# Complete Blow-up for a Degenerate Semilinear Parabolic Problem in the Sense of Semigroup Theory

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***Abstract***— Before blow-up occurs, under certain conditions, we establish a unique blow-up solution for a degenerate semilinear parabolic problem: $u_t - (a(x)u_x)_x = f(u)$ in $(0,1) \times (0,\infty)$ where $f$ is a specified function and $a(0) = 0$, $a(x) > 0$ on $(0,1]$ together with the Dirichlet boundary condition and the suitable initial condition. The blow-up set of such a blow-up solution $u$ is given. Furthermore, the sufficient condition to guarantee the occurrence for blow-up in finite time is shown.

***Keywords***—Blow-up problems, Semilinear parabolic problems, Semigroup theory, Semilinear evolution problems, Blow-up in finite time, Degenerate parabolic problems.

## I. Introduction

The subject of blow-up was posed in the 1940's and 50's in the context of Semenov's chain reaction theory, adiabatic explosion and combustion theory. There has been a tremendous amount of recent activities due to the subjects of solutions to various partial differential equations blowing up in finite time. Finite time blow-up occurs in situations in mechanics and other areas of applied mathematics. Studies of these phenomena have very recently been gaining momentum. In the following, we give examples of blow-up problems in the way of blow-up mathematical theory. In 1985, C.E. Mueller and F. B. Weissler [7] studied the semilinear heat equation:

$$\left.\begin{array}{c} u_t = \triangle u - \lambda u + f(u),\ (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0,\ x \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ x \in \Omega, \end{array}\right\} \tag{1}$$

where $\Omega$ is $\mathbb{R}^n$ or $\Omega$ is a smooth bounded subset of $\mathbb{R}^n$, $\partial\Omega$ denotes the smooth boundary of $\Omega$, $\triangle = \sum_{i=1}^{n} \partial_i^2$, $\lambda \geq 0$ and $f$ and $u_0$ are specified functions. Under suitable assumptions, they showed that the solution of (1) blows up in finite time and the blow-up set of blow-up solution consists of only one point. Further, in 2009, J. P. Pinasco [8] established the blow-up positive solutions of problems (2) with reaction terms of local and nonlocal type involving a variable exponent,

$$\left.\begin{array}{c} u_t = \triangle u + f(u),\ (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0,\ (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ x \in \overline{\Omega}, \end{array}\right\} \tag{2}$$

where $\Omega$ is a smooth bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and the source term is of the form $f(u) = a(x)u^{p(x)}$ or $f(u) = a(x)\int_\Omega u^{q(y)}(y,t)dy$ where $a, p$ and $q$ are given functions. For blow-up problems of the degenerate semilinear parabolic type, in 1999, C.Y. Chan and W. Y. Chan [3] studied the existence of a blow-up solution of the degenerate semilinear parabolic initial-boundary value problem

$$\left.\begin{array}{c} x^q u_t - u_{xx} = f(u),\ (x,t) \in \Omega \times (0,\infty), \\ u(0,t = 0 = u(1,t),\ t > 0, \\ u(x,0) = u_0(x),\ x \in [0,1], \end{array}\right\} \tag{3}$$

where $q \geq 0$, $f$ and $u_0$ are given functions. They proved existence and uniqueness of a blow-up solution of problem (3) by transforming problem (3) into the equivalent integral equation in terms of its associated Green's function. Furthermore, in 2006, C. Y. Chan and W.Y. Chan [4] showed that under certain condition on functions $f$ and $u_0$, a solution $u$ of problem (3) blows up at every point in $[0,1]$. After paper [3] published, in 2004, Y.P. Chen and C.H. Xie [6] considered the degenerate parabolic problem with the nonlocal term : for any $(x,t) \in (0,1) \times (0,\infty)$,

$$\left.\begin{array}{c} u_t - (x^\alpha u_x)_x = \int_0^1 f(u)dx, \\ u(0,t) = 0 = u(1,t),\ t > 0, \\ u(x,0) = u_0(x),\ x \in [0,1], \end{array}\right\} \tag{4}$$

with $\alpha \in [0,1)$ and $f$ and $u_0$ are given functions. They proved the local existence and uniqueness of a classical solution. Under appropriate hypotheses,

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they obtained the sufficient conditions for the global existence and for blow-up of a positive solution of problem (4). Additionally, in 2004, Y.P. Chen, Q. Liu and C.H. Xie [5] studied the degenerate nonlinear reaction-diffusion equation with nonlocal source: for any $(x,t) \in (0,1) \times (0,\infty)$,

$$\left.\begin{array}{r} x^q u_t - (x^\alpha u_x)_x = \int\limits_0^1 u^p dx, \\ u(0,t) = 0 = u(1,t),\ t > 0, \\ u(x,0) = u_0(x),\ x \in [0,1], \end{array}\right\} \quad (5)$$

They established the local existence and uniqueness of a classical solution of problem (5). Under appropriate hypotheses, they gave the sufficient conditions for a global existence and for blow-up of a positive solution. Furthermore, under certain conditions, they proved that the blow-up set of such a solution of problem (5) is the whole domain. In 2010, P. Sawangtong and W. Jumpen[10] showed, under certain condition, the existence of a blow-up solution of the degenerate parabolic problem: for any $(x,t) \in (0,1) \times (0,\infty)$,

$$\left.\begin{array}{r} x^q u_t - (x^\alpha u_x)_x = x^q f(u), \\ u(0,t) = 0 = u(1,t),\ t > 0, \\ u(x,0) = u_0(x),\ x \in [0,1], \end{array}\right\} \quad (6)$$

where $q \geq 0,\ \alpha \in [0,1)$ and $f$ and $u_0$ are suitable functions. Furthermore the sufficient condition to blow-up in finite time and the blow-up of such a solution of problem (6) are shown. Furthermore, in 2010, P. Sawangtong and W. Jumpen [11] extended problem (6) to more general form: for any $(x,t) \in (0,1) \times (0,\infty)$,

$$\left.\begin{array}{r} k(x)u_t - (a(x)u_x)_x = k(x)f(u), \\ u(0,t) = 0 = u(1,t),\ t > 0, \\ u(x,0) = u_0(x),\ x \in [0,1], \end{array}\right\} \quad (7)$$

where $k(0) = 0 = a(0)$, $k, a > 0$ on $(0,1]$ and $f$ and $u_0$ are given functions. They showed the existence and uniqueness of a blow-up solution of problem (7) by classical method, i.e., Greens'function method. As shown in [11], there are many conditions on functions $k$ and $a$ to obtain the existence of corresponding eigenvalues and eigenfunctions to problem (7) to use their properties in the part of existence of solution of problem (7). In 2010, P. Sawangtong and W. Jumpen [13] still study the same problem as in [11] by replacing the term $k(x)f(u)$ by the term $k(x)f(u(x_0,t))$ where $x_0$ is a fixed point in $(0,1)$. By method of Green's function, we obtain the existence and uniqueness of such a problem.

In this paper, we study the following degenerate semilinear parabolic problem closed to problem (7) via semigroup theory:

$$\left.\begin{array}{r} u_t - (a(x)u_x)_x = f(u),\ (x,t) \in (0,1) \times (0,\infty), \\ u(0,t) = 0 = u(1,t),\ t > 0, \\ u(x,0) = u_0(x),\ x \in [0,1], \end{array}\right\} \quad (8)$$

where $a, f$ and $u_0$ are given functions.

The objective of this article is to show the existence of a unique blow-up solution of the problem (8) before blow-up occurs by semigroup theory and the blow-up set of such a blow-up solution. Furthermore, the sufficient condition to guarantee the occurrence of blow-up in finite time is given.

## II. Setting out a degenerate problem

We next give the definition of blow-up in finite time.

*Definition 1:* A solution $u$ of the problem (8) is said to **blow-up** at the point $b$ in finite time $T_b$ if there exists a sequence $\{(x_n,t_n)\}$ with $(x_n,t_n) \in (0,1) \times (0,T)$ and $(x_n,t_n) \to (b,T_b)$ as $n \to \infty$ and $\lim\limits_{n\to\infty} u(x_n,t_n) = +\infty$. The point $b$ is called the **blow-up point** . The set consisting of all blow-up points of such a blow-up solution $u$ is called the **blow-up set**. Furthermore, if $u$ blows up at every point $x$ in $[0,1]$, then the **complete blow-up** occurs.

Because of the function $a$ which expresses the degeneracy we need to introduce a variant of the classical Sobolev space $H^1(0,1)$, namely $H^{1,a}(0,1)$. Throughout this paper, we make the following assumptions on $a$:

(A) $a \in C^0[0,1] \cap C^1(0,1]$, $a > 0$ in $(0,1]$ and $a(0) = 0$;

(B) $\exists K \in [0,1)$ such that $xa'(x) \leq Ka(x)$ for all $x \in [0,1]$.

We note that

1) an example of functions satisfies the conditions (A) and (B) is $x^\alpha$ with $\alpha \in [0,1)$,
2) the condition (B) implies that $\int\limits_0^1 \frac{1}{a(x)}dx$ is finite which is a sufficient condition to obtain that the space $H^{1,a}(0,1)$ is compactly embedded in $L^2(0,1)$.

If $u_x$ denote the derivative in the sense of distribution of the distribution $u$ in $\mathcal{D}'(0,1)$, then

$$H^{1,a}(0,1) = \{u \in L^2(0,1) \text{ possessing an absolutely continuous representative on [0,1] and} \sqrt{a}u_x \in L^2(0,1)\}.$$

It is known that equipped with the following inner product and norm

$$\langle u,v\rangle_{H^{1,a}(0,1)} = \int\limits_0^1 [u(x)v(x) + a(x)u_x(x)v_x(x)]\,dx$$

and

$$\|u\|_{H^{1,a}(0,1)} = \langle u, u\rangle^{1/2}_{H^{1,a}(0,1)} .$$

respectively. The space $H^{1,a}(0,1)$ is a Hilbert space. By due account of the fact that $\int_0^1 \frac{1}{a(x)}dx$ is finite,

$H_0^{1,a}(0,1) = \left\{u \in H^{1,a}(0,1) \text{ s.t. } u(0) = 0 = u(1)\right\}$, is a closed subspace of $H^{1,a}(0,1)$ with equivalent norm

$$\|u\|_{H_0^{1,a}(0,1)} = \left\|\sqrt{a}u_x\right\|_{L^2(0,1)} ,$$

and the injection of $H^{1,a}(0,1)$ and $C^0[0,1]$ is continuous. Eventually we will consider

$H^{2,a}(0,1) = \left\{u \in H^{1,a}(0,1) \text{ s.t. } au_x \in H^{1,a}(0,1)\right\}$

with its norm:

$$\|u\|^2_{H^{2,a}(0,1)} = \|u\|^2_{H^{1,a}(0,1)} + \|(au_x)_x\|^2_{L^2(0,1)} .$$

In order to obtain the existence of a blow-up solution $u$ of problem (8), we also make some hypothesis on functions $u_0$ and $f$:

(C) $u_0 \in H^{2,a}(0,1) \cap H_0^{1,a}(0,1)$, $u_0 \geq 0$ on $[0,1]$, $u_0(0) = 0 = u_0(1)$ and $u_0$ satisfies

$$\frac{d}{dx}\left(a(x)\frac{du_0(x)}{dx}\right) + f(u_0(x)) \geq 0$$

for $x \in (0,1)$.

(D) $f \in C^2[0,\infty)$ is convex with $f(0) = 0$, $f(s) > 0$ for $s > 0$.

We note that by [7], condition (D) implies that $f$ is increasing and $f$ is locally Lipschitz on $[0,\infty)$, that is, $\forall M > 0, \exists C_M$ such that $|f(a) - f(b)| \leq C_M |a-b|\ \forall a,b$ with $|a|, |b| \leq M$.

To apply a useful result in the semigroup theory [15], we transform problem (8) into the equivalent semilinear evolution problem:

$$\left.\begin{array}{c} u_t - Au(t) = F(u),\ t > 0, \\ u(0) = u_0, \end{array}\right\} \tag{9}$$

where $A$ is an operator mapping from $D(A)$, the domain of $A$, into $L^2(0,1)$ with

$$\begin{aligned} & D(A) \\ = & \Big\{u \in H_0^{1,a}(0,1) \text{ s.t. } \exists! w \in L^2(0,1) \text{ satisfies} \\ & \int_0^1 w(x)\varphi(x)dx = -\int_0^1 a(x)u_x(x)\varphi_x(x)dx, \\ & \text{for all } \varphi \in H_0^{1,a}(0,1)\Big\} \end{aligned} \tag{10}$$

and

$$Au = (au_x)_x = w \text{ for all } u \in D(A) \tag{11}$$

and where $F$ is an operator mapping from $D(A)$ into $L^2(0,1)$ defined by

$$F(u) = f(u) \text{ for all } u \in D(A). \tag{12}$$

## III. The Main Results

Here, we prove that problem (8) has a unique blow-up solution in the sense of semigroup theory.

*Theorem 2:* There exists a positive constant $T$ such that the equivalent evolution problem (9) has a unique solution $u \in C([0,T], D(A)) \cap C^1([0,T], L^2(0,1))$ defined by

$$u(t) = S(t)u_0 + \int_0^1 S(t-\tau)F(u(\tau))d\tau$$

where $S(t)$ is an analytic semigroup generated by the operator $A$.

*Theorem 3:* Let $[0, T_{max})$ be the maximal time interval in which a solution $u$ of problem (9) exists. If $T_{max}$ is finite, then $\lim\limits_{t\to T_{\max}} \max\limits_{x,\in[0,1]} |u(x,t)|$ is unbounded.

*Theorem 4:* Let $u$ be a blow-up solution of problem (8). Then the blow-up set of such a solution $u$ is $[0,1]$.

Let $\lambda_1$ be the first eigenvalue of a singular eigenvalue problem corresponding to problem (8) and let $\phi_1$ be its associating eigenfunction. Without loss of generality we assume

$$\int_0^1 \phi_1(x)dx = 1.$$

We then define the function $H$ by

$$H(t) = \int_0^1 \phi_1(x)u(x,t)dx.$$

*Theorem 5:* Suppose that

1) $f(\xi) \geq b\xi^p$ with $b > 0$ and $p > 1$,
2) $H(0) > \left(\frac{\lambda_1}{b}\right)^{\frac{1}{p-1}}$.

Then a solution $u$ of problem (8) blows up in finite time.

## IV. The Proof of Main Results

In this section we will give the proof of our main theorems by starting from the proof of theorem 2.

### A. The proof of theorem 2

In this section, we will first consider some properties of operators $A$ and $F$ defined by (11) and (12), respectively.

Let us state important properties of $A$:

*Proposition 6:* The operator $A$ defined by (11) is maximal dissipative and self-adjoint on $L^2(0,1)$ which, consequently, generate an analytic semigroup on $L^2(0,1)$.

**Proof:** To prove the maximal dissipative property of $A$, we have to show two conditions:

1) $\langle Au, u\rangle_{L^2(0,1)} \leq 0$ for all $u \in D(A)$ and
2) $R(I-\lambda A) = L^2(0,1)$ for any $\lambda > 0$ where $R(I-\lambda A)$ and $I$ denote the range of $I-\lambda A$ and the identity operator on $L^2(0,1)$, respectively.

Condition 1 follows directly from (10), the definition of $A$. Let $h \in L^2(0,1)$ and $\lambda$ be any positive constant. For verifying condition 2, we have to show that there exists a unique $u \in D(A)$ such that $u - \lambda Au = h$ which equivalent to show that there exists a unique $u \in D(A)$ such that the following equation holds:

$$\frac{1}{\lambda}\int_0^1 u(x)\varphi(x)dx + \int_0^1 a(x)u_x(x)\varphi_x(x)dx$$
$$= \int_0^1 h(x)\varphi(x)dx \text{ for all } \varphi \in H^1_a(0,1).$$

Such the existence is guaranteed by Lax-Milgram theorem. Hence, the operator $A$ is maximal dissipative on $L^2(0,1)$. Hence to show that $A$ is self-adjoint it suffices to prove that $A$ is symmetric: let $u, v \in D(A)$. We consider that, by (10),

$$\begin{aligned}\langle Au, v\rangle_{L^2(0,1)} &= -\int_0^1 a(x)u_x(x)v_x(x)dx \\ &= \langle u, Av\rangle_{L^2(0,1)}.\end{aligned}$$

□

The proof of next lemma is not difficult. We can prove directly and then we have:

*Lemma 7:* $D(A) = H^{2,a}(0,1) \cap H_0^{1,a}(0,1)$.

The next lemma is used to guarantee the existence of corresponding eigenvalues and eigenfunctions of $-A$ refereed to [1].

*Lemma 8:* The space $H_0^{1,a}(0,1)$ is compactly imbedded in $L^2(0,1)$.

**Proof:** See [1] □

Since the operator $(-A)^{-1}$ is a bounded well-defined operator on $L^2(0,1)$ with values in $H_0^{1,a}(0,1)$, lemma 8 implies that $(-A)^{-1}$ is compact operator on $L^2(0,1)$. The next lemma is the well-known results about the spectral theory of self-adjoint compact operator referred from [2].

*Lemma 9:* There exists a sequence $(\lambda_n, \phi_n) \subset (0, +\infty) \times H_0^{1,a}(0,1)$ such that

1) $A\phi_n = -\lambda_n\phi_n$ for all $n \geq 1$,
2) $\int_0^1 \phi_n(x)\phi_m(x)dx = \begin{cases} 0, & n \neq m, \\ 1, & n = m, \end{cases}$
3) $\int_0^1 a(x)\phi_n'(x)\phi_m'(x)dx = \begin{cases} 0, & n \neq m, \\ \lambda_n, & n = m, \end{cases}$
4) $v(x) = \sum_{n=1}^{\infty} \langle v, \phi_n\rangle_{L^2(0,1)} \phi_n(x)$ for any $v \in L^2(0,1)$,
5) $\|v\|^2_{L^2(0,1)} = \sum_{n=1}^{\infty} \langle v, \phi_n\rangle^2_{L^2(0,1)}$ for any $v \in L^2(0,1)$,
6) $Av = -\sum_{n=1}^{\infty} \lambda_n \langle v, \phi_n\rangle_{L^2(0,1)} \phi_n(x)$ for any $v \in D(A)$ with $D(A) = \{v \in L^2(0,1)$ such that $\sum_{n=1}^{\infty} \lambda_n^2 \langle v, \phi_n\rangle^2_{L^2(0,1)} < +\infty\}$.
7) $S(t)v = \sum_{n=1}^{\infty} e^{-\lambda_n t}\langle v, \phi_n\rangle \phi_n$ for all $(v,t) \in L^2(0,1) \times [0,\infty)$.

We now can define the domain of $(-A)^{1/2}$ by

$$D((-A)^{1/2}) = \left\{ v \in L^2(0,1) \text{ s.t. } \sum_{n=1}^{\infty} \lambda_n \langle v, \phi_n\rangle^2 < \infty \right\} \tag{13}$$

and the unbounded self-adjoint operator $(-A)^{1/2}$ in $L^2(0,1)$ by

$$(-A)^{1/2}v = \sum_{n=1}^{\infty} \lambda_n^{1/2} \langle v, \phi_n\rangle \phi_n \tag{14}$$

for any $v \in D((-A)^{1/2})$. We then have the following:

*Lemma 10:* $D((-A)^{1/2}) = H_0^{1,a}(0,1)$ and $\|v\|_{D((-A)^{1/2})} = \left\|(-A)^{1/2}v\right\|_{L^2(0,1)} = \|v\|_{H_0^{1,a}(0,1)}$ and consequently $D((-A)^{1/2}) \hookrightarrow C^0[0,1]$.

**Proof:** See [9] □

In order to prove lemma 12, we have to use a fact referred to [1]:

*Lemma 11:* The space $D(A)$ is completely imbedded in $D((-A)^{1/2})$.

**Proof:** See [1] □

We next state and prove some properties of $F$.

*Lemma 12:* The operator $F$ defined by (12) is local Lipschitz.

**Proof:** Let $u, v \in D(A)$. It follows form lemmas 20 and 10 that there exists a positive constant $M$ such that $|u| \leq M$ and $|v| \leq M$. Locally Lipschitz condition of $f$ and lemma 10 imply that there exists a positive constant $L_M$ depending on $M$ such that

$$\begin{aligned} & \|F(u)-F(v)\|^2_{L^2(0,1)} \\ = & \int_0^1 |F(u)(x)-F(v)(x)|^2\,dx \\ = & \int_0^1 |f(u)-f(v)|^2\,dx \\ \leq & L_M^2 \int_0^1 |u(x)-v(x)|^2\,dx \\ \leq & L_M^2 \|u-v\|^2_{C^0[0,1]} \\ \leq & C_0^2 L_M^2 \|u-v\|^2_{D((-A)^{1/2})} \\ \leq & C_1^2 L_M^2 \|u-v\|^2_{D(A} \end{aligned}$$

where $C_0$ and $C_1$ are the constants involved in the Sobolev embedding $H_0^{1,a}(0,1) \hookrightarrow C^0[0,1]$. and $D((-A)^{1/2}) \hookrightarrow D(A)$, respectively. □

Moreover, we show that the operator $F$ defined by (12) is Hölder continuous of exponent $\alpha \in (0,1)$. Before going to that point, we give the definition of mild solution of the equivalent semilinear evolution problem (9).

*Definition 13:* A solution $u$ is said to be a **mild solution** of the equivalent semilinear evolution problem (9) if there exists $u \in C([0,\infty), H_a^1(0,1))$ such that

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))d\tau$$

with $u_0 \in H_a^1(0,1)$.

Based on the proof of theorem 2.5.1 of [16], we have the following.

*Lemma 14:* The equivalent semilinear evolution problem (9) has a unique mild solution $u$ on the time interval $[0,T]$ for some positive constant $T$. Moreover, let $u(t)$ and $\widetilde{u}(t)$ be mild solutions corresponding to $u_0$ and $\widetilde{u}_0$, respectively. Then for all, $t \in [0,T]$, the following estimate holds

$$\|u(t)-\widetilde{u}(t)\|_{H_a^1(0,1)} \leq \|u_0-\widetilde{u}_0\|_{H_a^1(0,1)} e^{C_1 T^{1/2}},$$

for some positive constant $C_1$.

By modifying the proof of corollary 2.5.1 of [16], we establish the following lemma.

*Lemma 15:* The mild solution $u$ of the equivalent semilinear evolution problem (9) is Hölder continuous of exponent $\alpha = (1/2)$ in $t$ for any $u_0 \in D(A)$.

*Proposition 16:* The operator $F$ defined by (12)is Hölder continuous of exponent $\alpha = (1/2)$ in $t$.

**Proof:** Since $F$ satisfies the locally Lipschitz condition and $u$ is Hölder continuous of exponent $\alpha = (1/2)$ in $t$, $F$ is also Hölder continuous of exponent $\alpha = (1/2)$ in $t$. □

Now we are in a position to prove theorem 2.

**Proof of theorem 2:** It follows directly from proposition 6 and 16. □

### *B. The proof of theorem 3*

Let us modify the proof of theorem 2.5.5 of [16] to obtain the following result.

*Lemma 17:* Let $[0, T_{max})$ be the maximal time interval in which the mild solution $u$ of the equivalent semilinear evolution problem (9) exists. If $T_{max}$ is finite, then the solution $u$ of the semilinear parabolic problem (8) blows up in finite time $T_{max}$, i.e.,

$$\lim_{t\to T_{\max}} \|u(t)\|_{H_0^{1,a}(0,1)} = \infty.$$

Before proving theorem 3, we have to find some useful properties of the analytic semigroup $S(t)$ generated by operator $A$. By modifying the proof of proposition 2.3.1.4 and 2.3.1.5 in [9], we obtain two results

*Lemma 18:* If $v \in D((-A)^{1/2})$, then $S(t)v \in D((-A)^{1/2})$ and $\left\|(-A)^{1/2}S(t)v\right\|_{L^2(0,1)} = \left\|S(t)(-A)^{1/2}v\right\|_{L^2(0,1)} \leq \left\|(-A)^{1/2}v\right\|_{L^2(0,1)}$.

*Lemma 19:* There exists a position $C_2$ such that $\left\|(-A)^{1/2}S(t)v\right\|_{L^2(0,1)} = \|S(t)v\|_{H_0^{1,a}(0,1)} \leq \frac{C_2}{t^{1/2}} \|v\|_{L^2(0,1)}$ for any $(v,t) \in L^2(0,1) \times (0,+\infty)$.

We next prove theorem 3.

**Proof of theorem 3:** We will prove theorem 3 by contradiction argument. Suppose that there exists a positive constant $M$ such that $\max_{x\in[0,1]} |u(x,t)| \leq M$ as $t \to T_{\max}$. It follows from $u(t) = S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))d\tau$ that

$$\begin{aligned} \|u(t)\|_{H_0^{1,a}} \leq & \|S(t)u_0\|_{H_0^{1,a}} \\ & + \int_0^t \|S(t-\tau)F(u(\tau))\|_{H_0^{1,a}}\,d\tau. \end{aligned}$$

By lemmas 18 and 19, we obtain

$$\begin{aligned}\|u(t)\|_{H_0^{1,a}} &\leq \|u_0\|_{H_0^{1,a}} + C\int_0^t \frac{\|F(u(\tau))\|_{L^2(0,1)}}{(t-\tau)^{1/2}}d\tau \\ &\leq \|u_0\|_{H_0^{1,a}} + Cf(M)\int_0^t \frac{1}{(t-\tau)^{1/2}}d\tau \\ &= \|u_0\|_{H_0^{1,a}} + 2Cf(M)t^{1/2},\end{aligned}$$

for some positive constant $C$. So, as $t \to T_{\max}$, $\|u(t)\|_{H_0^{1,a}(0,1)}$ is bounded which contradicts to lemma 17. Hence the proof of this theorem is complete. □

### C. The proof of theorem 4

To show the blow-up set of such a blow-up solution $u$ of problem (8), we need following lemmas.

*Lemma 20:* Let $v \in L^2(0,1)$ and $v \geq 0$ a.e. on $(0,1)$. Then, for all $t > 0$, we have $S(t)v \geq 0$ a.e. on $(0,1)$.

**Proof:** Suppose that there exists a $(x_1,t_1) \in (0,1)\times(0,\infty)$ such that $(S(t_1)v)(x_1) < 0$. By continuity of $S(t)$ for $t > 0$, there exists a positive constant $\varepsilon$ such that $(S(t)v)(x) < 0$ for any $(x,t) \in (x_1-\varepsilon, x_1+\varepsilon)\times(t_1-\varepsilon, t_1+\varepsilon)$. Let us consider the following auxiliary problem:

$$\left.\begin{array}{c} w_t - (a(x)w_x)_x = 0,\ (x,t)\in(0,1)\times(0,\infty), \\ w(0,t) = 0 = w(1,t),\ t>0, \\ w(x,0) = v(x),\ x\in[0,1]. \end{array}\right\} \tag{15}$$

We remark that if the solution $w$ of problem (15) exists, then by maximum principle for parabolic type, $w \geq 0$ on $[0,1]\times[0,\infty)$. We transform problem (15) into the equivalent evolution problem:

$$w_t - Aw = 0,\ t>0 \text{ and } w(0) = v, \tag{16}$$

where $A$ is an operator mapping from $L^2$ to $L^2$ defined by (11). As in previous discussion, the evolution problem (16) has a solution $w(t) = S(t)v$. for $t \geq 0$ From remark, we have that $(S(t)v)(x) \geq 0$ for any $(x,t) \in [0,1]\times[0,\infty)$. We therefore get a contradiction. □

*Lemma 21:* Let $u$ be a continuous solution of problem (8). Then $u(x,t) \geq u_0(x)$ for any $(x,t) \in [0,1]\times[0,T_{max})$ and $u$ is a nondecreasing function in $t$.

**Proof:** Let $w(x,t) = u(x,t) - u_0(x)$ for any $(x,t) \in [0,1]\times[0,T_{max})$. Let us consider:

$$w_t - (a(x)w_x)_x = f(u) + \frac{d}{dx}\left(a(x)\frac{du_0(x)}{dx}\right).$$

for any $(x,t) \in [0,1]\times[0,T_{max})$. Condition (C) yields that

$$\begin{aligned} w_t - (a(x)w_x)_x &\geq f(u) - f(u_0) \\ &= f'(\xi_1)w(x,t) \end{aligned}$$

where $\xi_1$ is some constant between $u$ and $u_0$. We moreover have that $w(0,t) = 0 = w(1,t)$ for $t \in [0,T_{max})$ and $w(x,0) = u(x,0) - u_0(x) = 0$ for $x \in [0,1]$. Maximum principle for parabolic type implies that $w \geq 0$ on $[0,1]\times[0,T_{max})$, that is, $u \geq u_0$ on $[0,1]\times[0,T_{max})$.

Let $h$ be any positive constant with $0 < t+h < T_{max}$. Let $v(x,t) = u(x,t+h) - u(x,t)$ for any $(x,t) \in [0,1]\times[0,T_{max}-h)$. We then consider:

$$\begin{aligned} v_t - (a(x)v_x)_x &= f(u(x,t+h)) - f(u(x,t)) \\ & \quad f'(\xi_2)f(w(x,t)) \end{aligned}$$

where $\xi_2$ is some constant between $u(x,t+h)$ and $u(x,t)$ for any $(x,t) \in (0,1)\times(0,T_{max}-h)$. We furthermore have that $v(0,t) = 0 = v(1,t)$ for $t \in (0,T_{max}-h)$ and it follows from $u \geq u_0$ on $[0,1]\times[0,T_{max}-h)$ that $v(x,0) = u(x,h) - u_0(x) \geq 0$ for any $x \in [0,1]$. Maximum principle for parabolic type implies that $v \geq 0$ on $(0,1)\times(0,T_{max}-h)$. Therefore the proof of this lemma is complete. □

We next give the proof of theorem 4.

**Proof of theorem 4:** Let $(x,t) \in (0,1)\times(0,T_{\max})$. We then consider that

$$\begin{aligned} u(x,t) &= u(t)(x) \\ &= (S(t)u_0)(x) + \int_0^t (S(t-\tau)F(u(\tau)))(x)d\tau. \end{aligned}$$

We then have

$$\begin{aligned} |u(x,t)| &\leq |(S(t)u_0)(x)| \\ &\quad + \int_0^t |(S(t-\tau)F(u(\tau)))(x)|\,d\tau. \end{aligned}$$

Since there exists a positive constant $C_2$ such that $|S(t)u_0(x)| \leq C_2$, we obtain that

$$|u(x,t)| \leq C_2 + \int_0^t |(S(t-\tau)F(u(\tau)))(x)|\,d\tau$$

or

$$\max_{x\in[0,1]}|u(x,t)| \leq C_2 + \int_0^t |(S(t-\tau)F(u(\tau)))(x)|\,d\tau.$$

By theorem 3, we have

$$\int_0^t |(S(t-\tau)F(u(\tau)))(x)|\,d\tau \to \infty \text{ as } t \to T_{\max}.$$

It then follows from lemmas 20 and 21 that

$$\begin{aligned}&\int_0^t \left|\left(S(t-\tau)F(u(\tau))\right)(x)\right| d\tau \\ = &\int_0^t \left(S(t-\tau)F(u(\tau))\right)(x) d\tau.\end{aligned}$$

and then we obtain that

$$\int_0^t \left(S(t-\tau)F(u(\tau))\right)(x) d\tau \to \infty \text{ as } t \to T_{\max}.$$

On the other hand, lemma 20 and condition (C) imply that

$$u(x,t) \geq \int_0^t \left(S(t-\tau)F(u(\tau))\right)(x) d\tau.$$

Then, we have that $u(x,t)$ approaches to infinity for all $x \in (0,1)$ as $t$ converges to $T_{max}$. Moreover, for $x \in \{0,1\}$, we can find a sequence $\{(x_n,t_n)\}$ in $(0,1)\times(0,T_{\max})$ such that $\lim\limits_{t\to T_{\max}} u(x_n,t_n) \to \infty$. Therefore, the proof of theorem 4 is complete. □

### *D. The proof of theorem 5*

We next show the proof of theorem 5.

**Proof of theorem 5:** Multiplying problem (8) by $\phi_1$ and integrating problem (8) with respect to $x$ from 0 to 1 yield

$$\begin{aligned}\frac{dH(t)}{dt} &= -\lambda_1 H(t) + \int_0^1 f(u)\phi_1(x)dx \\ &\geq -\lambda_1 H(t) + b\int_0^1 u^p(x,t)\phi_1(x)dx.\end{aligned}$$

Hölder inequality implies that

$$\int_0^1 \phi_1(x)u(x,t)dx \leq \left(\int_0^1 \phi_1 u^p dx\right)^{\frac{1}{p}} \left(\int_0^1 \phi_1(x)dx\right)^{\frac{p-1}{p}}$$

or

$$\left(\int_0^1 \phi_1(x)u(x,t)dx\right)^p \leq \int_0^1 \phi_1(x)u^p(x,t)dx.$$

We obtain that

$$\frac{dH(t)}{dt} \geq -\lambda_1 H(t) + bH^p(t)$$

and then we can rewrite such a inequality in the form

$$H^{p-1}(t) \geq \frac{1}{\frac{b}{\lambda_1} + \left[H^{1-p}(0) - \frac{b}{\lambda_1}\right] e^{-\lambda_1(1-p)t}}.$$

The second assumption of theorem 5 implies that there exists a finite time $T$ such that $H$ tends to infinity as $t$ converges to $T$. By definition of $H$, we obtain

$$H(t) \leq \int_0^1 \phi_1(x)dx \max_{x\in[0,1]} |u(x,t)| = \max_{x\in[0,1]} |u(x,t)|.$$

Therefore, the solution $u$ of problem (8) blows up in finite time. □

## V. Conclusion

As shown in [11], if we would like to prove the existence and uniqueness of a blow-up solution by Green's function method, we have to make many assumptions on functions $k$ and $a$ to guarantee the existence of eigenvalues and eigenfunctions of such a problem which contrast to method in semigroup theory. But the difficulty of applying semigroup theory is to construct the suitable Banach spaces.

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