Stability analysis and Hopf Bifurcation in an IMC structure

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***Abstract*—In this paper we analyze the dynamic of an IMC structure system. We prove the existence of a closed trajectory (limit cycle) that is a solution of a differential system that represent the ICM structure, to do this, we compute the first Lyapunov Coefficient using the Kuznetsov Theorem obtaining a Hopf bifurcation. Numerical simulation and experimental results are included to give a full comprehension of the system. The existence of a limit cycle obtained in the results suggests that the implementation of advanced control laws are required. In this paper we analyze the dynamic of an IMC structure system. We prove the existence of a closed trajectory (limit cycle) that is a solution of a differential system that represent the ICM structure, to do this, we compute the first Lyapunov Coefficient using the Kuznetsov Theorem obtaining a Hopf bifurcation. Numerical simulation and experimental results are included to give a full comprehension of the system. The existence of a limit cycle obtained in the results suggests that the implementation of advanced control laws are required.**

***Index Terms*—Internal Model Control; Stability analysis; Andronov-Hopf bifurcation; First Lyapunov coefficient.**

I. Introduction

In the past years, several control techniques has been developed in order to solve the problem of variables manipulation in processes. However, limitations like the non-linearity of the system, poor modeling and the use of linear controllers have to be solved before selecting the controlling technique. One of the method that had increased in popularity is the Internal Model Control (IMC) (see Ref.[1]). The reason of this is because the lack of expensive elements for its implementation, its analogy to the traditional PID controller and the simplicity of the system, makes it a viable low-cost option for the budget of a non-lineal controller. Despite of its benefits, the lack of robustness of this structure makes it inadequate for certain processes and it is often combined with other controllers to compensate its limitations.

According with some experimental results the main issue of using only the IMC method is that after reaching the set point it can turn very sensitive to perturbations and changes in the reference value (see Ref.[2]). This would be seen in form of maintained oscillations, but not necessarily in the output of the process. The idea of the performing oscillations suggest that Hopf Bifurcations are existing within the system.

The purpose of the following paper is to analyse the dynamics and describe the stability of a non-linear system with an Internal Model Control. The equation of the plant in this work is given, but the importance of study lies in the structures of the IMC.

Hopf Bifurcation control has been vastly studied and applied in different situations (See Ref.[3]). For example, in Power Systems the Hopf Biffurcation Control Techniques are used to damp oscillations related to instabilities (See Ref.[4]). Another application is on Biological systems, where these techniques are used in parasite control on crops (See Ref.[5]). An additional usage of these, exists on mechanical systems, where they have been used to stabilize maglev trains (See Ref.[7]). The idea of studying the stability in the IMC structure consists in, understanding the dynamics of the process, in order to evaluate its potential to use the Bifurcation Control later on.

The outline of this paper is as follows. In section II is provided the description of the system with the equations that models it. The local stability and Hopf bifurcation analysis is computed on section III. The qualitative analysis of the Hopf bifurcation iis presented using the Kuznetsov theorem (See Ref.[9] pp. 177-180). The section IV is devoted to the numerical simulations and experimental results, including the detection of the limit cycle with its phase portrait and his time series graphs. Finally, we provide a conclusion of obtained results in the last section V.

II. System Modelling

The structure of an IMC can be seen in the figure 1. As it was mentioned before, it became popular because of the analogy of this structure and the classical closed loop. The reason of this is that when the model is perfect, both methods are equivalent.

The traditional procedure to develop this arrangement consists in the next steps:

- **Find the transfer function of the model, usually a linear one.**
- **Obtain the controller using the inverse transfer function of the model.**
- **Design for a filter in order to make the controller to be pole dominant i.e. proper.**

So, following this steps and knowing that a the ecuation of a first order transfer function of the following form $\frac{c}{\tau s+1}$ is $\dot{x} = -\frac{1}{\tau}x + \frac{c}{\tau}u$ the system can be written like:

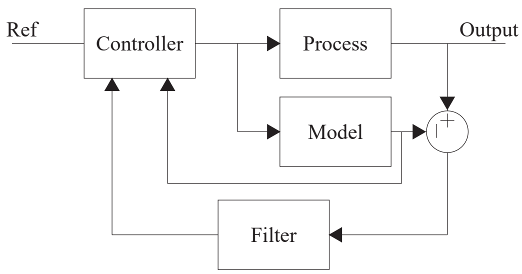

Fig. 1. IMC classic structure.

$$
\begin{aligned}
\dot{X}_p &= \mu X_p - X_p^3 + C_p U, \\
\dot{X}_m &= -K X_m + C_m U, \\
\dot{X}_f &= -a_f X_f + a_f (X_p - X_m), \\
U &= \frac{K X_m + a_f X f - a_f (X_p - X_m) - a(X_p)}{C_m}
\end{aligned} \qquad (1)
$$

Where X_p, X_m and X_f are the states of the process, the model and the filter. The control law is represented by U. The constants C_p and C_m are the input gain of the controller to the real process and the model. The parameter μ is the intrinsecal parameter of the process. K and a_f are the time constant of the model and the filter. Finally C_m and a are the output and the input gain of the controller.

III. Local stability and Hopf bifurcation analysis

In order to analyze the system, some mathematical considerations were applied. First, the equations were translated at the origin. Using the following transformation.

$$
\begin{aligned}
\xi_1 &= X_p - X_m \\
\xi_2 &= X_f \\
\xi_3 &= X_m - X_f - y*
\end{aligned} \qquad (2)
$$

The advantages of this transformation is that a few of this coordinates represent different elements of the structure that can be measured. For example, ξ_1 is the model error, i.e. the inaccuracy of the model equation to describe the process and ξ_2 is the output of the filter.

$$
\begin{aligned}
\dot{\xi}_1 &= \mu(\xi_1 - \xi_2 + \xi_3) - (\xi_1 - \xi_2 + \xi_3)^3 + \frac{C_p K}{C_m}(-\xi_2 + \xi_3) + \\
&\quad + a_f \xi_2 \left(\frac{C_p}{C_m} - 1 \right) + a_f \xi_1 \left(1 - \frac{C_p}{C_m} \right) + a \xi_3 \left(1 - \frac{C_p}{C_m} \right) \\
\dot{\xi}_2 &= -a_f \xi_2 + a_f \xi_1 \\
\dot{\xi}_3 &= -a \xi_3
\end{aligned} \qquad (3)
$$

For all the values of the parameters, the transformated system has the only equilibrium at $\xi = (0,0,0)$. With this result the the jacobian evaluated at the origin has the form

$$
\begin{pmatrix}
a_f - \frac{C_p a_f}{C_m} + \mu & \frac{C_p(a_f - K) - C_m(a_f + \mu)}{C_m} & \mu + \frac{C_p(K-a)+a}{C_m} \\
a_f & -a_f & 0 \\
0 & 0 & -a
\end{pmatrix} \qquad (4)
$$

with the characteristic equation

$$
\lambda^3 + \left(\frac{C_p a_f}{C_m x} - \mu + a \right) \lambda^2 + \left(\frac{C_p a_f (K + a) - C_m a \mu}{C_m} \right) \lambda + \frac{C_p a_f K a}{C_m} = 0 \qquad (5)
$$

To find the relationship between the parameters and the Hopf Bifurcation critical frequency it was used the Routh-Hurwitz criterion. Solving for μ, two options were obtained.

$$
\begin{aligned}
\mu_1 &= \frac{C_p a_f}{C_m} \\
\mu_2 &= \frac{C_p a_f (K + a) + C_m a^2}{C_m a}
\end{aligned} \qquad (6)
$$

If μ_2 is subsituted in the characteristic equation, the obtained factorization gives two positive roots that leads to exponential growing, hence this solution is discarded.

On the other hand, μ_1 gives the hopf bifurcation factoriazation, in which critical frequency can be easily obtained, giving the following value.

$$
\omega^2 = \frac{C_p K a_f}{C_m} \qquad (7)
$$

Now with the obtained result is possible to check the Liu conditions(See Ref.[8]). First the coefficients of the caracteristic polinomial evaluated on the critical value of the parameters must be positive.

$$
\begin{aligned}
P_0(\mu_1) &= 1 \\
P_1(\mu_1) &= \frac{2 C_p a_f}{C_m} + a \\
P_2(\mu_1) &= \frac{K C_p a_f}{C_m} \\
P_3(\mu_1) &= \frac{K C_p a_f a}{C_m}
\end{aligned} \qquad (8)
$$

Second, the derivative of the Hurwitz determinants, evaluated on the critical value of the parameters must be non-zero.

$$
\frac{d H_2(\mu_1)}{d\mu} = -\frac{K C_p a_f + C_m a^2}{C_m} \qquad (9)
$$

Lemma. For $\mu = \frac{C_p a_f}{C_m}$ and $\omega^2 = K\mu$ there is a set of three parameters for which the linealization has a couple of pure imaginary proper values and a non-zero real one.

Finally, acording to Kuznetsov theorem (See Ref.[9])the first Lyapunov coefficient is.

$$
L_1(0) = -\frac{3\sqrt{K\mu}}{2(2a_f^2 + K\mu)} \qquad (10)
$$

From the results is concluded that, the Lyapunov coeficient is clearly negative for all positive value of the parameters. Thus, a Hopf bifuration exists and it is nondegenarate and always supercritical.

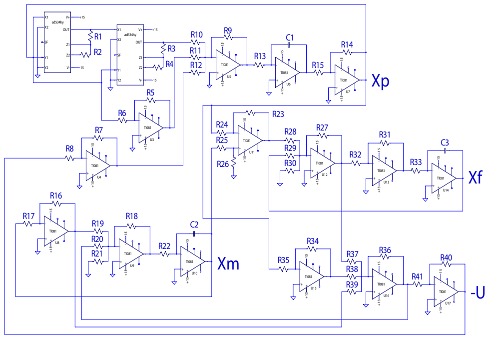

Fig. 2. Circuit Implementation.

IV. Numerical Simulation and Experimental Results

The system described by the equations (1) can be developed with electronic circuits. The implemented circuit can be seen on figure 2.

The structure of the circuit is such as every block is equivalent to an equation. Having the state defined at the output of these. The relationship between the component values and they labels can be seen in the following table.

Value	Label
2k Res	2, 4, 6, 8, 9, 10, 11, 12, 14, 15, 17, 18, 19, 20, 21, 23, 24, 25, 26, 28, 29, 30, 31, 32, 34, 36, 37, 38, 39, 41
90k Res	1, 3
10M Res	13, 22, 33
10K Pot	5, 7, 16, 27, 34, 40
0.1 μ Cap	1, 2, 3

TABLE I
Component Table

All the components above have a tolerance of 5%.The integrated circuits used on the circuit were tl081 for operational amplifiers and ad534 for the multiplier. The last one had an error of 0.25%.

Before executing the experiment, a first numerical run was made with $\mu = 2$, $C_p = 2$, $C_m = 1$ $K = 0.5$, $a_f = 1$ and $a = 4$.

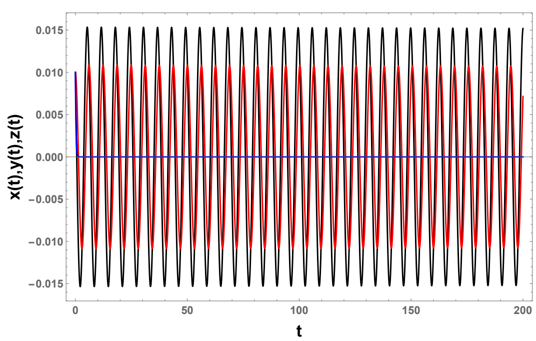

Fig. 3. First Run: Time Response.

For this parameters values it was expected to obtain a limit cicle of small amplitude. Instead, with the experiment the obtained results were somewhat different.

There are many things to explain about figure 4. First of all, the parameters used in the experiment are different from the

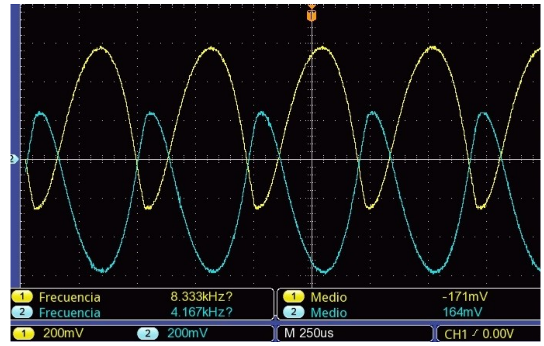

Fig. 4. Experiment: Time Response.

first run. Later, it was made a second run with the experimental values. The tolerances of the components added a DC component to the output. Also, it is suspected that the amplifiers added a little distorsion to the signal. Despite of these limitations, the qualitative characteristics were kept.

After this results, it was made a second run of the numerical simulation with the experimental values. These were: $\mu = 0.2$, $C_p = 0.33$, $C_m = 0.3$ $K = 0.7$, $a_f = 0.05$ and $a = 0.3$. With this values the following graphs were obtained.

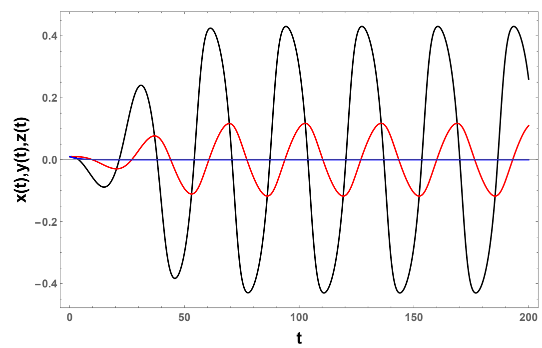

Fig. 5. Second Run: Time Response.

The importance of the discussion here is that the magnitude of the response is indeed constant but, is bigger than expected.

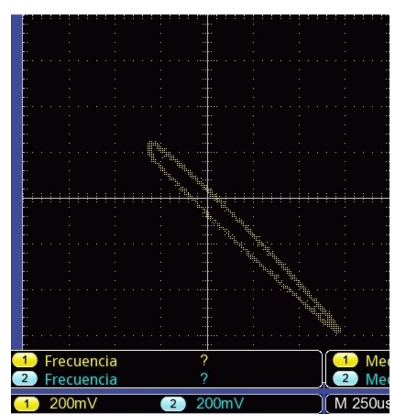

Fig. 6. Experimental: Phase Portrait.

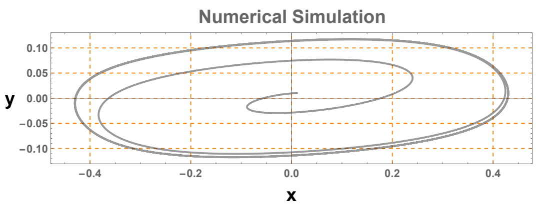

Fig. 7. Second Run: Phase Portrait.

V. Results

The system stability was successfully predicted by the model. The description was accurate enough to understand the qualitative response of the system. Now the importance of study of this is to know the set of parameters that generates this change in stability. This analysis is usually relevant for design, because oscillatory response can generate several damage to the systems elements. Whereby, it is of interest to known the region where this effect does not appear.

The damage generated by the oscillations depends of the system where they are presented. For example in electrical systems, this can lead to overdriving electronic components, just like the experiment above, which can also provoke the breakage or failure in the signal and power elements (See Ref[3]). Also, a similar results can be seen on mechanical systems where the vibrations generates fatigue of the components (See Ref[10]), leading to the decrease in life of the linkages.

In control theory is often seen that a lack of robustness of the controller is the reason for which oscillations are generated at closed loop. The problem in the controller design is to know the gains that gives a good tracking of the reference. As a consequence, the election of the control technique is vital at the moment of design. Now with the results it can be seen that the implemented structure was not enough to ensure the robustness of the systems. So other techniques should be implemented. This is relevant because this means that the IMC structure is not capable to manage a Hopf Bifurcation or even it can change the dynamics of a system leading to this Bifurcation.

Appendix

In this section we provide the Routh-Hurwitz criterion for polynomials of grade three, this with the purpose to show how to determine the local stability of the equilibrium points in three-dimensional systems. For example, if we have the next Jacobian matrix evaluated at an equilibrium point P_0

$$J(P_0) = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix} \tag{11}$$

then the characteristic polynomial associated with this matrix is

$$P(\lambda) = A_0\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 \tag{12}$$

where the coefficients are

$$\begin{aligned} A_0 &= 1, \\ A_1 &= -J_{11} - J_{22} - J_{33}, \\ A_2 &= -J_{12}J_{21} - J_{13}J_{31} - J_{23}J_{32} + J_{22}J_{33} + J_{11}(J_{22} + J_{33}), \\ A_3 &= J_{13}(J_{22}J_{31} - J_{21}J_{32}) + J_{12}(J_{21}J_{33} - J_{23}J_{31}) + J_{11}(J_{23}J_{32} - J_{22}J_{33}). \end{aligned}$$

Under the previous assumptions, the Routh-Hurwitz criterion is summarized in the following theorem

Theorem A.1. *Given the polynomial*

$$P(\lambda) = A_0\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3$$

with associated Hurwitz matrix

$$H = \begin{pmatrix} A_1 & A_3 & 0 \\ A_0 & A_2 & 0 \\ 0 & A_1 & A_3 \end{pmatrix} \tag{13}$$

If it satisfy the following conditions

$$H_1 > 0, \quad H_2 > 0 \quad and \quad H_3 > 0,$$

where H_1, H_2 and H_3 are the principal diagonal minors of (13)*, then the matrix* (13) *is Hurwitz stable.*

The theorem A.1 implies that, if the characteristic equation of 11 is cubic, this is, $P(\lambda) = A_0\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3$, then $A_0 > 0$, $A_1 > 0$, $A_2 > 0$, $A_3 > 0$ and $A_1A_2 - A_0A_3 > 0$ are the necessary and sufficient conditions that the roots of the equation $P(\lambda) = 0$ are negative or have negative real parts. In the case of polynomials of grade two is very simple, the conditions are: $A_0 > 0$, $A_1 > 0$ and $A_2 > 0$, where A_0, A_1 and A_2 are the coefficients of the terms of grade two, one and zero, respectively. In general, a polynomial $P(\lambda) = A_0\lambda^n + A_1\lambda^{n-1} + \cdots + A_n$ of grade n has a Hurwitz matrix

$$H = \begin{pmatrix} A_1 & A_3 & A_5 & \cdots & 0 \\ A_0 & A_2 & A_4 & \cdots & 0 \\ 0 & A_1 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_n \end{pmatrix} \tag{14}$$

For an applied treatment about the Routh-Hurwitz criterion see for instance [?].

Fig. 8. A three level trophic chain.

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