A revisit solution for non homogeneous heat equation by Adomian decomposition method

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Abstract

In this paper, a revisit solution of Adomian decomposition method based on series Fourier is proposed to solve the homogenous and non-homogenous initial and boundary value problem of heat equation, leading to the same solution as the one obtained by the separation of variables method. A numerical example is thus given to prove that the presented method is reliable, efficace and can be employed to derive successfullt analytical approximate solutions of heat equation.

Key-words : Heat equation- Adomian decomposition method- Separation of variables method.

1 Introduction

The Adomian decomposition method ([3],[7],[2]) is used for solving the linear and nonlinear systems (differential, partial, algebraic, integral ,...). The solution is an analytical function given in series form explicitly dependent on the parameters of the system. This method is based on the decomposition of the nonlinear part of the system, using special polynomials called Adomian polynomials. These polynomials are calculated by recursive formulas ([3]). The numerical methods, like the differences method or elements method give a numerical solution that depends implicitely on those parameters.

In its original form, the Adomian method method (ADM) does not take into account the boundary conditions for solving most partial differential equations (PDEs). In [7] the authors proposed a new scheme of the ADM to solve a non linear PDE : Fisher equation with Neumann boundary conditions. [8] has presented an efficient modification of the ADM that facilitates the calculations. Benabidallah and Y. Cherruault [1] have given a solution of some class of linear PDEs with the Dirichlet boundary conditions. Serdal Pamuk [10] have given a solution of linear and nonlinear heat equations by ADM. In ([9]), the authors have proposed a revised scheme of the ADM applied to parabolics and hyperbolics (heat and wave) equations with inhomogeneous boundary conditions. We are motived to give a revised solution of the ADM in order to avoid this inconvenience for solving a heat equation with Dirichlet boundary conditions.

The resolution of the homogenous initial and boundary value problem of heat equation with separation of variables method ([5],[6]) need of solving ordinary differentials equations, to avoid these computations we propose a revisit solution of the ADM and that take into account the boundary conditions.

This article is organised as follows, in the second section we apply the separation variables method (Fourier method) to heat equation. We developed in section 3, a solution of the ADM, to solve the equation. Section 4 is devoted to solve a non-homogeneous heat equation with two methods, namely separation variables method and the ADM. The examples illustration are given.

2 The separation of variables method : homogeneous heat equation :

In this section we apply the separation of variables method to solve the homogeneous initial boundary value problem (IBVP) of heat equation.

We denote by $\Omega = [0,l], l > 0$, $Q =]0,l[\times]0,+\infty[$.The heat equation ([10],[5]) is given as follows :

(1)
$$\begin{cases} \partial_t V(x,t) = \alpha \partial_{xx} V(x,t) + qV(x,t); & (x,t) \in Q \\ V(x,0) = f(x); & x \in \bar{\Omega} \\ V(0,t) = V(l,t) = 0; & t \geqslant 0 \end{cases}$$

where q, l are the positive constants. $f(x)$ is a function $C([0,l])$. $V(x,t)$ is an unknown function.

We seek out solution to equation (1) applying the Fourier method as follows :

(2) $$V(x,t) = X(x)T(t)$$

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After substituting (2) in (1) and separating the variables, we get :

$$X(x)T^{'}(t) = \alpha X^{''}(x)T(t) + q\ X(x)T(t)$$

If $T(t) \neq 0$ and $X(t) \neq 0$,we deduce two possible superpositions of ordinary differentials equations (ODEs) that each side is equals a constant, therefore we set :

$$\begin{aligned} \frac{T^{'}(t) - qT(t)}{T(t)} &= \frac{\alpha X^{''}(x)}{X(x)} = -\lambda \\ \frac{T^{'}(t)}{T(t)} &= \frac{\alpha X^{''}(x) + qX(x)}{X(x)} = -\lambda \end{aligned}$$

with the following conditions, we get : $T(0) = f(x)$, $X(0) = 0$, $X(l) = 0$.

By resolving the ODEs established from the second superposition,we get :

(3) $$\frac{T^{'}(t)}{T(t)} = -\lambda \quad \text{and} \quad \frac{\alpha X^{''}(x) + qX(x)}{X(x)} = -\lambda$$

Thus the solution obtained is :

(4) $$V(x,t) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{l}\right) e^{t\left(q - \alpha\left(\frac{m\pi}{l}\right)^2\right)}$$

where :

(5) $$b_m = \frac{2}{l}\int_0^l f(\xi)\sin\left(\frac{m\pi\xi}{l}\right)d\xi, \quad m = 1, 2, ...$$

The following are worth mentioning :

- The same exact solution is obtained by choosing arbitrarily any superpositions.
- Applying the Fourier method, two ODEs have to be solved : one is a simple equation and the other being a second degree equation. However, the Adomian method can be used to spare us the harassment of computation.

3 Revisit solution of the ADM for homogeneous heat equation

We seek to solve the homogeneous (IBVP) of heat equation (1) presented in section 2.

The decomposition method consists in writing V of equation (1) into a series form :

(6) $$V(x,t) = \sum_{n=0}^{\infty} V_n(x,t)$$

Applying the inverse operator $L_t^{-1}(.) = \int_0^t (.)d\tau$, to both side of equation (1) we get :

(7) $$V(x,t) = V(x,0) + \alpha L_t^{-1} L_{xx} V(x,t) + q L_t^{-1} V(x,t)$$

where $L_{xx}(.) = \partial_{xx}(.)$

Identifying the first component to be the initial condition, and using the recursive algorithm, we can write the solution as follows :

(8)
$$\begin{cases} V_0(x,t) = V(x,0) = f(x) \\ \vdots \\ V_n = \alpha L_t^{-1} L_{xx} V_{n-1}(x,t) + q L_t^{-1} V_{n-1}(x,t); \quad n \geqslant 1 \end{cases}$$

We get :

(9)
$$\begin{cases} V_0(x,t) = V(x,0) = f(x) \\ V_1(x,t) = \alpha L_t^{-1} L_{xx} V_0 + q L_t^{-1} V_0 \\ \qquad = \alpha L_t^{-1} L_{xx} f(x) + q L_t^{-1} f(x) \\ V_2(x,t) = \alpha L_t^{-1} L_{xx} V_1 + q L_t^{-1} V_1 \\ \qquad = [\alpha^2 \left(L_t^{-1} L_{xx}\right)^2 + 2\alpha q \left(L_t^{-1}\right)^2 L_{xx} + q^2 \left(L_t^{-1}\right)^2] f(x) \\ \quad ..... \end{cases}$$

This gives the relation of V_n :

$$\begin{aligned} V_n(x,t) &= \sum_{p=0}^{n} C_n^p \alpha^{n-p} q^p \left(L_t^{-1}\right)^n (L_{xx})^{(n-p)} f(x) \\ &= \left(L_t^{-1}\right)^n (\alpha L_{xx} + q)^n f(x) \end{aligned} \tag{10}$$

$\left(L_t^{-1}\right)^n = \int_0^t ... \int_0^r \int_0^\tau (.) ds d\tau ... dr$ is the n th integration.

V is Adomian solution given as follows :

$$\begin{aligned} V(x,t) &= \sum_{n=0}^{\infty} \left(L_t^{-1}\right)^n (\alpha L_{xx} + q)^n f(x) \quad (11) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} (\alpha L_{xx} + q)^n f(x) \end{aligned}$$

In practice we seek for a truncated solution of the order 's' :

$$\begin{aligned} V(x,t) &= \sum_{n=0}^{s} \left(L_t^{-1}\right)^n (\alpha L_{xx} + q)^n f(x) \\ &= \sum_{n=0}^{s} \frac{t^n}{n!} (\alpha L_{xx} + q)^n f(x) \end{aligned} \tag{12}$$

If $V(x,t)$ satisfied the PDE , then V is an exact solution to (1). Else, we expand $f(x)$ in an infinite series, thus :

$$f(x) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{l}\right) \tag{13}$$

where the coefficients b_m are given by :

$$b_m = \frac{2}{l}\int_0^l f(\xi)\sin\left(\frac{m\pi\xi}{l}\right)d\xi, \quad m = 1, 2, ... \tag{14}$$

Substituting (13) in (11) yields :

$$V(x,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} C_n^k \alpha^{n-k} q^k \left(L_t^{-1}\right)^n (L_{xx})^{(n-k)} \sum_{m=1}^{\infty} b_m \sin\left(\tfrac{m\pi x}{l}\right)$$
$$= \sum_{m=1}^{\infty} b_m \sum_{n=0}^{\infty} \tfrac{t^n}{n!} \sum_{k=0}^{n} C_n^k \alpha^{n-k} q^k \left(L_{xx}\sin\left(\tfrac{m\pi x}{l}\right)\right)^{(n-k)}$$
$$= \sum_{m=1}^{\infty} b_m \sum_{n=0}^{\infty} \tfrac{t^n}{n!} \sum_{k=0}^{n} C_n^k \alpha^{n-k} q^k \left(\tfrac{m\pi}{l}\right)^{2(n-k)} (-1)^{n-k} \sin\left(\tfrac{m\pi x}{l}\right)$$
$$= \sum_{m=1}^{\infty} b_m \sin\left(\tfrac{m\pi x}{l}\right) \sum_{n=0}^{\infty} \tfrac{t^n}{n!} \left(q - \alpha\left(\tfrac{m\pi}{l}\right)^2\right)^n$$
$$= \sum_{m=1}^{\infty} b_m \sin\left(\tfrac{m\pi x}{l}\right) e^{t\left(q-\alpha\left(\frac{m\pi}{l}\right)^2\right)} \tag{15}$$

which is the Fourier expansion of the solution $V(x,t)$ derived by the rearrangement of the decomposition solution

In practice, an approximate solution is satisfied, when the upper board of m is fixed.

The fonction $V(x,t)$; sum of series (15) is continuous on $[0,l] \times [0,+\infty[$,

$V(x,t) \in C^2(]0,l[\times]0,+\infty[), V(x,t)$ satisfies (IBVP) of heat equation.

3.1 Example

Let : $l = 1$, $f(x) = 300$, $\alpha = 0.01$ and $q = 0.1$.

In this example, we see that, the solution is given by the ADM for $s = 5$, does not satisfy the boundary conditions :

$V(0,t) = V(l,t) = 300 + 300.qt + 150qt^2 + 50\ q^2.t^3 + 12.5\ q^4.t^4 + 2.5\ q^5t^5 \neq 0$

Consequently, we give a Fourier expansion of the solution $V(x,t)$.

The solution $V(x,t)$ is given by the following figures, for $m = 10$:

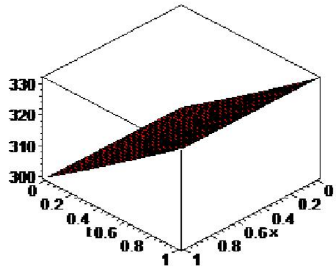

Fig 1: Curve of $V(x,t)$ by ADM

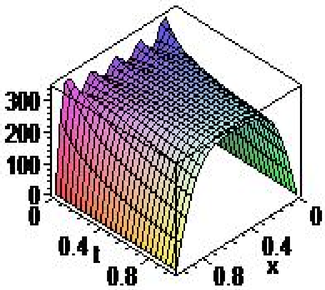

Fig 2: Curve of $V(x,t)$ by revisit ADM

We note that, more m is great, then we approach to the exact solution.

4 Resolution of non-homogeneous heat equation with Fourier method

In this section we consider the non-homogeneous IBVP of heat equation :

$$\begin{cases} \partial_t u(x,t) = \alpha\partial_{xx}u(x,t) + qu(x,t) + h(x,t) & \\ u(x,0) = f(x), & 0 < x < l \\ u(0,t) = u(l,t) = 0, & t \geq 0 \end{cases} \tag{16}$$

$h(x,t)$ is $C^1([0,l])$.

We seek out solution to equation (16) applying the Fourier method as follows :

$$u(x,t) = V(x,t) + W(x,t) \tag{17}$$

where $V(x,t)$ is solution of homogenous problem studied in section (2) and $W(x,t)$ is a particular solution.

We writing the solution $W(x,t)$ into a series form :

$$W(x,t) = \sum_{m=1}^{\infty} T_m(t)\ \sin\left(\frac{m\pi x}{l}\right) e^{t\left(q-\alpha\left(\frac{m\pi}{l}\right)^2\right)} \tag{18}$$

After substituting (18) in (16), we get :

$$\sum_{m=1}^{\infty} T_m^{'}(t)\sin\left(\frac{m\pi x}{l}\right) e^{t\left(q-\alpha\left(\frac{m\pi}{l}\right)^2\right)} = h(x,t) \tag{19}$$

We expand $h(x,t)$ in an infinite series, thus :

$$h(x,t)=\sum_{m=1}^{\infty}\delta_m(t)\sin\left(\frac{m\pi x}{l}\right) \tag{20}$$

where

$$\delta_m(t)=\frac{2}{l}\int_0^l h(\xi,t)\sin\left(\frac{m\pi\xi}{l}\right)d\xi, m=1,2,... \tag{21}$$

Substituting (20) in (19) yields :

$$T_m^{'}(t)e^{t\left(q-\alpha\left(\frac{m\pi}{l}\right)^2\right)}=\delta_m(t) \quad , \quad m=1,2,... \tag{22}$$

The solutions of equations (22) are as follows :

(23)

$$T_m(t)=\int_0^t\delta_m(\tau)\ e^{-\tau\left(q-\alpha(\lambda_m)^2\right)}d\tau\ ,\quad m=1,2,...$$

Substituting $T_m(t)$ in the serie (18) yields the solution $W(x,t)$:

$$W(x,t) = \sum_{m=1}^{\infty}\left[\int_0^t\delta_m(\tau)\ e^{-\tau\left(q-\alpha(\lambda_m)^2\right)}d\tau\right] \sin(\lambda_m x)\,e^{t\left(q-\alpha(\lambda_m)^2\right)} \tag{24}$$

where : $\lambda_m=\frac{m\pi}{l}$,

$$\delta_m(t)=\frac{2}{l}\int_0^l h(\xi,t)\sin(\lambda_m\xi)\,d\xi, m=1,2,... \tag{25}$$

The function $u(x,t)=V(x,t)+W(x,t)$ will be the solution of (16).

5 Revisit solution of the ADM for non-homogeneous heat equation

We seek to solve the non-homogeneous heat equation (16) presented in section 4.

We seek a solution $V(x,t)$ into series form :

$$V(x,t)=\sum_{n=0}^{\infty}V_n(x,t) \tag{26}$$

Applying the operator $L_t^{-1}(.)=\int_0^t(.)d\tau$, we get :

$$V(x,t)=V(x,0)+\alpha L_t^{-1}L_{xx}V(x,t)+qL_t^{-1}V(x,t)+L_t^{-1}h(x,t)$$

where $L_{xx}(.)=\partial_{xx}(.)$

The ADM defines the components V_n, $n\geqslant 0$ by the following recursive relationship :

(27)

$$\begin{cases} V_0(x,t)=V(x,0)+L_t^{-1}h(x,t)=f(x)+L_t^{-1}h(x,t) \\ \vdots \\ V_n(x,t)=\alpha L_t^{-1}L_{xx}V_{n-1}(x,t)+qL_t^{-1}V_{n-1}(x,t); \quad n\geqslant 1 \end{cases}$$

The components V_n, $n\geqslant 0$ of (26) can be obtained as follows :

(28)

$$\begin{cases} V_0(x,t)=V(x,0)=f(x)+L_t^{-1}h(x,t)=f+g \\ V_1(x,t)=\alpha L_t^{-1}L_{xx}V_0+qL_t^{-1}V_0 \\ \qquad =(\alpha L_t^{-1}L_{xx}+qL_t^{-1})(f+g) \\ V_2(x,t)=\alpha L_t^{-1}L_{xx}V_1+qL_t^{-1}V_1 \\ =[\alpha^2\left(L_t^{-1}L_{xx}\right)^2+2\alpha q\left(L_t^{-1}\right)^2L_{xx}+q^2\left(L_t^{-1}\right)^2](f+g) \\ \quad .... \end{cases}$$

with $g=L_t^{-1}h(x,t)$.

We deduce a formula to calculate V_n :

$$\begin{aligned} V_n(x,t) &= \sum_{p=0}^{n}C_n^p\alpha^{n-p}q^p\left(L_t^{-1}\right)^n\left(L_{xx}\right)^{(n-p)}[f(x)+g(x,t)] \\ &= \left(L_t^{-1}\right)^n(\alpha L_{xx}+q)^n[f(x)+g(x,t)] \end{aligned} \tag{29}$$

where

$\left(L_t^{-1}\right)^n=\int_0^t\int_0^r\int_0^\tau(....)dsd\tau dr$ is the n th integration.

$(L_{xx})^n$ represent the n th derivation.

Consequently V is the Adomian solution given as follows :

$$V(x,t)=\sum_{n=0}^{\infty}\left(L_t^{-1}\right)^n(\alpha L_{xx}+q)^n[f(x)+g(x,t)] \tag{30}$$

If $V(x,t)$ satisfied the PDE and the boundary conditions, then V is an exact solution to (16).

Else, we expand $f(x)$ and $g(x,t)$ in an infinite series, thus :

$$f(x)=\sum_{m=1}^{\infty}b_m\sin\left(\frac{m\pi x}{l}\right) \tag{31}$$

$$g(x,t)=\sum_{m=1}^{\infty}\delta_m(t)\sin\left(\frac{m\pi x}{l}\right) \tag{32}$$

where :

$$b_m=\frac{2}{l}\int_0^l f(\xi)\sin\left(\frac{m\pi\xi}{l}\right)d\xi,\quad m=1,2,...$$

$$\delta_m(t) = \frac{2}{l}\int_0^l g(\xi,t)\sin\left(\frac{m\pi\xi}{l}\right)d\xi, m = 1,2,...$$

Substituting (31) and (32) in (30) yields :

$$\begin{aligned} V(x,t) &= \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{l}\right) e^{t\left(q-\alpha\left(\frac{m\pi}{l}\right)^2\right)} + \\ &\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left(L_t^{-1}\delta_m(t)\right)^n \sin\left(\frac{m\pi x}{l}\right)\left(q-\alpha\left(\frac{m\pi}{l}\right)^2\right)' \end{aligned}$$

In order to illustrate the technique discussed above, we shall give in the following an example in which we can apply our technique and Fourier method.

5.1 Example : non-homogeneous heat equation

Consider the non-homogeneous initial and boundary conditions value problem of heat equation :

$$\begin{cases} \partial_t u(x,t) = 0.01\ \partial_{xx} u(x,t) + 0.1\ u(x,t) + x(x-1)t \\ u(x,0) = x(x-1), \quad 0 \leqslant x \leqslant l \\ u(0,t) = u(l,t) = 0, \quad t \geq 0 \end{cases}$$

The solution of above problem given by the initial form of ADM does not take into account the bounary conditions, we have used the revisit solution of ADM. Let $l = 1.m = 6$; $n = 4$.

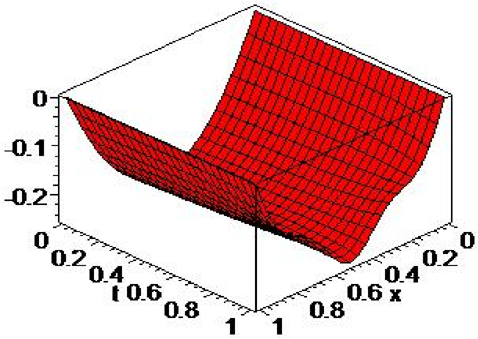

Fig 3: Curve of $u(x,t)$ by a revised scheme of Adomian

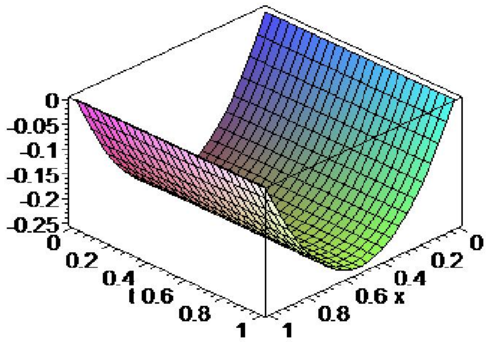

Fig 4: Curve of $u(x,t)$ by Fourier method

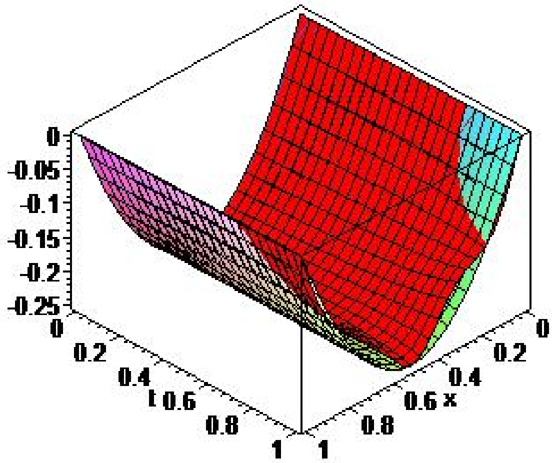

Fig 5 : Superposition of curves solutions obtained by (revisit ADM : Red, Fourier : +colors) methods

It's clear that we have a good superposition of the curves solutions.

6 Homogeneous eqaution with inhomogeneous boundary conditions: Functions

Now let us consider the equation (1) with non-homogeneous boundary conditions. As seen in previous section, the result (11)-(15) does not satisfy the non-homogeneous boundary conditions. Hence it important to consider the problem :

(34)
$$\begin{cases} \partial_t V(x,t) = \alpha\partial_{xx} V(x,t) + qV(x,t), \quad 0 < x < l,\ t > 0 \\ V(x,0) = f(x), \quad 0 \leqslant x \leqslant l \\ V(0,t) = g_1(t), t \geq 0 \\ V(l,t) = g_2(t), t \geq 0 \end{cases}$$

Let :

(35) $$V(x,t) = v(x,t) + w(x,t)$$

where :

$$w(x,t) = \frac{x}{l}g_2(t) + g_1(t)\frac{(l-x)}{l}$$

Substitution of $V(x,t)$ in problem (34) yields :

$$\begin{cases} \partial_t v(x,t) = \alpha\partial_{xx} v(x,t) + qv(x,t) + h(x,t) \\ v(x,0) = F(x) \\ v(l,t) = 0 \\ v(0,t) = 0 \end{cases}$$

where :

$$h(x,t) = qw(x,t) - \partial_t w(x,t), \qquad F(x) = f(x) - w(x,0).$$

Solving for $L_t\ v$, we have :
$v_0 = F + L_t^{-1} h$
$v_n = \frac{t^n}{n!}(\alpha L_{xx} + q)^n\ v_0\ , n \geq 0.$
$v = \sum_{n=0}^{\infty} v_n$

In cases where v_0 is ill defined, we consider the Fourier expension of v_0.Thus, v can be found as above and (35) yields the solution V to the corresponding non-homogenous problem.

7 Conclusion

We have considered the ADM to solve non-homogeneous initial and boundary conditions value problem of heat equation. The ADM gives a solution explicitly depending on the parameters of the equation. The advantage of this method is that, it solves the problem without linearization or discritisation of space or time variables. The main difficulty in applying the ADM to heat equation equation lies in the fact that the solution series does not take into account boundary conditions. We have proposed a revisit solution of the Adomian method for solve the heat equation that takes into account all the boundary conditions and we have compared the solution with the traditionnel separation of variables method. The work shows that Adomian has signifiant advantages over the existing techniques. The method does not require restrictive assumption or transformation formulae.

Our future work is to propose the new scheme to solve a non linear parabolic partial differential equation [4]. The idea can be extented to solve many other kinds of equations.

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