From D. Bernouilli to M. Allais: a new paradigm in the making?

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***Abstract*—**This study revives an approach to elicit human preferences based on the stimuli-response procedure long forgotten. Fifty years ago a new theory, Elementary Catastrophe Theory,(E.C.T.), unfolding a unique Potential in our brain, provided the underlying dynamics needed to fulfill all the desiderata of the so-called school of Psycho-Physics(Weber-Fechner, 1860), seeking to make mathematical sense of the procedure above. The axiomatization of a self-measurement process brings a rationale to the empirical data away from any “a priori” assumption about human purpose. Besides fitting the major landmark criteria in the fields of Value and Utility this 5th degree symmetric polynomial exhibit a characteristic (Negative Schwarz’ Derivative) going a long way to solve controversies and remove roadblocks in the progress of Portfolio Theory. In the Annex we provide an algorithm finding the roots of polynomials of any degree based on the same feature. Finally with this newly-found Human Scale (Cardinal), grounded solely on the axiom of Non-Satiation, Free Energy could dislodge Entropy as a paradigm in this field.

***Keywords*—**Elementary Catastrophe Theory, Synapses, Poincare – Dulac, Psycho-Physics**,** Gradient of a Potential, Negative Schwarz’ derivative**.**

I. Introduction

One major controversy of the XX th century in Economic theory is clearly the shape of the Value function. The ones advocating a cardinal one i.e. a polynomial shape included J. Von Neumann, M. Allais and R. Aumann. The mainstream adopted the concave shape, based on the Logarithmic function initiated by D. Bernouilli, taken up again by the Psycho-Physicists Weber and Fechner. The subjective factor entering the representation took the form of an initial value Xo showing as a denominator of X convenient to make the argument of the Log homogeneous and of degree zero. That the latter went nowhere comes from the state of Micro-economics reached nowadays (see for example S. Keen “The debunking of Economics”, 2000). Many approaches tried to infer specific aim via utility (Expected utility or Maximization under constraints), to no avail all the while putting a veto on any use of polynomials since WWII. Finally out of despair, it took the shape of “Prospect Theory”, imagining superadditive weights taking the place and role of the value function, based on the reference point and a change of concavity that our polynomial exhibits too.

The recent Nobel laureate in Econ, Pr Thaler, claimed that he intended to use his prize money “irrationally”, probably quoting A. Greenspan with his” irrational exhuberance” of the marketplace, but for sure indicating that Academia had failed to explain rationally human behavior. Finally what’s more disturbing to me is the insistance of the M. Allais Foundation in following the precepts of “prospect theory” for the last 30 years, while M. Allais distanced himself from anything resembling the “expected Utility” hypothesis and approved only the “Pratt’s approach” because it was apprehending Utility alone by letting probability go to zero. I showed since 1987(TIMS/ORSA, Anaheim) that this lead rigorously to a polynomial representation. The use of probability has been likened to the “Aether”. In our approach we do not need the support of any Aether since the Gradient of potential is propagated through the infrastructure of our Physiology or Anatomy

Drawing on an old philosophical idea equating value with energy (Jean Ville, 1946), we identify the physiological potential as predicted (in the region of the Pons) at the level of the junction between the brain and the spinal cord with the intensity of the output to an external stimulation in a stimuli-response model. Since the novel approach of ECT takes to the letter the introduction from the outset of the observer as participant in the measurement process, we stress here the unique aim of the endeavor: To represent the interaction of the human being with his environment and letting him express it, therefore blending the subject with the object and completing what was missing from Quantum theory. In our polynomial the subjective factor Xo shows up in the numerator and leads to an objective valuation since of a finite degree.

II. The Measurement Process

The basic question we pose is: ‘How does the individual process two different inputs in a row, i.e., what, if any, is the scaling function taking the same value either when its argument is z or when we add the respective values of x and y such that $\boldsymbol{z=x+y}$?

$$\boldsymbol{v(z) = v(x+y) = v(x) + v(y)} \quad (1)$$

A special case of Cauchy Equation:

$$\boldsymbol{v(xoy) = v(x) + v(y)} \quad (2)$$

The fundamental equation that mathematicians have tried to solve in different contexts, constituting the basic functional equation for a theory of measurement. This formulation put the mental process into a materialistic frame known as the Grassmanian. The important point is that the addition rule on the argument side could be different from the one on the functional space. If we look for a physical process supporting such a algebraic view, we notice that, for about 50 years now, there exists a physiological theory axiomatizing a potential function as mentioned in the Introduction and processed throughout the brain by a network of about 10^{14} synapses predicting as the only stable functional a ratio scale represented by a fifth degree potential:

$$\boldsymbol{v = \frac{1}{5}x^5 + ax^3 + bx^2 + cx} \quad (3)$$

Corresponding to dimension 3 of the control space (basic rods or cones for visual perception) and dimension 1 (x) of the state space (input stimulus) in the terminology of Elementary Catastrophe Theory; the conditions for structural stability, together with homogeneity, give the functional:

$$\boldsymbol{v = \frac{1}{5}x^5 - \frac{2}{3}x_0^2x^3 + x_0^4x} \quad (4)$$

It has been a widespread request to introduce the observer into the observation process in order to do away with the uncertainty principle of quantum theory. In our case this is translated by a self-measurement process giving rise to an emergent unfolding of a specific thermo-dynamical potential known as Free Energy. Under the shape of a function called Swallowtail it is found in Physics in the theory of wave fronts of plasmas and as a singularity of Landau surfaces associated with certain Feynman diagrams. In classical geometric optics it is realized as a singularity of caustics. In regard to embryology, R. Thom propose to consider the extremities of the blastopore furrow as swallowtails:"...anatomy of the nervous system. First, the retraction of swallowtail, defining the motor horns, must occur in the spinal cord at the level of its junction with the brain; might this be the origin of the interlacing of the motor and sensory fibers at the level of the Pons? Second, an interpretation of the neurocoele as the 'support of infinity' of the external world is curiously confirmed in the cephalic extremity of this cavity. It is known to divide into two horns in the brain, the vestiges of the interior cavities of the peduncles of the optic vesicles during eye formation; therefore we can say that the neurocoele ends at its cephalic end in the retinas of the eyes, exactly that nervous zone specialized in a particularly precise simulation of distant phenomena." pp. 199 in Structural Stability& Morphogenesis.

This shows the striking difference between E.C.T. (Elementary Catastrophe Theory) and Fechner's law based solely on empirical data defined only on the positive quadrant of the diagram and for a limited range away from the origin. The closest attempt at representation by the simple square integrable function, without the symmetry toward the origin, is based on the so-called Neural-Quantum hypothesis. As a special case of the functional equation, let us consider:

$$\boldsymbol{f(x + y.f(x)) = f(x).f(y)} \quad (5)$$

Recursive or self-calling.

It is the source of involved work due to its relation to continuous groups (Sophus Lie) and geometric object.

Assuming the commutation of the multiplication on the R.H.S. and the convertibility of f(x) we should get, switching the roles of x and y:

$$\boldsymbol{x + yf(x) = y + xf(y) \rightarrow \frac{f(x)-1}{x} = \frac{f(y)-1}{y}} \quad (6)$$

Assuming continuity, the late ratio is a constant

$$\boldsymbol{C = 1/xo} \quad (7)$$

$$\boldsymbol{f(x) = 1 + x/xo} \quad (8)$$

Taking the logarithm of f(x) gives us a special solution of Cauchy Equation:

$$\boldsymbol{V(x) = log(1 + x/xo)} \quad (9)$$

Basis of theorizing in this field from Bernouilli to M. Allais passing by Fechner.

We will compare in the sequel the two functional, our quintic with the Fechner Law, showing their intersection in two different points on the positive quadrant, besides the origin, in order to suggest the small difference in their respective interpretation of the empirical data generated at the time of Fechner.

In the small, the big surprise is that the very same function, the Swallowtail is predicted to occur at the level of the synapse by the same theory (E.C.T.).We quote R. Thom directly:

"I would not have carried these very hypothetical considerations so far if they did not give a good representation of the behavior of nervous activity in the nerve centers, where an excitation (called a stimulus in physiology) remains relatively canalized until it results in a well-defined motor reflex; here the role of diffusion seems to be strictly controlled, if not absent. It is known(the Tonusthal theorem of Uexkull) that, when the associated first reflex of a stimulus is inhibited by artificially preventing the movement, there is a second reflex which, if inhibited, leads to a third reflex, and so forth. This seems to suggest that diffusion of the excitation is, in fact, present, but that, as soon as the excitation find an exit in an effective reflex, all the excitation will be absorbed in the execution of this reflex. This gives a curious analogy with the mysterious phenomenon of the reduction of a wave packet in wave mechanics."Pp.149-150.

Briefly, considering the general 5^{th}degree polynomial, we see on its graph in the positive quadrant that a necessary condition for self-duality is that the local Max and Min, respectively Min and Max to each other, are confounded at the same point. It is an analog to von Neumann's Mini Max concept. With symmetry toward the origin this condition becomes a sufficient one.

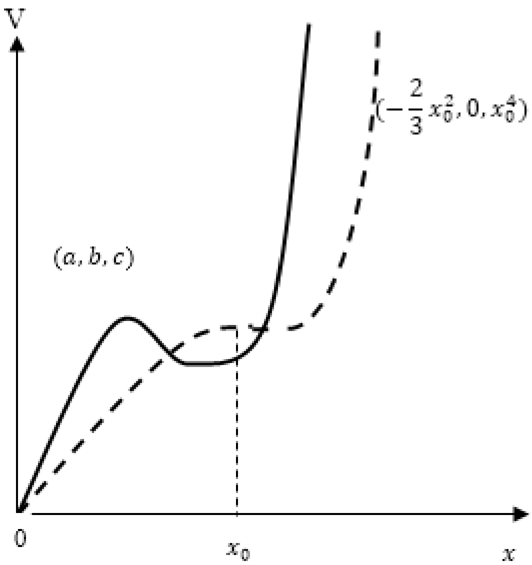

Fig.0 the general quintic polynomial

This was written over 44 years ago. By now many Quantum Scientists like Henry Stapp firmly believe the following empirical facts related in Jose R. Dos Santos latest book "La clé de Salomon"pp. 452.

"En général, ces sauts quantiques ne sont possibles que dans des espaces d'une largeur équivalente à sept atomes, mais, dans des cas rares, ils peuvent se produire dans des largeurs pouvant aller jusqu'a cent quatre-vingts atomes au maximum. Et bien, il se trouve, par coïncidence, ou peut-être pas, que la largeur de la fente synaptique est justement de cent quatre-vingts atomes. Or, comme les électrons sont constamment en mouvement, ils peuvent tenter des milliards de fois de traverser la membrane synaptique pendant le millième de seconde que met une synapse électriquement polarisée à s'activer, ce qui porte a 50% leur taux de réussite dans le tunnel quantique pour cette largeur. En étudiant attentivement la structure d'une synapse, on s'est aperçu que son architecture, encore une fois par une étrange coïncidence, est parfaite pour exploiter un effet de tunnel quantique. Lorsqu'une impulsion arrive à la synapse, la fente devient électriquement polarisée et c'est ce puissant champ électrique qui permet l'effet de tunnel quantique. C'est pourquoi on peut supposer que la fonction d'onde s'effondre dans les synapses lorsqu'une pensée se produit, et c'est de ce phénomène qu'émerge la conscience."

III. A Formalist Approach

Taking the derivative of our potential gives us our expression of a force field as follows:

$$\boldsymbol{v' = x^4 - 2x_0^2x^2 + x_0^4 = (x^2 - x_0^2)^2} \quad (10)$$

This algebraic picture is seen to be compatible with a Machian view of the force as the result of inertia there inducing inertia here; or in other words, depicting value as the expression of the subject's attractiveness in the presence of the object of his(her) attention.

The simplest way to make the result intuitively appealing without any recourse to E.C.T. axiomatizing is to consider the derivative of value as a mapping:

$$\boldsymbol{v' : R_e \to R_e^+}$$

Which has to reach its minimum, 0, being the absolute one in absence of any restrictive assumptions, at some point X_0. But at this point by its non-decreasing nature, its derivative, i.e. v" has also to cancel. Therefore X_0 is double root of v". By the further assumption of symmetry toward the origin, by the principle of insufficient reason to the contrary, its symmetric $-x_0$ has the same properties: therefore

$$\boldsymbol{v' = (x - x_0)^2(x + x_0)^2 = (x^2 - x_0^2)^2} \quad (11)$$

$$\boldsymbol{v = \frac{1}{5}x^5 - \frac{2}{3}x_0^2x^3 + x_0^4x} \quad (12)$$

Since

$$\boldsymbol{v(x) + v(-x) = 0 \Rightarrow v(0) = 0} \quad (13)$$

The black box of the brain is subject to determinism due to its interaction with its physical milieu, but the point of reference to which the input is compared, representing an aggregate of the psychic phenomenon and parameterizing completely the control space of dimension 3, is an unfathomable island of free will or volition.

In computer terminology, the likely picture is one of software built in the hardware. A relevant theory of the brain as a holographic process has been advanced by K.Pribram and D.Bohm, compatible with the singularity concept, an essential ingredient to E.C.T. The second essential ingredient being the concept of analytical continuation. They are essential in the sense of allowing the representation to go from the local to the global, hence to be complete.

It is not the appropriate place here to elaborate on the ambitions that the author (R.Thom) nourished toward his theory (E.C.T.). Nevertheless, the view I am unfolding constitutes, I think, the closest interpretation of its author and hence the best chance it has to fulfill its promises. To that effect, I will only mention the crucial new feature of self-duality valid with our functional because the control space (regular physical one) has dimension 3, compatible with the tunneling effect in quantum theory and the collapse of the wave function in the Von Neumann measurement process. Also compatible with it is the hidden parameter theory revived by D. Bohm.

Before leaving the domain of physics, we should mention the representation of free energy in thermo-dynamical scaling where the exponent 5 of the state variable around the critical point has been identified, corresponding to dimension 3 of the parameter space. This again is an added feature of E.C.T.: Connecting its ultimate purpose of geometrizing thermodynamics with the well-known scaling hypothesis in critical phenomenon. Finally, a succinct picture of the brain, functioning like a superconductor at 0°K (Kelvin), provides an additional impetus to the overall conjecture that such a theory may have closed the gap between the physical and the mathematical continua.

For our purpose, it is pertinent to notice that starting from a fifth degree polynomial the observation process ends up with a shape symmetric toward the origin and affording a simple interpretation under the necessary assumption of a non-decreasing function: the non-satiation axiom.

Last but not least the symmetrical shape put our formula within the scope of the theorem (Poincare – Dulac) on normal forms deployment around the critical point in case of resonance, completing the picture of “Morphic Resonance” à la R Sheldrake and the linkage of thermodynamics with the quantum level Hence the importance of verifying the empirical grounding of our quintic. This shows the ultimate appeal of such a functional, which could be guessed at, due to its simplicity, ever since the XIX century.

Although this polynomial checks the most well known criteria for utility functions, like the S.Ross’ one, and the cardinal impossibility result corresponding to Arrow famous Theorem both presented at Oslo in 1982 these are seen as sufficient conditions to qualify such a value function. I then returned to Pratt’s approach since it was a seminal achievement bringing to the fore the concepts of certain equivalent and risk premium together with the unique blessing of M .Allais although it represented the cornerstone of the belief in the concavity as a necessary condition for risk aversion. By a rigorous pursuit of Pratt’s approximation the XX th century controversy about the shape of the utility function was resolved to the benefit of a polynomial representation with the characterization of the negative Schwarz’ derivative as the measure of systematic risk aversion instead of concavity.

IV. Comparison With The State Of The Art

We focus our attention at the positive quadrant where we take the Taylor expansion of

$$\boldsymbol{log}\left(\mathbf{1}+\left|\frac{x}{x_0}\right|\right)=\frac{x}{x_0}-\frac{x^2}{2x_0{}^2}+\frac{x^3}{3x_0{}^3}-\frac{x^4}{4x_0{}^4}+\frac{x^5}{5x_0{}^5} \qquad \dots$$

Approximation valid for small $I\boldsymbol{x}I < Ixo I$. We seek its intersection with our quintic:

$\frac{x^5}{5}-\frac{2x^3x_0{}^2}{3}+\boldsymbol{x_0^4x}$ Setting $\boldsymbol{xo}=\mathbf{1}$ for sake of simplicity, we get:

$\frac{x^4}{4}-\boldsymbol{x}^3+\frac{x^2}{2}=\mathbf{0}$ Whose first solution $\boldsymbol{x}'<1$ is $\boldsymbol{x}'=\mathbf{2}-\sqrt{\mathbf{2}}=\mathbf{0.6}$ so the two curves, starting tangential from their common origin zero, have one dominating the other until $\boldsymbol{x}'=\mathbf{0.6}$ then invert their dominance until their next intersection

$$\boldsymbol{x}''=\mathbf{2}+\sqrt{\mathbf{2}}=\mathbf{3.4} \qquad (14)$$

Beyond 1, limit of validity of the Taylor expansion. It is pertinent at this point to recall that Fechner was influenced in picking the logarithmic by the choice of Bernouilli!

This could be compared to the early attempts of Cramer or the so-called the Engineering transformation curve. This latter is the subject of representation by A.C.M.S. (Arrow, Chenery, Minhas and Solow) in 1961.

Although the empirical attempts to find a fit stopped at the level of the cubic, even if they had guessed the fit with the symmetric quintic, a result which was possible a century earlier, there would have been no explanation by any measurement process nor behavioral interpretation neither to any link to the early attempts of the logarithmic shape of Psycho-Physics. Only the presence of E.C.T. could provide the underlying structure leading to the emergence of the quintic polynomial as an observable response to a stimulus to our senses, as sought after a century earlier by Weber and Fechner themselves. This constitutes a right turn of things since Psycho-Physics finally returned to D. Bernouilli the gift they borrowed from him.

V. The Missing Link

Going back to the last attempt by Pr Maurice Allais to axiomatize the utility function, I noticed 2 new axioms that Pr Allais added, after reviewing his 1952 experiments and analyzing the diverse reactions for over a generation, more precisely in “The expected utility hypotheses and the Allais paradox”,1979:

Axiom (VI): Axiom of invariance and homogeneity of the index of psychological value and

Axiom (VII): Axiom of cardinal is o variation.

After analyzing two cases, the log linear and the non log linear approximation one, he finds an excellent fit with a behavior verifying his axioms, up to approximation to the errors due to psychological introspection. About the same time, he was made aware of my first paper on the subject” The essential Tension”, which he approved wholeheartedly.

To make a long story short, there are two ingredients worth noticing:

First: the new conceptualization of ECT brings an essential feature under the name of critical point or Catastrophe point, fitting the bill for a reference point, but the novelty being that it shows on the numerator side, instead of the denominator one, retained since the time of Weber-Fechner.

Second: the basic problematic, also from the time of Weber-Fechner, as to the search for an origin and scale of the logarithmic shape inherited from the time of Bernouilli, precisely represented by his two new axioms, was addressed with success by ECT, by the hypothesis of “diffeomorphism”. This latter is a “smooth” and “reversible” feature of the representation. Hence the relevance of ECT to the problem at hand .Now, the final punch line to relate it to our problem resides in the methodology brought about by ECT. Indeed, one recalls from classical Analysis, two basic results:

1. Any function could be approximated to any degree (of approximation) by a polynomial.
2. The Taylor series expansion cannot be sure to converge. And even when it does, it’s not sure the convergence will be to the original function which led to the local Taylor expansion.

ECT formalism had, a decade earlier, solved this problem in an original fashion.

Inspired by his thorough correspondence with Waddington and Zeeman about physiology, (chreods…) R.Thom embarked on a research work leading to the emergence of potential functions in finite number, the famous seven catastrophes, corresponding to a combination of the dimensions of two spaces, the control and state spaces. It was the first time the control space was introduced from the outset in conjunction with the state space and I identified it with the physical space corresponding to the brain wiring. Hence I focused on the dimension 3 in order to have a chance to have a measurement by the human brain corresponding to an observable result.

The basic procedure of ECT states that, starting from the “germ”, highest term of the Taylor expansion at which one wishes to stop the approximation, a diffeomorphic change of coordinates leads to a UNIQUE representation with “determinacy”, i.e. exact polynomial. And this is possible only by the introduction of the catastrophe point which insures both the “structural stability” and the “analytical continuation” i.e. the passage from the local to the global. It was the striking answer to the problem dormant since the beginning.

How could this connect with the logarithmic shape? Taking the Taylor expansion of the neural quantal function:

Log(1+x)= x $-x^2/2 + x^3/3 - x^4/4 + x^5/5 + \ldots$

Retaining the “germ” $x^5/5$, corresponding to the dimension 3 by definition,since the germ has to have two degrees higher than the dimension of the control space, ECT leads to a function, called “self intersection curve”, defined by the cancellation of its first and second derivatives at the critical(catastrophic)point. This was the perfect topological translation of a non-decreasing function with its derivative reaching its strict minimum (zero) at the reference point, parameterizing the control space, by definition. Symmetry towards the origin completes the representation from – to + infinity.

With one stroke, we have an exact polynomial representing an observable phenomenon at the level of the brain. This was exactly what Pr Allais was predicting since before WWII, coming from Psycho-Physics. By the same token, it was immune to all the criticism that ECT had been exposed to, since they dealt with the applications in the social sciences, but never to the rigor of its mathematics. Of course, R. Thom had to pioneer new terminology, since there were many other definitions of stability. His theory was dubbed semi-open. He used Gateaux rather than Frechet Derivative and other concepts I will not encumber the reader with.

The Annex of the paper deal with an algorithm looking for the real roots of all polynomials of any degree solely based on the negative Schwarz derivative, in order to show the unifying power of such a concept.

The identification of the same quintic, the Swallowtail, as emerging at the level of the synapse opens new vistas to the functioning of the brain, and more importantly the basis of the case of resonance. In the meantime, the overall picture unlocks the door to an alternative to Weber-Fechner’s Just Noticeable Difference as defined in P.Buser last book to be the central problem of neurophysiology.

It’s remarkable that R.Thom himself thought that Consciousness emerges from the linking together of the right and left hemispheres of the brain and P. Buser still labelled the brain response as a “subjective” outcome.

VI. Extensions and Conclusions

My attempt should be seen in light of L. Brentano ultimate aim to criticize the objective theories of Value (Smith,Ricardo…Marx) after synthesizing the history of such studies since Aristotle. In a nutshell, this alternative to the logarithmic function put to rest the dichotomy entertained since WW2 between the “descriptive” approach and the normative, ”prescriptive” one, since the latter was built on the explicit premise that the former “doesn’t exist, and could not, even in Plato’ heavens”. This is typically illustrated by the prominence of the Prospect Theory seeking to explain people behavior thanks to hypothetical supper additive weights “in spite of the value function” having given up on the usefulness of any such function. The same highest order (the third) differential invariant, the negative Schwarz’ derivative, has been known as “extrinsic risk aversion” since J-M. Grandmont in his analysis of business cycles. It was well known to characterize the frontier of chaos in dynamical systems. Moreover, an alternative theory of general equilibrium in economy has been devised since the 70s’ of last century by Y. Balasko, using specifically the mathematics of ECT for the society as a whole, without the need for concavity. This should put to rest any discomfort that these new concepts are stand alone and on the contrary this shows that they bring a solid grounding to the new burgeoning macro-economics of the last generation.

Having an expression with a finite degree, our polynomial brings a unit of measurement, common to the consumer and producer, therefore letting the exchange process take place by value equivalence, not permitted under any concave representation. Even if one could handle only pathological cases, one would need still a rod of rationality to which one could compare his observations. See my latest paper presented in London 2017: “A synthetic approach to value as a standard of reference”.

The same Mathematics of ECT brings about another curve, again from cognitive introspective mode, tackling the 2-dimensional input space, known as the Umbilic Hyperbolic, again fitting the empirical data with 2 corresponding reference points, to replace the well-known hypothetical Cobb-Douglass curve in classical Economics

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Because the most salient points of E.C.T. stress that there is no need to the (1) Law of excluded middle, (2) Law of contradiction.

We have shown that it fulfilled Einstein intuition: “God does not play dice”. Indeed this stems directly from the message of ECT: “And the word was made flesh” which underlies the semantic trend of Thom’s endeavor. Specifically the paradox of the double slit experiment from Quantum Mechanics is dealt with at the synaptic junction, whose function in ECT terminology is precisely “to slice”. We have thus shown how people “speak their mind up”. No wonder then that the linguistic aspect, proper to humans, has been taken up by Jean Petitot for more than 40 years now. Moreover, the force unfolded, being the perfect square of the difference of two squares the one of the state variable with the one of the control variable has all the ingredients to fulfill another dream of Einstein, joining the basic forces of nature in one unified view, one where no scalars or constants interfere, in an expression of the force directly proportional to the distance of the two prime entities, coming from a deterministic formalization of Thermodynamics. Indeed, the independence of E.C.T. from its substratum, together with the resonance phenomenon, unused until now even in Physics, and the logic we uncovered, could provide an answer to the “curse of dimensionality” nagging all recent attempts for unification.

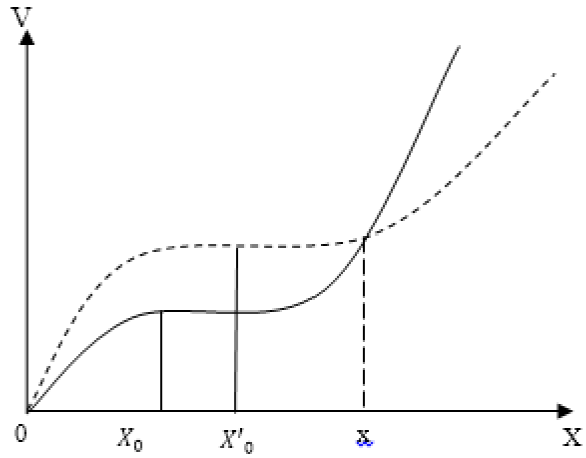

Fig.6 The exchange process by value equivalence

Always exists on convex slope given by:

$$\frac{1}{5}x^5 - \frac{2}{3}x_0{}^2x^3 + x_0{}^4x = \frac{1}{5}x^5 - \frac{2}{3}x'^2_0x^3 + x'^4_0x \qquad (4)$$

Or $\frac{2}{3}x^2 = x_0^2 + x'^2_0$:

The 2-dimensional (state space) value function known as the Umbilic Hyperbolic (in E.C.T):

$$x^3 + y^3 + axy - bx - cy = v(x, y)$$

(self-dual, also) with

$$a = 6\sqrt{x_0y_0}$$
$$b = 3\left(x_0^2 + 2y_0\sqrt{x_0y_0}\right)$$
$$c = 3(y_0^2 + 2x_0\sqrt{x_0y_0})$$

The curve down below represents the projection XOY of the intersection of the surface $V(x, y)$with the plane parallel to XOY at the level $V(x_0, y_0)$ hence an indifference curve between x and y . Since (25) $x^3 + y^3 = (x + y)(x^2 + y^2 - xy)$, one way to decrease the degree is to cancel $x + y$, thus it is the asymptotic direction.

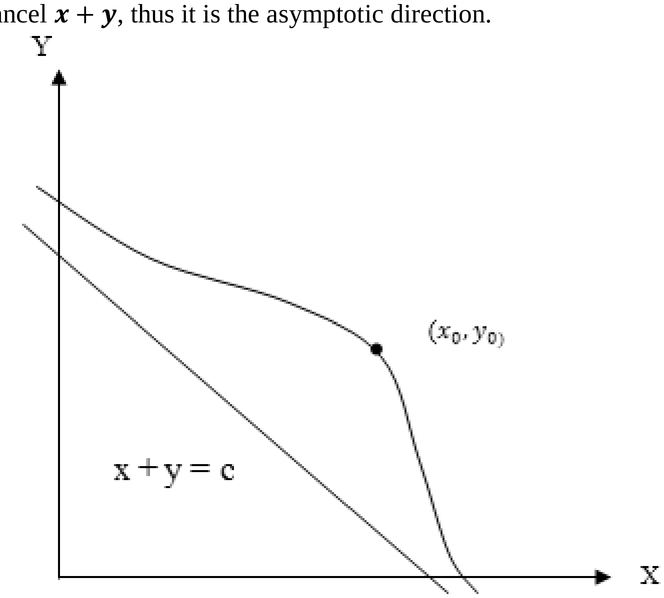

Fig.7 Indifference curve between X&Y

Annex:
Order bordering on chaos

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***Abstract*—**Following Newton's method for finding the zeros of a rational function, we prove that negative Schwarz derivative is intrinsic to the Taylor series' approach. Based on this finding we develop a new algorithm giving all real roots for a polynomial of arbitrary degree at any level of approximation. The implications of this approach for the complex roots are elaborated upon.

***Keywords*—**Newton's Method, Schwarz derivative, real and complex polynomials.

I. INTRODUCTION

It has been shown recently that, in general, for a real polynomial Newton' s method can and will yield (Hurley, [12]) several attractors besides the roots of the polynomial. The question by Smale whether there exists a "generally convergent iterative" algorithm for finding the roots of a polynomial of degree higher than three has been answered negatively by McMullen in 1987. These results have refocused the debate on the importance of the choice of the initial points and the location and orbits of the critical (roots of the derivative) ones.

Indeed it has been known, since Barna [1951-1961] that the set of "bad" starting points for a polynomial with real roots, and real coefficients is a Cantor set with Lebesgue measure zero. This set includes the above mentioned critical points. Furthermore on has (see Hurley and Martin [13]) that there are real polynomials whose Newton Method has an attractive "cycle" of any order thereby they must have non-real roots. Moreover we know from P. Fatou [8] that for a rational function R of degree>=2, if R has an attractor, then its immediate basin of attraction has at least one critical point of R. Since we also know that for real roots polynomial, its derivative' s roots are also real and alternate with those of the initial polynomial, we decided to investigate the matter from first principles i.e. to study the local properties of the Taylor expansion around any root to reach some necessary conditions to be fulfilled by any iterative algorithm.

An immediate consequence from the above is the necessity to reduce the set of polynomials and/or to device non-purely iterative methods insuring the judicious choice of initial points. The result we obtain is that the very definition of convergence leads us to the subset of real polynomials of any degree having all real roots and to a general algorithm finding every root. Of course, the convergence of Newton Method, "almost everywhere", has been established for the subset considered above by (Cosnard and Masse,[6]) confirming Barna earlier constructive proof but no general result with systematic implementation had been proposed that we know of. Thanks to the medium (Schwarz derivative) by which we derive our result a straightforward explanation of the geometry of the roots becomes transparent, namely that the critical line is the Boundary of the basins of attraction of the roots of the polynomial.

This has the advantage of shedding some light on the method of conjugation routinely adopted, for example by Henrici and McMullen, most recently, although it started with Cayley and Schroder in the 19thcentury mainly as a tool to obtain a convenient form for calculations.

II. A NECESSARY CONDITION

Starting from the definition of convergence near the root of a function F : $|FF''| \prec |F'|^2$ we consider the two cases possible:

$FF'' > 0$. In this case it is necessary that

$\left(\frac{F''}{F}\right) \prec \left(\frac{F'}{F}\right)^2 \prec {}^3/_2 \left(\frac{F'}{F}\right)^2$ i.e. Schwarz derivative of $\int$ F has to be negative.

$FF'' < 0$. In such case the same result obtains trivially.

Since the root could be located anywhere this local requirement is also a global one. As it turns out this is the only property needed to prove Singer conjecture equating the negativity of the Schwarz derivative with the reality of the roots of the derivative (see the appendix).

We show in the following that the consideration of higher levels in the Taylor expansion around the root to take into account all terms in the third derivative of the function adds only one requirement namely that the Schwarz derivative of F itself be negative, i.e. that its derivative polynomial has real roots. This recoups what we already know about the concatenation of the respective roots from classical analysis.

Let us do the following classical expansion of F around the root:

$$F(\alpha+\varepsilon) \equiv 0 = F(\alpha) + \varepsilon F'(\alpha) + \frac{\varepsilon^2}{2} F''(\alpha) + \cdots \tag{1}$$

Or $\frac{\varepsilon^2}{2} F''(\alpha) + \varepsilon F'(\alpha) + F(\alpha) = 0$ with

$\Delta = F'^2 - 2F''F$.

Assuming $\Delta > 0$, we have 2 roots.

$$\varepsilon_1 = \frac{-F' + \sqrt{F'^2 - 2FF''}}{F''} \approx -\frac{F}{F'} \text{ the trivial one: } \varepsilon_1 = 0 = F(\alpha) \tag{2}$$

$$\varepsilon_2 = \frac{-F' - \sqrt{F'^2 - 2FF''}}{F''} \approx +\frac{F}{F'} - \frac{2F'}{F''}$$

The reader could verify the ordering of the roots by the norm of the Taylor series:

$|\varepsilon_2| > |\varepsilon_1| \forall$ signs of F, F', F''.

The same logic of convergence for Newton $\left(e \prec 0 \Rightarrow \frac{|FF''|}{F'^2} \prec 1\right)$ applies toε_2 .

For any increase in α should correspond a decrease (corresponding) in ε_2 therefore $\varepsilon'_2 = \frac{d\varepsilon_2}{d\alpha}$has to be negative or:

$$\varepsilon'_2 = \frac{F'^2 - FF''}{F'^2} - 2\frac{F''^2 - F'F'''}{F''^2} = 1 - \frac{FF''}{F'^2} - 2 + \frac{2F'F'''}{F''^2} = -1 - \frac{FF''}{F'^2} + 2\frac{F'F'''}{F''^2} \tag{3}$$

But$\Delta \succ 0 \Rightarrow F'^2 \succ 2FF'' \Leftrightarrow \frac{-FF''}{F'^2} \succ -\frac{1}{2}$

Then

$$\varepsilon'_2 \succ -1 - \frac{1}{2} + 2\frac{F'F'''}{F''^2} = \frac{F'^2}{F''^2}\left[-\frac{3}{2}\left(\frac{F''}{F'}\right)^2 + \frac{F'''}{F'} + \frac{F'''}{F'}\right]$$

If the first 2 terms in the bracket were positive i.e. the Schwarz derivative of F positive, then $\frac{F'''}{F'}$ would also be positive and the whole bracket and the r.h.s of the inequality positive. This would mean than $\varepsilon'_2 \succ 0$ which contradicts the hypothesis of convergence to the root therefore Schwarz derivative of$F$ is negative implying that all roots of F' are real and alternate with those of F, following classical analysis.

Since the analysis extends by induction to all subsequent derivatives and they are in finite number, we have exhausted the set of necessary conditions. In order to see how this could be sufficient for our purpose to build an algorithm converging on each and every solution we have the following diagram:

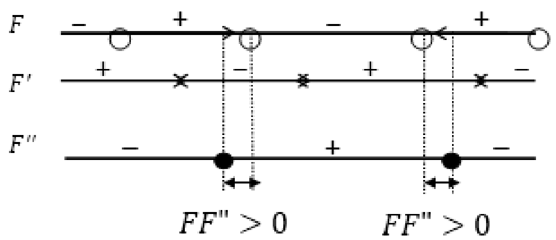

Fig. 1 Step by step view of the algorithm

III. THE ALGORITHM

Starting from the derivative before last we arrive at (binomial), we see that its roots constitute the boundaries of the segment within which lies one root of its primitive polynomial, i.e.the cubic. By elementry methods (secant or Newton or otherwise) this root can be converged to as closely as needed. The remaining two roots of the cubic could then be computed easily by plugging back the value of the iterated root into the expression of the cubic and solving the resulting quadratic expression. It is clear now how we can proceed recursively to arrive at all the roots of all the derivatives because of the alternation of the roots between each polynomial and its derivative and primitive polynomials.

While this might seem as a tedious work nevertheless this proves constructively the sufficiency of the procedure. Notice that we can take a big shortcut and devise a smarter algorithm by remarking that the roots of F and F'' involved in the procedure belong two by two to the same segment namely the one constituted by the roots of F'. Therefore the roots ofF'' constitute a convergent initial point following Newton Method for obtaining the roots of$F$. The question whether to start at the right or the left of the roots ofF'' is easily solved by taking the direction given by $FF'' > 0$ as the diagram shows. The remaining two roots of F at the edge of its spectrum are disposed of as suggested earlier by solving the binomial arrived at when we plug back the

values of the iterated roots. While the comparison of ε_2 and ε_1 might have raised the hope of ε_2

being an alternative to ε_1, the expression $\varepsilon'_2 = -1 - \frac{FF''}{F'^2} + 2\frac{F'F'''}{F''^2}$ is the sum of 3 negative terms: $FF'' > 0$ by choice, $F'F''' < 0$ by being close to thee roots of $F'' = 0$. Therefore $\varepsilon'_2 \prec -1$ which is the characteristics of a repellent point, hence unstable.

Now then we displaced the problem from F to one concerning F''. We can repeat the process to reach either the binomial or the monomial and work the process backward.

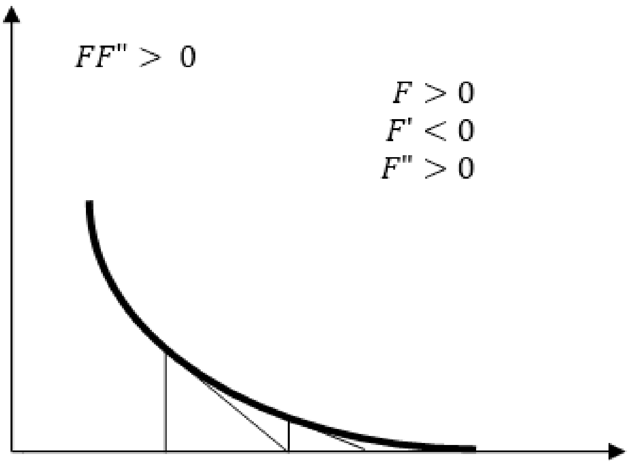

Fig.2 Convergence case 1

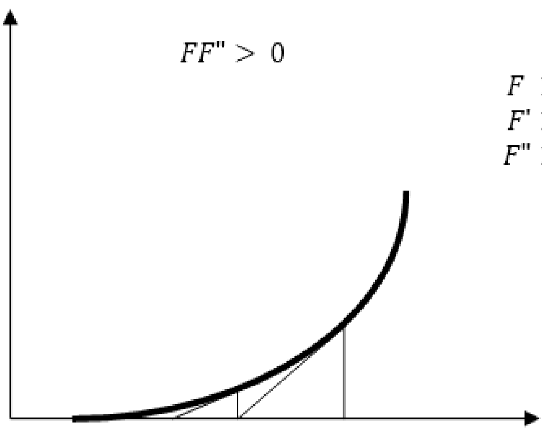

Fig.3 Convergence case 2

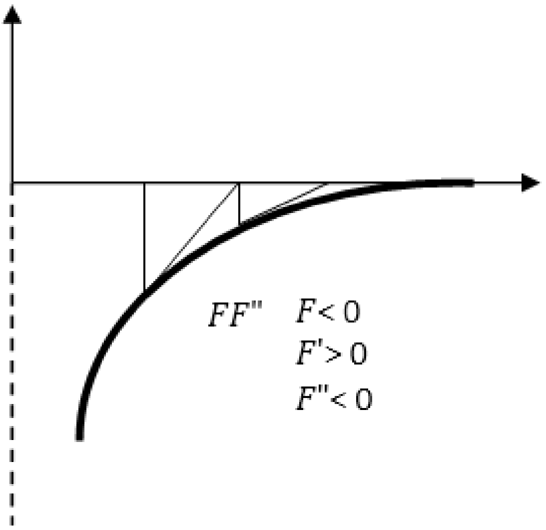

Fig.4 Convergence case 3

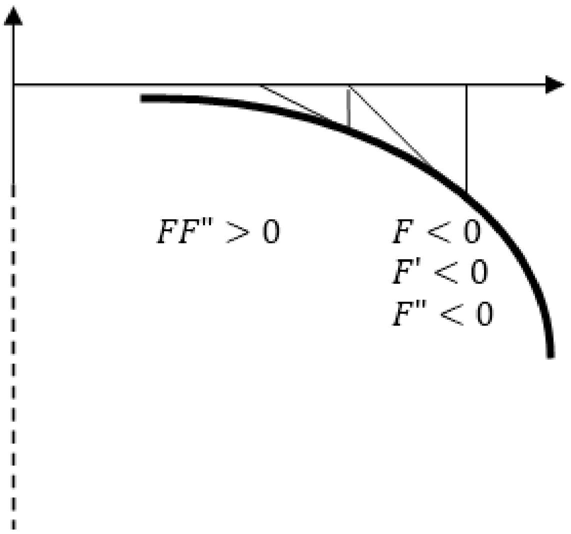

Fig.5 Convergence case 4

IV. DISCUSSION AND CONCLUSION

While we made all the above seems elementary, because of the features of the polynomial (proper and entire) it is interesting to notice that for entire functions E with an infinite number of roots, the same alternation and reality of the roots obtains only for a subset whose "order" is within a certain limit. In this context there is a conjecture dating back to the beginning of the century and proven recently [17] whereby the roots of E ″ might not be all real while those of E ′ and other derivatives could be still all real at the same time. However if the roots of E ″ are all real then all the roots of all other derivatives will also be real and alternating, i.e. behaving exactly like polynomials. While the practical use of entire functions with an infinite number of roots might be questionable, although the Bessel functions are a prominent example to the contrary in Physical Sciences, we mentioned the conjecture to show the key role the roots of the second derivative play and the natural generality of the procedure outlined above. This should come as no surprise for the reader familiar with the literature since the critical points of the Newton Method include the zeros of E ″ and their iteration may lead to a periodic attractor of the Method. This shows us also the caution we should show in the proof of Singer conjecture for polynomials when the number of roots tend to infinity. One other area where the same roots are involved is the general algorithm that McMullen [15] found for the cubic polynomial which he replaced with a transformed one, whose change of concavity coincide with the roots of the original polynomial, to which he then applies Newton Method. The proof of the limit of his method to the third degree polynomial actually relies on the invariance of the cross ratio due to the conjugation by Moebius transformation. The connection with our method is precisely that the Schwarz derivative expression is invariant to a Moebius transformation of its argument. But in our case it comes naturally from the inception of the Taylor procedure to define the set of functions amenable to the limits of its power of application. In the general literature conjugation by conformal mapping is applied either to the polynomial or to the algorithm and certainly on the complex domain. In other words we have shown that the reality of the roots depends solely on the reality of the coefficients of the polynomial considered. This contains the promise that we can extend the above results to general complex polynomials and their roots as soon as we devise the complex equivalent to the negative Schwarz derivative since the framework does not involve more than the Taylor expansion. It is good to remember at this juncture that Moebius transformation is defined in general on the complex domain and that Gauss result extends the concatenation of the roots to the convex hull. Also more dimensions of the argument can be tackled within our present frame due to the natural extension of the concepts at work, see Ahlfors [1].

APPENDIX

The following are excerpts basically from the famous paper by David Singer about "Stable orbits and bifurcation of maps of the interval" as reported in Collet &Eckmann [5]. Among the best understood endomorphisms are the rational functions. In particular it is shown that, for differentiable maps of class at least C^3, there are a finite number of periodic sinks, i.e. periodic orbits which attract iterates of nearby points, and that each attracts some critical point of the endomorphisms.

The above assumes the hypothesis of everywhere negative Schwarzian derivative. To prove its necessity, D. Singer brings out a counter-example of an endomorphism exhibiting two distinct attracting periods. This example, a polynomial of degree four, contradicts assertions made in the literature about bifurcations. The explanation for this lies in the fact that there is a type of bifurcation of periodic orbits not commonly encountered in simple examples, which can nevertheless not be generically avoided without restricting the class of allowable endomorphisms. The assumption of negative Schwarzian derivative is sufficient to exclude this bifurcation. As seen in the main text, this condition is a necessary implication, intrinsic to any Taylor series expansion for entire functions.

While restrictive, it is satisfied by all the commonly studied models of population dynamics. It replaced successfully the assumption of concavity widespread until recently not only in biological models but also in Business cycles, in the definition of risk aversion in decision utility functions, in dynamic programming as well as in Julia Conjecture [14] as D. Singer has shown in his paper. One can only measure its importance fully by the pervasiveness of the use of Taylor's series in applied mathematics.

When applied to the notoriously important case of polynomials, this translates into the necessity of all roots for the polynomial and its derivatives to be real. The algorithm is made possible by the finite number of roots involved.

To the extent that chaos theory has found a paradigm in the divergence of Newton's Method, we have put an order into a process bordering on chaos, since our algorithm insures a local and global convergence by a judicious choice of initial points. The invariance of the expression of Schwarz derivative to Mobius transformation is a good omen for the prospect of extending the approach to the complex field.

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BIBLIOGRAPHY

[1] L.V. Ahlfors, Lectures on Quasiconformal Mappings, Van Nostrand, 1996.

[2] I.N. Baker, Some entire functions with multiply connected wanderings domains, Erg. Th. Dyn. Sys. 5, 1985, pp. 163 -169.

[3] B. Barna, Uber das Newtonsche Verfahren zur Anncherung von Wurzeln algebraischer Gleichungen, Publ. Math. Debrecen, 2, 1951, pp. 55-63.

[4] B. Barna, Uber die Divergenzpunkte des Newstonschen Verfahrens zur Bestimmung von Wurzeln algebraischer Gleichungen ,I, II, III, Publ. Math. Debrecen, 3, 1953, 109 -118, Publ. Math. Debrecen, 4, 1956, 384 - 397, Publ. Math. Debrecen, 8, 1961, pp. 193-207.

[5] P. Collet and Eckmann, Iterated map on the interval as Dyn. Systems, Birkhauser, Basel, Boston 1980.

[6] M. Cosnard, C. Masse, Convergence presque partout de la méthode de Newton, CRAS Paris, t. 297 (14 novembre 1983), Série I – 549.

[7] A. Douady, J. H. Hubbard, on the dynamics of polynomial-like mappings, Ann. Sci. Ecole Norm. Sup.4^{e} série, t. 18, 1985, pp. 287-343.

[8] P. Fatou, Sur l'itération des fonctions transcendantes entières, Acta Math. 47, 1926, pp. 337-370.

[9] J. Guckenheimer, Endomorphisms of the Rieman sphere, Proc. Of the Symp. Of Pure Math. V, XIV (1968), pp. 95-123.

[10] J. Guckenheimer, On bifurcation of maps of the interval, Invent. Math. 39 (1977) pp. 165-178.

[11] P. Henrici Applied & computational complex analysis, vol. 1, 1998, wiley Interscience.

[12] M. Hurley, Multiple attractors in Newton's Method, Ergod. Th & Dynam. Sys. 6, 1986, pp. 561-569.

[13] M. Hurley, C. Martin, Newton's Algorithm and chaotic Dynamical systems, SIAM J. Math. Anal. Vol. 15 N° 2, March 1984, pp. 238-252.

[14] G. Julia, Mémoire sur l'itération des fonctions rationnelles, J. out Math. pures et appliquées, 4 C. (1918), pp. 47-245.

[15] C. McMullen, Families of rational maps and iterative root-finding algorithms, Annals of Math. 125, 1987, pp. 467-493.

[16] C. H. Pommerenke, On conformal mappings and iteration of rational functions, Complex Variables, 1986, Vol. 5, pp. 117-126.

[17] T. Sheil-Small, On the zeros of the derivatives of real entire functions and Wiman's conjecture. Annals of Math. 129 (1989), pp. 179-193.

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REFERENCES

[1] André Orléan: L'empire de la valeur, Points, Économie 2013 Ed. du seuil
[2] D.W.Griesinger: The Physics of Motivation and Choice. IEEE Transactions on Systems, Man and Cybernetics, vol.SMC-8, n°12, Dec.1978.
[3] Francois Jacob: La logique du vivant, Gallimard,Paris 1970.
[4] G.L. Schackle : Decision, order and time/ Epistemics and Economics, Cambridge 1972.
[5] G.Menges: Information, Inference and Decision. (Theory and Decision Library, vol. 1) Dordrecht, Boston 1974.
[6] Guy Orban:La Vision, mission du cerveau, le consinaugurales du College de France 2007.
[7] Henry P. Stapp: Mindful Universe, Quantum Mechanics and the participating observer, Springer, 2014.
[8] J.M. Grandmont :On endogeneous business cycles.IMSSS Tech. Report # 438.1983
[9] Jean Petitot : Logos et Theorie des Catastrophes, Colloque de Cerisy sur l'œuvre de R.Thom, Ed. Patino, 1988.
[10] Jean Petitot: in "Issues in contemporary Phenomenology and Cognitive Science" Stanford University Press 1999.
[11] Jean Petitot: in Mathématiques et Sciences Humaines Numéro Spécial linguistique et Mathématiques – III, 1982.
[12] Jose Rodrigues Dos Santos : La clé de Salomon, Ed. Herve Chopin, 2014.
[13] Joseph Delboeuf : Etude critique de la loi psychophysique de Fechner, l'Harmattan, 2006.
[14] K. Pribram: Languages of the Brain. Englewood Cliffs, N.J.Prentice Hall, '70
[15] Keen Steve, Fullbrook Edward, "Improbable, incorrect or imposible? The persuasive but flawed mathematics of icroeconomics", in A guide to what's wrong with Economics, London, Anthem Press, p.209-222. 2004
[16] Keeney Ralph: Value-focused thinking. Harvard U.Press,Cambridge, Massachusetts. London, England 1992
[17] L.C. Jeffrey: The logic of decision. 2 nd Ed. University of Chicago Press 1983
[18] Leinfellner W. & Booth E.: The naturalistic versus the intuitionistic school of values in Allais & Hagen op.cit. 303-332, 1979.
[19] Lujo Brentano,"Die Entwickelung der Wertlehre" in Sitzungsberichte der Koniglich Bayerischen Akademie der Wissenschaften.Philossophisch-philologische und historische Klasse, Band XXIV, 3. 1908
[20] M. Allais and O. Hagen: Expected Utility Hypothesisand Allais Paradox. (Theory and Decision Library, Vol. 21) Dordrecht, Reidel 1979.
[21] Paul Davies: The mind of God. Penguin Book,1993
[22] Philip Kitcher: The nature of mathematical knowledge.Oxford University Press, 1984.
[23] Pierre Buser:Neurophilosophie de l'Esprit, Odile Jacob, 2013.
[24] R.Thom :Structural stability and Morphogenesis, Addison-Wesley.1989
[25] R.Thom: Esquisse d'une Semiophysique, InterEditions, 1991.
[26] Roger Penrose : Les cycles du temps, une nouvelle vision de l'univers. Odile Jacob, 2013.
[27] Roger Penrose : Les ombres de l'Esprit.A la recherche d'une science de la conscience, InterEditions.1995.
[28] Rupert Sheldrake: The Science Delusion Coronet 2013.
[29] S.C. Tsiang:"The rationale of mean-standard deviation analysis,skewness preference & the demand for money. A.E.R. p:354-371, june1972
[30] S.Ross: Some Stronger Measures of Risk Aversion in the Small and the Large with Applications, Series n°39. Yale University 1979.
[31] W. M. Saade:"A synthetic approach to value as a standard of reference", in Proceedings of FTC conference. London july 18-20, 2017
[32] W. M. Saade:"Toward a convergence between utility and value through a polynomial representation", Proceedings of FTC conference in San Francisco, dec 2016. http:ieeexplore.ieee.org/xpl/most Recentissue.jsp?punumber7802033
[33] W.M.Saade:"An alternative vision to Weber-Fechner Psycho-Physics, WSEAS, NAUM joint conference ICANCM, Malta..SLIEMA, August 2015.
[34] W. M. Saade:" The identification of Negative Schwarz' derivative with systematic risk aversion", same publication as following reference below.
[35] W.M.Saade:"A Phenomenological approach to Value", Post conference proceedings of the 18th World Multi-Conference on Systemics, Cybernetics and Informatics, Orlando, Florida, 2014.
[36] Weibull J.W.: A dual to the Von-Neumann-Morgenstern theorem. Journal of mathematical Psychology 26 :191-203, 1982
[37] Y. Balasko (1978): Economie et Théorie des Catastrophes, in Cahiers du Séminaire d'Econométrie, #19.