat{r}(\varphi),\varphi,\hat{z}) = \frac{\omega r}{V^*}, \\ \hat{V}_z(\hat{r}(\varphi),\varphi,\hat{z}) = 0, \end{cases} \quad \begin{cases} \hat{V}_\rho(\hat{R}(\hat{z}),\varphi,\hat{z}) = 0, \\ \hat{V}_\varphi(\hat{R}(\hat{z}),\varphi,\hat{z}) = 0, \\ \hat{V}_z(\hat{R}(\hat{z}),\varphi,\hat{z}) = 0. \end{cases} \quad (5)$$

For pressure function $\hat{P}$ on the ends of canal:

$$\hat{P}(\hat{\rho},\varphi,0) = 1, \ \hat{P}(\hat{\rho},\varphi,1) = 0. \quad (6)$$

Because the fluid flow canal is closed form the tangential coordinate direction the periodical conditions can be described as follows:

$$\hat{F}_i(\hat{\rho},0,\hat{z}) = \hat{F}_i(\hat{\rho},2\pi,\hat{z}), \ \frac{\partial \hat{F}_i(\hat{\rho},0,\hat{z})}{\partial \varphi} = \frac{\partial \hat{F}_i(\hat{\rho},2\pi,\hat{z})}{\partial \varphi}. \quad (7)$$

B. Analyses of model

As was said before, the flows in noncontact seals and journal bearings are investigated, so the flow thickness is very small. The set of main parameters and theirs order of magnitudes are presented in table I.

I the main parameters order of magnitudes

Parameter	Lower level	Upper level
r, m	10^{-2}	10^{-1}
h_{01}, m	10^{-5}	10^{-4}
β	0	10^{-1}
L, m	10^{-2}	10^{-1}
n, rpm	10^{1}	10^{5}
ΔP, Pa	10^{5}	10^{7}
ν, m^2/s	10^{-6}	10^{-3}
μ, Pa·s	10^{-5}	10^{0}
ρ_f, kg/m^3	10^{0}	10^{3}

Using order-of-magnitude analysis it is easy to determining which terms in the equations are very small relative to the other terms [1], [4], [7], [8]. The values of the terms of equations (4) are presented in table II, the geometry parameter δ coefficient domain is 10^{-4} to 10^{-1}. In order to δ coefficient two cases available, firstly, if the conicity parameter β has the same magnitude with the relative gap η, and secondly, if the conicity parameter β exceeds the relative gap η by one or more orders of magnitude. Also the Euler number and the Reynolds number orders of magnitude are considered in follow conclusions:

- if the conicity parameter β is less than 10^{-3} and the Reynolds number is less than 10^{0}, then the velocity radial component, the inertial term and the velocity components derivatives in the tangential and axes directions are negligible;

- if the Reynolds number is more than 10^0, then the inertial terms and the viscosity terms has the same magnitude, and if the conicity parameter β is more or equal to 10^{-3}, all the velocity components has significant values.

II the equation (4) terms order of magnitudes

The Navier-Stokes equation										
Inertial terms $\nabla \cdot \left(\vec{V} \otimes \vec{V}\right)$				∇P	Dissipative terms $\nabla \cdot D$					
δ^2	δ^2	δ^2	δ	Eu	$\frac{\delta}{Re}$	$\frac{\delta^2}{Re}$	$\frac{\delta^3}{Re}$	$\frac{\delta^3}{Re}$	$\frac{\delta^3}{Re}$	$\frac{\delta^2}{Re}$
1	1	1	δ	Eu	$\frac{1}{\delta\, Re}$	$\frac{1}{Re}$	$\frac{\delta}{Re}$	$\frac{\delta}{Re}$	$\frac{\delta}{Re}$	$\frac{1}{Re}$
1	1	1	-	Eu	$\frac{1}{\delta\, Re}$	$\frac{1}{Re}$	$\frac{\delta}{Re}$	-	-	$\frac{\delta}{Re}$
The continuity equation										
1			δ				1		1	

According to the order-of-magnitude analysis, widely used assumptions of hydrodynamic theory of lubrication [4], [7] are acceptable if the conicity parameter β is less than 10^{-3}, and that is the flow between two cylinders actually. In this study it is necessary to consider the Navier-Stokes equation in its complete form.

Thus, the mathematical model of the researched process has a look (4)-(7) and consists of four nonlinear partial differential equations with four unknown functions.

III. NUMERICAL CALCULATIONS

Numerical calculations of equations (4)-(7) are based on the finite volume method (F.V.M.). By means of the F.V.M. it is possible to get an adequate solution even for a crude mesh, because of guaranteed fulfilling of the fundamental laws of conservation [3], [8].

Equations (4) in tensor form look as follows:

$$\begin{cases} \hat{\nabla} \cdot \left(\hat{\vec{V}} \otimes \hat{\vec{V}} \right) = -Eu\hat{\nabla}\hat{P} + \dfrac{1}{Re}\hat{\nabla} \cdot \left(\hat{\nabla} \otimes \hat{\vec{V}} + \hat{\vec{V}} \otimes \hat{\nabla} \right), \\ \hat{\nabla} \cdot \hat{\vec{V}} = 0. \end{cases} \tag{8}$$

According to the flow region geometry the element size by Oρ direction is variable and depends on the Oφ coordinate. The element sizes measured with the Oφ and Oz coordinates are constant. See fig. 2 as the discretization principle visualization in case of the coaxial flow region.

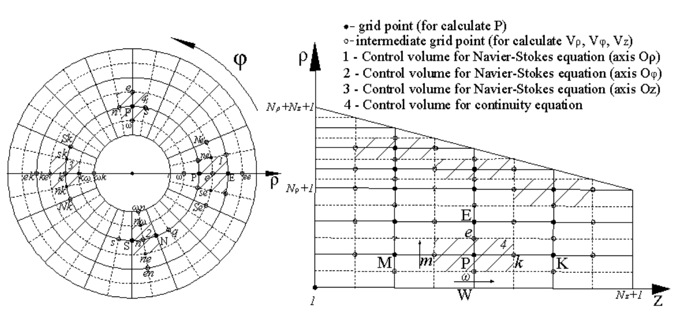

Fig. 2 Flow region discretization

According to approach [3], [9] the following operation is the volume integration of equations (8) in each finite volume (FV). Using the Ostrogradski formula it is possible to decrease the digit of the derivatives of the velocity vector:

$$\begin{cases} \int\limits_{\hat{S}_i} \hat{n}_i \cdot \left(\hat{\vec{V}} \otimes \hat{\vec{V}} \right) d\hat{S}_i = -\int\limits_{\hat{\Omega}} Eu\hat{\nabla}\hat{P}d\Omega + \\ + \dfrac{1}{Re} \int\limits_{\hat{S}_i} \vec{n}_i \cdot \left(\hat{\nabla} \otimes \hat{\vec{V}} + \hat{\vec{V}} \otimes \hat{\nabla} \right) d\hat{S}_i, \\ \int\limits_{\hat{S}_i} \vec{n}_i \cdot \hat{\vec{V}} d\hat{S}_i = 0. \end{cases} \tag{9}$$

where $\vec{n}_i$ is a unit normal vector on the respectively surface of FV.

When we calculate surface integrals on each FV surface (fig. 2) and use the mean-value theorem, system (9) turns to:

$$\begin{cases} -\hat{a}_e\hat{V}_{\rho_e} + \hat{a}_E\hat{V}_{\rho_{ee}} + \hat{a}_P\hat{V}_{\rho_\omega} + \hat{a}_{ne}\hat{V}_{\rho_{Ne}} + \hat{a}_{se}\hat{V}_{\rho_{Se}} + \\ + \hat{a}_{ke}\hat{V}_{\rho_{Ke}} + \hat{a}_{me}\hat{V}_{\rho_{Me}} = Eu\Delta\varphi\Delta\hat{z}(\hat{\rho}_e + \psi)\hat{P}\Big|_{P}^{E} + \dfrac{\Delta\hat{\rho}\Delta\hat{z}}{Re}\dfrac{\hat{V}_\varphi}{\hat{\rho}+\psi}\Big|_{se}^{ne}, \\ -\hat{a}_n\hat{V}_{\varphi_n} + \hat{a}_{ne}\hat{V}_{\varphi_{en}} + \hat{a}_{n\omega}\hat{V}_{\varphi_{\omega n}} + \hat{a}_N\hat{V}_{\varphi_q} + \hat{a}_P\hat{V}_{\varphi_S} + \\ + \hat{a}_{nk}\hat{V}_{\varphi_{kn}} + \hat{a}_{nm}\hat{V}_{\varphi_{mn}} = Eu\Delta\hat{\rho}\Delta\hat{z}\hat{P}\Big|_{P}^{N} + \dfrac{\delta\Delta\hat{\rho}\Delta\hat{z}}{Re}\dfrac{\hat{V}_\rho}{\hat{\rho}+\psi}\Big|_{P}^{N}, \\ -\hat{a}_k\hat{V}_{z_k} + \hat{a}_{ke}\hat{V}_{z_{ek}} + \hat{a}_{k\omega}\hat{V}_{z_{\omega k}} + \hat{a}_{nk}\hat{V}_{z_{Nk}} + \hat{a}_{sk}\hat{V}_{z_{Sk}} + \\ + \hat{a}_K\hat{V}_{z_{kk}} + \hat{a}_P\hat{V}_{z_m} = \delta Eu\Delta\varphi\dfrac{(\hat{\rho}+\psi)^2}{2}\Big|_{k\omega}^{ke}\hat{P}\Big|_{P}^{K}, \\ \hat{b}_e\hat{V}_{\rho_e} - \hat{b}_\omega\hat{V}_{\rho_\omega} + \hat{b}_n\hat{V}_{\rho_n} - \hat{b}_s\hat{V}_{\rho_s} + \hat{b}_k\hat{V}_{\rho_k} - \hat{b}_m\hat{V}_{\rho_m} = 0, \end{cases} \tag{10}$$

System (10) is a discrete analogue of (8), and some of its coefficients include unknown functions discrete solution. To solution of system (10) may be found by means of the iteration procedure of calculating the sum of the unknown function of the last iteration and some increment value: $F^{S+1} = F^S + h_F^{S+1}$.

The results of the zero iteration can be taken as the solution of some asymptotic problem. The increment values matrix equation looks like:

$$\begin{pmatrix} A_V & A_P \\ B_V & 0 \end{pmatrix} \begin{pmatrix} \hat{\vec{h}}_V^{s+1} \\ \hat{h}_P^{s+1} \end{pmatrix} = \begin{pmatrix} 0 \\ f(\hat{\vec{V}}^S) \end{pmatrix}, \tag{11}$$

where A_V – coefficients before increments of velocity components in the Navier-Stokes equation in all nodes of discrete flow region, A_P – coefficients before increments of pressure in the Navier-Stokes equation, B_V – coefficients before increments of velocity components in the continuity equation.

Matrix (11) includes zeroes block because of the continuity equation, so this matrix determinant approaches zero and its inversion is difficult to reach. As the result, the system of these equations may be solved as follows: find the vector of velocity increments in the first equation of (11) and set it in the second equation (11). Hereby we can, in the first place, find pressure increments and then - velocity components increment.

IV. DISCUSSION

Further you can see some simulated results for the viscous incompressible fluid enforced and shear flow in the eccentric gap between the outer cone and the inner cylinder with input data presented in table III.

III input data definition

Pressure drop ΔP (Pa)	Frequency n (rpm)	Radius r (m)
3.5×10^5	400	0.1
Gap h_{01} (m)	Length L (m)	Eccentricity e (m)
2×10^{-4}	0.1	$0.2h_{01}$
Conicity β	Density ρ_f (kg/m^3)	Viscosity μ (Pa·s)
1.0×10^{-3}	894.5	0.62

The field of velocity axial component through lengthwise and through thickness both in the region of the maximal gap is presented in fig. 4. As we can see on figure 4 the maximum velocity value is reached on the lip of the channel.

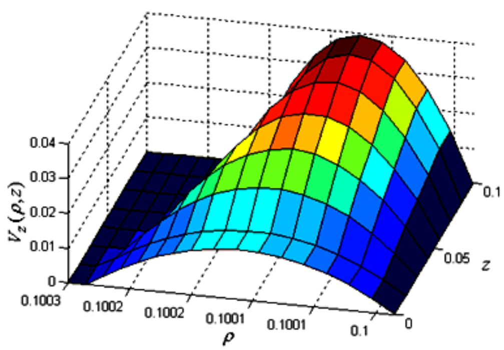

Fig. 4 field of the velocity axial component

Under constrains of the axial pressure difference and the inner cylinder rotation movement pressure in the axial direction has is nonlinear with extremum point as shown in figure 5(a). Also, in figure 5b you can see the pressure appearance in the tangential direction with the maximum point in the thinnest gap region, which defines the bearing capacity of lubricating layer.

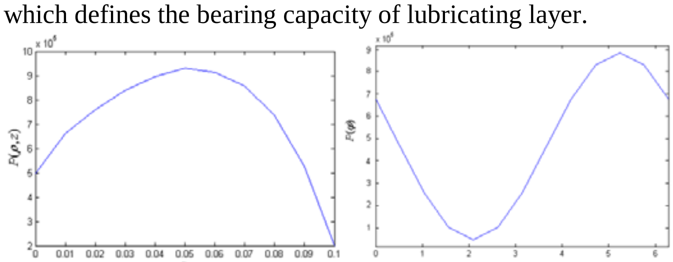

Fig.5 pressure function in the axial (a) and tangential (b) direction appearance

Reasoning from simulation results it is established that the eccentricity growth leads to nonlinear leakage growth. The conicity parameter growth in order of fixed gap on inlet of the channel leads to leakage drop.

Also, for the case of the enforced flow in the small nonzero conicity and the zero eccentricity region simulation the results were compared with the approximate solution of G. Nikitin [10], [11]. The result of this comparison is about 1% error in 1 degree region conicity, however, the error increases as the apex angle of the cone increase.

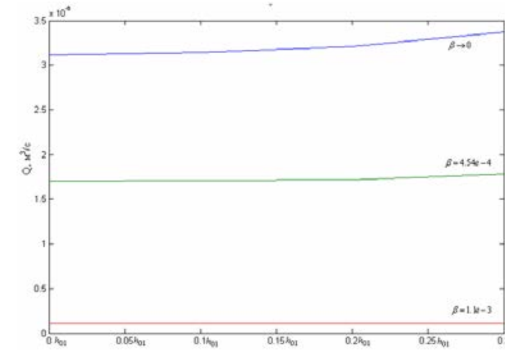

Fig. 6 leakage and conicity and eccentricity

For the case of coaxial and zero conicity flow region the results were compared with a well-known analytical solution and with other results simulated by the finite element method and the finite difference method specialized programs. Obviously, the FV method has smallest percentage of error and adequate results even on a crude mesh.

Thus the present mathematical model of three dimensional fluid flow was observed. On the basis of order-of-magnitude analysis the availability limits of widely used assumptions of hydrodynamic theory of lubrication [8] was determined: if the value of Reynolds number is more than 100 the inertial terms and the viscosity terms orders of magnitude are considered, plus if the conicity parameter β is more or equal to 10-3 all the velocity components has significant values which are depended on all three coordinates.

The simulation model is based on the finite volume approach. It was demonstrated that the finite volume method has smallest percentage of error and adequate results even on a crude mesh.

The adequacy of simulation model was confirmed in order of compare the simulation results with the results the particular results calculated by other authors and with analytical solutions for asymptotic cases.

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