Study of non-stationary heat transfer in two-layer plate

Hana Charvátová, Dagmar Janáčová, Vladimír Vašek, Pavel Mokrejš, Jan Hrabovský

***Abstract*—**In the paper we deal with study of unsteady heat transport process in solids. Especially we focused on a problem of non-stationary conduction of heat in a two-layer plane plate. For this purpose we formulated mathematical model describing heating or cooling of a semi-infinite region. The analytical solution of this model we used for computer modeling of the mentioned process by use of mathematical software Maple.

In the second part of the paper we demonstrate modeling of computing of heating or cooling of the two-layer plane plate by use of the software application that we programmed for automatic computing of temperature fields in the solids during heating or cooling of the two-layer plane plate. We also verified validity of the formulated problem by comparison of the computed data with computer simulation of the process by use of commercial software Comsol Multiphysics. Finally, we described main parameters that influence heating or cooling process course and described mathematical model use for economical costs of the studied process computing.

***Keywords*—**Mathematical model, Non-stationary heat conduction, Temperature field, Two-layer plate

I. INTRODUCTION

MANY technological operations of the synthetic materials treatment are based on procedures of heat effect on the processed body. These operations are generally energy demanding. Therefore we deal with their optimization [1], [2], [3], [4]. For finding of an appropriate optimization method, it is necessary to get information about given process course. But these data are often hardly experimentally determined and in addition time-consuming. Therefore mathematical modeling and computer simulation of the studied process can be only alternative how this information to obtain [5].

In many cases, the mathematical model of given process is very specific and in addition depending on concrete conditions of the process [8], [9], [10]. In this paper we will focus on a case of unsteady conduction of heat in the solid material. We will study problem of non-stationary conduction of heat in the two layers plate. In the following text we will formulate mathematical model of the mentioned process. Next we will present software application that we programmed for computing of unsteady temperature fields in the two-layer plate during its one-sided heating or cooling.

Finally we will show results that we obtained by comparison of the computed data with computer simulation of the process by use of the commercial software Comsol Multiphysics.

II. MATHEMATICAL DESCRIPTION OF THE SOLVED PROBLEM

In this section we will formulate mathematical model in the case of one-sided heating or cooling of two-layers semi-infinite wall in the region $0 < b < \infty$ [11]. Geometrical sketch of the problem you can see in Fig. 1.

The plate with the initial constant temperature tp will be exposed to the sudden one-sided heat action. Mathematical model of the process can be described by Fourier equation of the heat conduction (1) with initial and boundary conditions (2) – (7) [5][8] :

$$\frac{\partial t_1}{\partial \tau}(x,\tau) = a_1 \frac{\partial^2 t_1}{\partial x^2}(x,\tau),\ \tau > 0,\ 0 < x < b \tag{1}$$

$$\frac{\partial t_2}{\partial \tau}(x,\tau) = a_2 \frac{\partial^2 t_2}{\partial x^2}(x,\tau),\ \tau > 0,\ b < x < \infty \tag{2}$$

$$t_1(x,0) = t_2(x,0) = t_p \tag{3}$$

$$t_1(0,\tau) = t_o \tag{4}$$

$$\frac{\partial t_2(\infty,\tau)}{\partial x} = 0\ \ t_1(0,\tau) = t_o \tag{5}$$

$$t_1(b,\tau) = t_2(b,\tau) \tag{6}$$

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$$\lambda_1 \frac{\partial t_1(b,\tau)}{\partial \tau} = \lambda_2 \frac{\partial t_2(b,\tau)}{\partial \tau} \qquad (7)$$

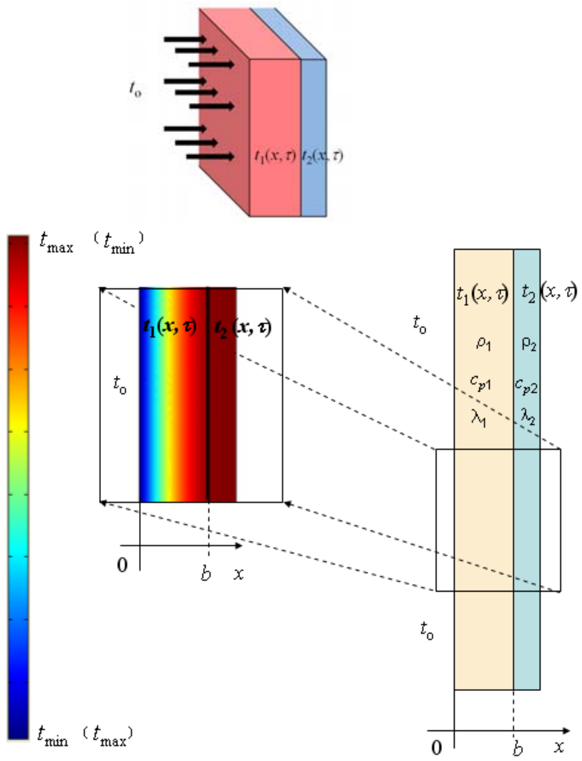

Fig. 1 Geometrical sketch of the model of unsteady heat conduction in two-layer plane plate

where $a_1 = \dfrac{\lambda_1}{\rho_1 \cdot c_{p1}}$ or $a_2 = \dfrac{\lambda_2}{\rho_2 \cdot c_{p2}}$ are thermal conductivity of the first or second layer of the heated (cooled) solid.

The condition (3) is assumption of the initial constant temperature in the plate. The boundary condition (4) is assumption of the time independent temperature in the boundary of the first layer and surrounding fluid.

The boundary condition (5) is assumption of semi-infinite region. The boundary conditions (6) and (7) are assumptions of perfect contact of the layers. The relations (8) and (9) describe analytical solutions of the model [5],[8].

The equation (8) describes temperature fields in the first layer $t_1(x,\tau)$:

$$\frac{t_1 - t_p}{t_o - t_p} = \left[\sum_{n=0}^{\infty} h^n \left\{ \operatorname{erfc}\frac{(2n+1)b + x}{2\sqrt{a_1\tau}} - h \cdot \operatorname{erfc}\frac{(2n+1)b - x}{2\sqrt{a_1\tau}} \right\}\right] \qquad (8)$$

Distribution of temperature in the second layer $t_2(x,\tau)$ can be described by equation (9):

$$\frac{t_2 - t_p}{t_o - t_p} = \frac{2K_\varepsilon}{1+K_\varepsilon} \sum_{n=1}^{\infty} h^{n-1} erfc \left(\frac{x - b + (2n-1)K_a^{-\frac{1}{2}} b}{2\sqrt{a_2 \tau}} \right) \qquad (9)$$

where

$$h = \frac{1-K_\varepsilon}{1+K_\varepsilon} \qquad (10)$$

$$K_a^{-\frac{1}{2}} = \sqrt{\frac{a_2}{a_1}} = \sqrt{\frac{\lambda_2 \rho_1 c_{p1}}{\lambda_1 \rho_2 c_{p2}}} \qquad (11)$$

$$K_\varepsilon = \frac{\lambda_1}{\lambda_2} \sqrt{\frac{\lambda_2 \rho_1 c_{p1}}{\lambda_1 \rho_2 c_{p2}}} \qquad (12)$$

III. LIST OF SYMBOLS

a - thermal diffusivity, $m^2 \cdot s^{-1}$;
c_p - specific thermal capacity, $J \cdot kg^{-1} \cdot K^{-1}$;
t - temperature of the heated (cooled) body, °C;
t_o - ambient temperature, °C;
t_p - initial temperature of the heated (cooled) body, °C;
x - space coordinate, m;
b - thickness, m;
λ - thermal conductivity, $W \cdot m^{-1} \cdot K^{-1}$;
ρ - density, $kg \cdot m^{-3}$;
τ - time, s;
1 - properties of the first layer;
2 - properties of the second layer.

IV. COMPUTER MODELING OF THE PROCESS

We computed temperature fields (8) and (9) in the mathematical software Maple user environment. In the Fig. 2 we show course the temperature field in the plate during heating. In the Fig. 3 - 4 you can see course the temperature field in the plate during cooling.

For simplification of computation of the temperature fields we also programmed interactive application for modeling of the above described process of the heat action course in the mathematic software Maple environment [13]. We made the application in the Maplet form which enables us to insert required input parameters, automatic compute and display temperature fields as both 3D graphics $t(x, \tau)$ and 2D graphics $t(x)$ in the required time of the process. Our software application can also compare the temperature fields in various time of the process and export the displayed graphics. In the Fig. 2 we present user interface of the application.

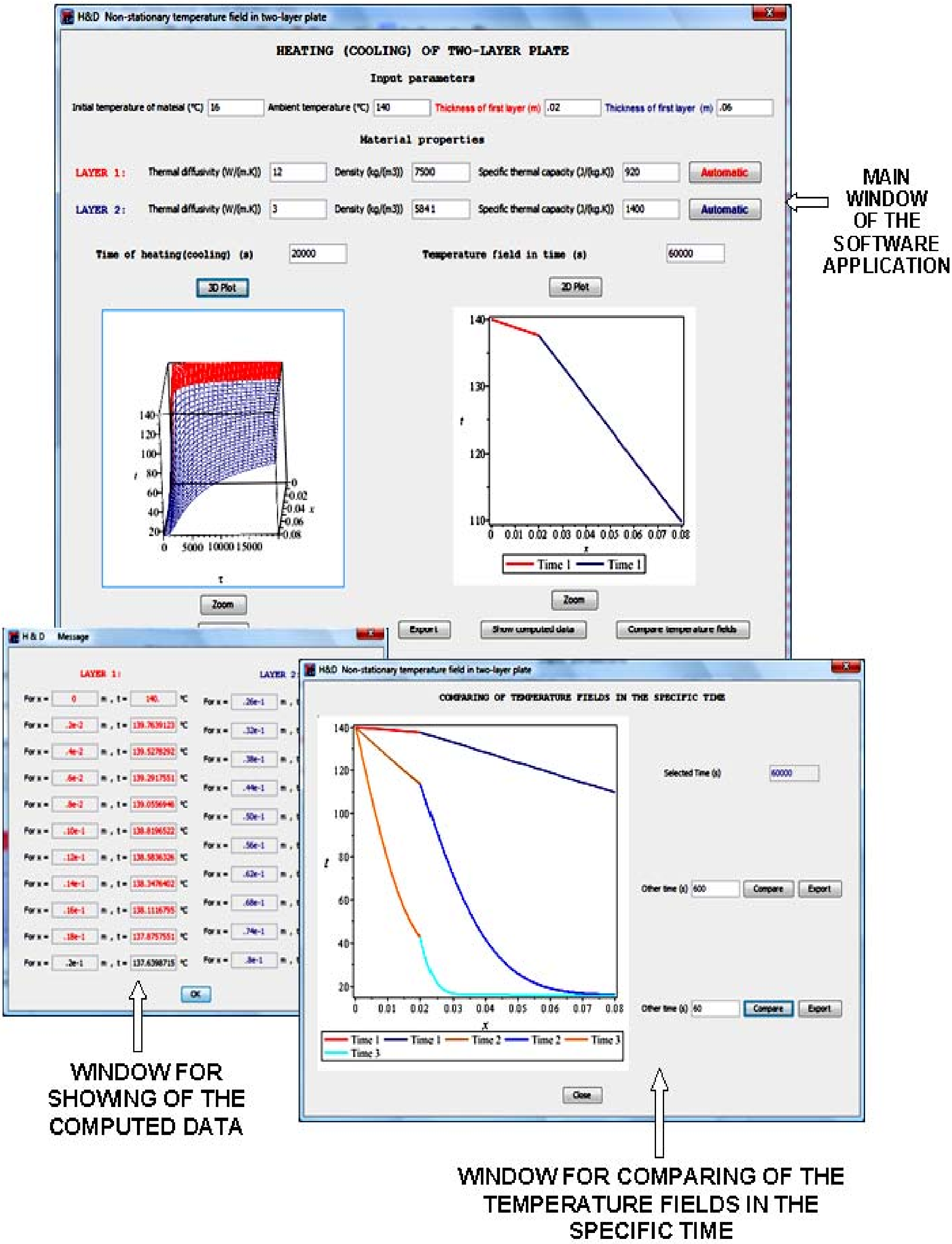

Fig. 2 Show of the programmed software application user environment

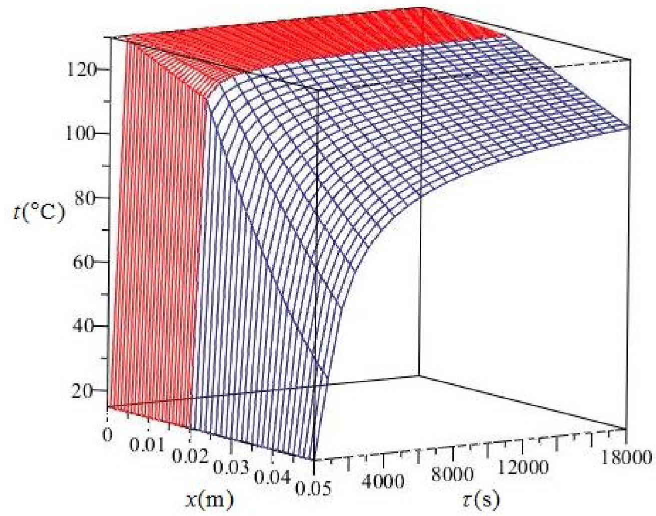

Fig. 3 Temperature field in the two-layer plate during heating
t_p = 15 °C, t_o = 130 °C, b_1 = 0.03 m, b_2 = 0.05 m,
a_1= 2.0·10^{-6} m^2.s^{-1}, a_2 = 5.8·10^{-6} m^2.s^{-1}

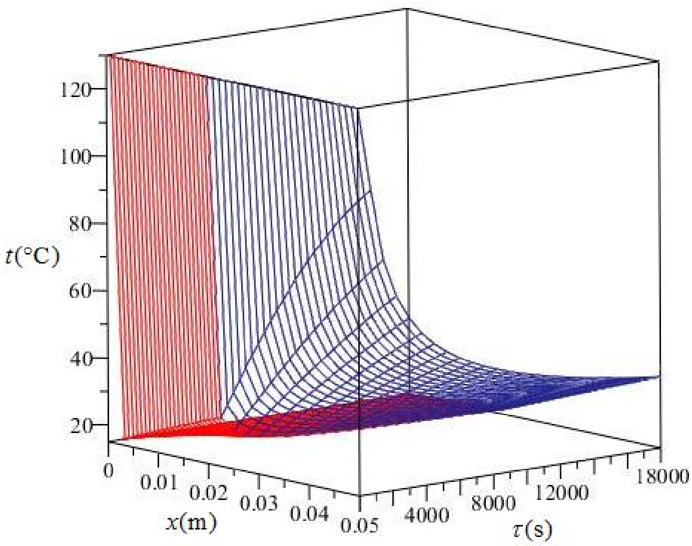

Fig. 4 Temperature field in the two-layer plate during cooling
t_p = 130 °C, t_o = 15 °C, b_1 = 0.03 m, b_2 = 0.05 m,
a_1= 2.0·10^{-6} m^2.s^{-1}, a_2 = 5.8·10^{-6} m^2.s^{-1}

V. Comparing of the Computed Data with Computer Simulation of the Process

Nowadays, the computer simulation is essential part of science and engineering. Digital analysis of components, in particular, is important for development of new products or for optimizing desing [19]. Therefore, we compared temperature fields computed by Maple with computer simulation by use of the commercial software Comsol Multiphysics for verification of the formulated model validity.

In the Fig. 5 – 8 we show simulated and computed data that we obtained by testing of heating of two-layer plate under the same conditions. Initial temperature of the plate is 20 °C, ambient temperature is 200 °C. Thickness of the first layer is 0.03 m, thickness of the second layer is 0.04 m. Thermal conductivity of the first layer is 2.15·10^{-6} m^2.s^{-1}. Thermal conductivity of the second layer is 3.06·10^{-6} m^2.s^{-1}.

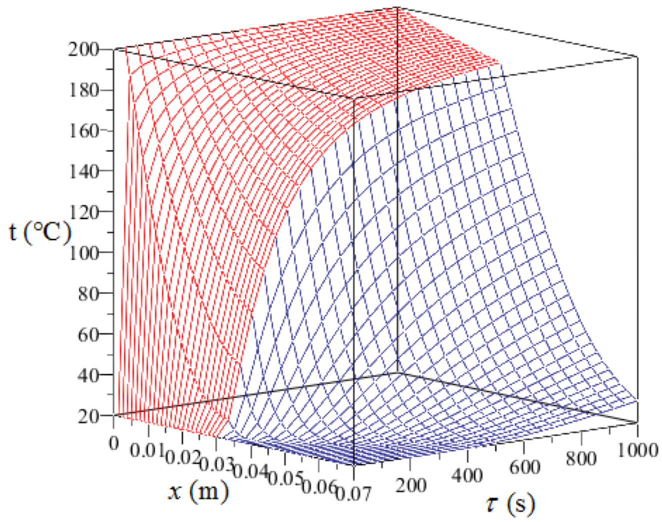

Fig. 5 Course of temperature field in the plate computed by software application programmed in Maple environment

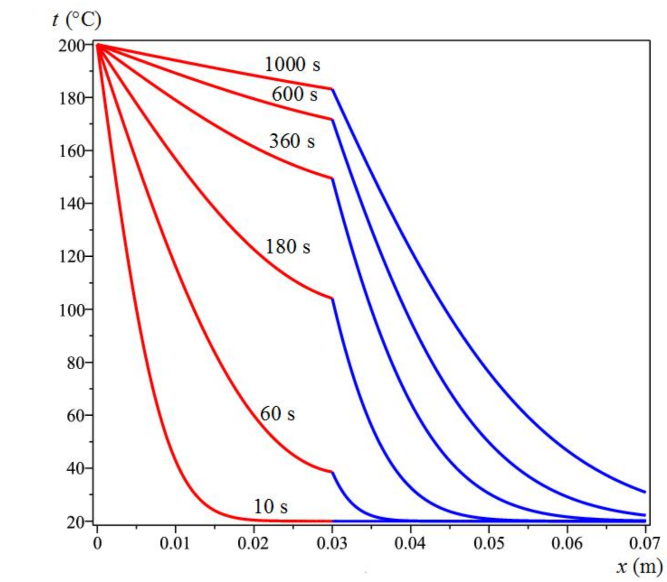

Fig. 6 Temperature fields in the plate, computed by the software application programmed in Maple environment in time 10 s, 60 s, 180 s, 360 s, 600 s and 1000 s.

In the Fig. 5 and Fig. 6 you can see temperature fields that we have computed by use of our software application programmed in Maple environment. In the Fig. 5 we show course of the temperature field for 1000 seconds of the heating. The Fig. 6 represents temperature fields in time 10 s, 60 s, 180 s, 360 s, 600 s and 1000 s.

In the Fig. 7 - Fig. 11 we show temperature fields in time 10 s, 60 s, 180 s, 360 s, 600 s and 1000 s simulated in the Comsol Multiphysics.

It is evident, that in the both cases, the temperature fields have a similar course, which confirms possibility to use our software application for modeling of the mentioned process.

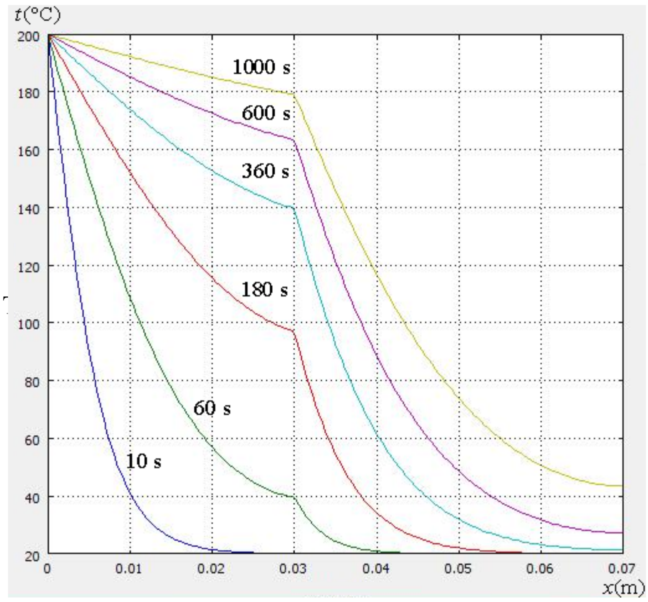

Fig. 7 Temperature fields in the plate simulated by Comsol Multiphysics in time 10 s, 60 s, 180 s, 360 s, 600 s and 1000 s

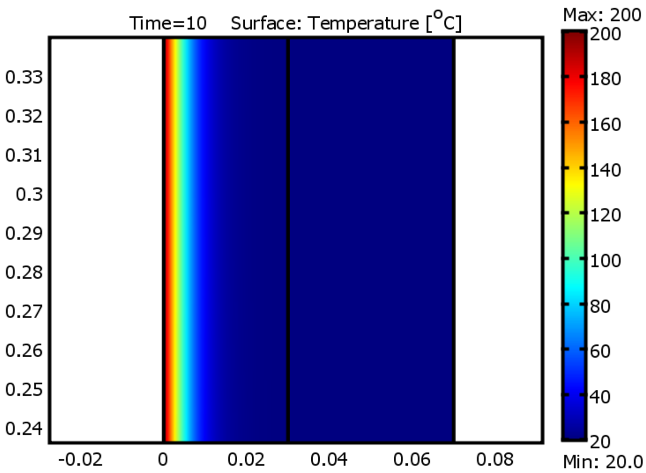

Fig.8 Distribution of temperature in section of the plate simulated by Comsol Multiphysics in time 10 seconds

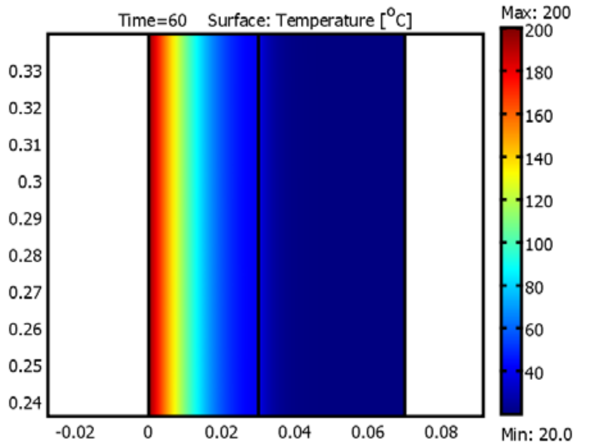

Fig. 9 Distribution of temperature in section of the plate simulated by Comsol Multiphysics in time 60 seconds

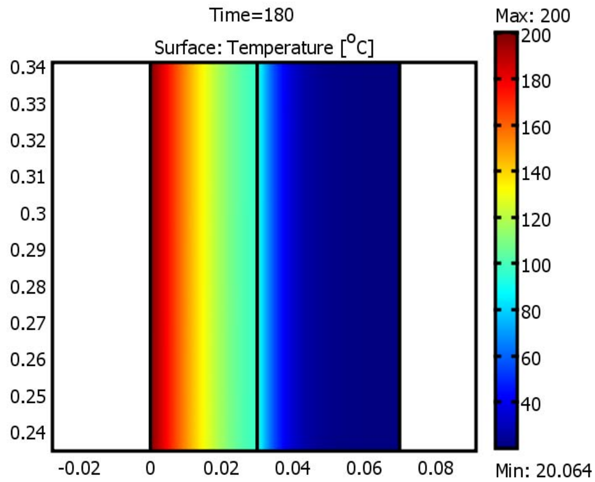

Fig. 10 Distribution of temperature in section of the plate simulated by Comsol Multiphysics in time 180 seconds

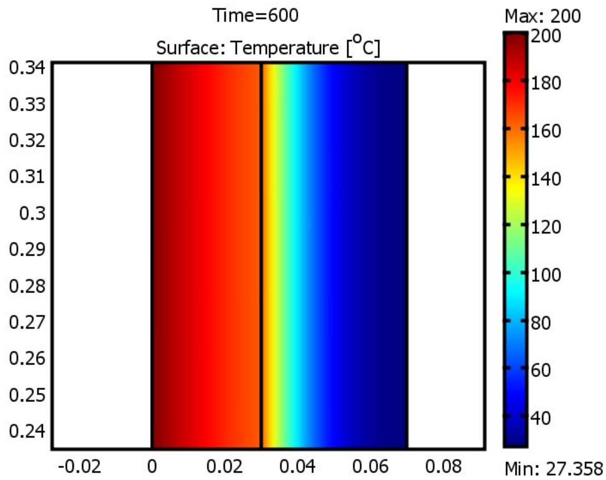

Fig. 11 Distribution of temperature in section of the plate simulated by Comsol Multiphysics in time 600 seconds

VI. Evaluation of the Parameters That Influence Non-Stationary Conduction of Heat in Two-Layer Plate

Process of heating or cooling of the plate be aimed at steady of the heat flow through the wall of plate. Therefore, time of the process belongs to the main parameters that influence temperature field course in the plate as you can see in the Fig. 12.

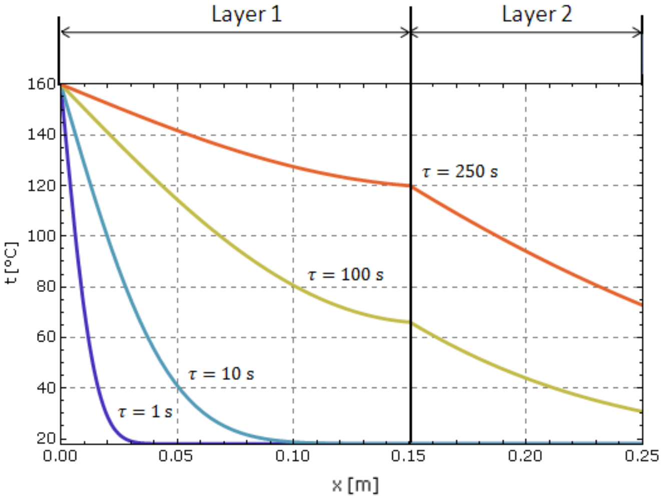

Fig. 12 Temperature fields in dependence on time of the process
$t_p = 18$ °C, $t_o = 160$ °C, $b_1 = 0.15$ m, $b_2 = 0.10$ m,
$\tau =$ variable, $\rho_1 = 7810$ kg.m^{-3}, $\rho_2 = 1900$ kg.m^{-3},
$c_{p1} = 600$ J.kg^{-1}.K^{-1}, $c_{p2} = 740$ J.kg^{-1}.K^{-1},
$\lambda_1 = 300$ W.m^{-1}.K^{-1}, $\lambda_2 = 20$ W.m^{-1}.K^{-1}

Thermal conductivity, specific thermal capacity and density of the layers also belong to the parameters that influence course of unsteady conduction of heat in two-layer plate.

In the figure 13 we show temperature fields in the plate in case of various thermal conductivity of the first layer. Low values of thermal conductivity are characteristic for the heat insulating materials as are felt, expanded polystyrene, cork etc. On the contrary, metals belong to the materials that have high thermal conductivity. During heating, the value of higher thermal conductivity causes that in the given place and time, the temperature will be higher in comparison with materials of lower thermal conductivity [15].

Specific thermal capacity determines quantity of the heat energy needed for heating of 1 kg of the material about 1 °C degree. Density of material determines mass of volume unit of the solid material. In case of higher value of specific thermal capacity or density of material, speed of the process decreases. In the Fig. 14 we show temperature fields in the plate for various specific thermal capacity of the first layer (layer 1) [16], [17].

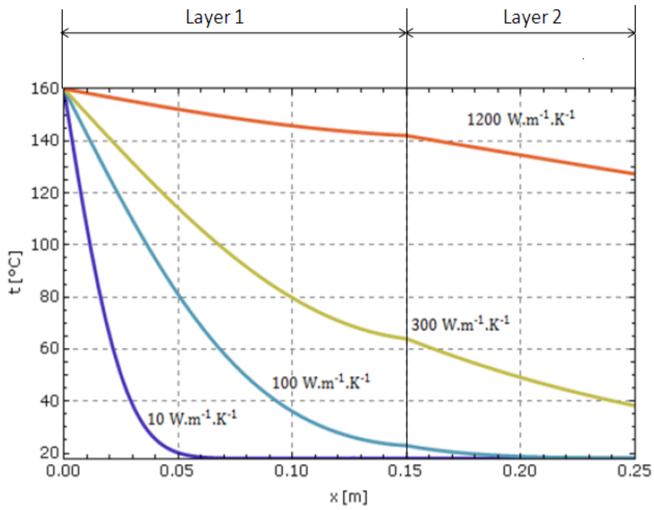

Fig. 13 Temperature fields in the plate for various thermal conductivity of the first layer (layer 1)
$t_p = 18$ °C, $t_o = 160$ °C, $b_1 = 0.15$ m, $b_2 = 0.10$ m,
$\tau = 100$ s, $\rho_1 = 7810$ kg.m^{-3}, $\rho_2 = 1900$ kg.m^{-3},
$\lambda_1 =$ variable, $\lambda_2 = 20$ W.m^{-1}.K^{-1},
$c_{p1} = 600$ J.kg^{-1}.K^{-1}, $c_{p2} = 740$ J.kg^{-1}.K^{-1}

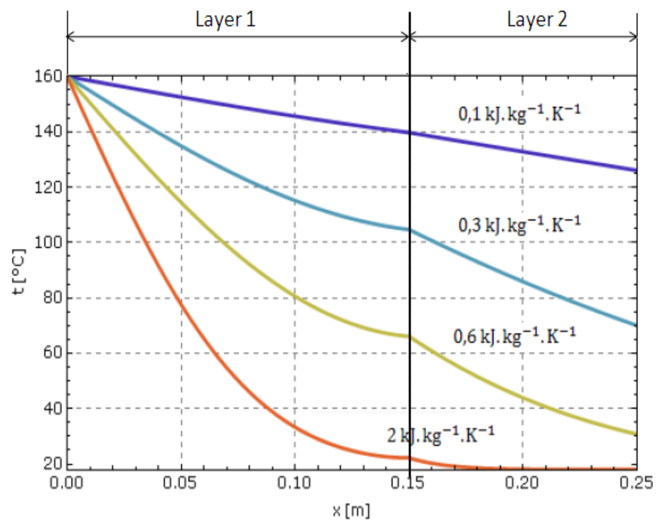

Fig. 14 Temperature fields in the plate in case of various specific thermal capacity of the first layer (layer 1)
$t_p = 18$ °C, $t_o = 160$ °C, $b_1 = 0.15$ m, $b_2 = 0.10$ m,
$\tau = 100$ s, $\rho_1 = 7810$ kg.m^{-3}, $\rho_2 = 1900$ kg.m^{-3},
$\lambda_1 = 300$ W.m^{-1}.K^{-1}, $\lambda_2 = 20$ W.m^{-1}.K^{-1},
$c_{p1} =$ variable, $c_{p2} = 740$ J.kg^{-1}.K^{-1}

As you can see in Fig. 15, course of the temperature field during heating or cooling also depends on difference of initial temperature of the plate and ambient temperature.

In the Fig. 16 are shown temperatures in the plate for distance 10 cm form the left margin of the plate. In this case we computed temperature fields for four various values of the ambient temperature (50 °, 160 °C, 300 °C and 400 °C). The initial temperature of the plate was 0 °C. The operating time was 100 s.

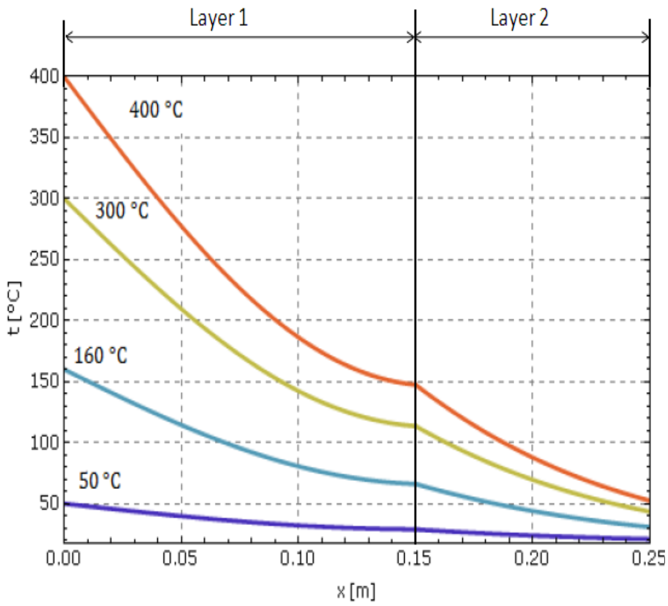

Fig. 15 Temperature fields in the plate for various ambient temperature
$t_p = 0$ °C, t_o =variable,
$b_1 = 0.15$ m, $b_2 = 0.10$ m,
$\tau = 100$ s, $\rho_1 = 7810$ kg.m^{-3}, $\rho_2 = 1900$ kg.m^{-3},
$c_{p1} = 600$ J.kg^{-1}.K^{-1}, $c_{p2} = 740$ J.kg^{-1}.K^{-1},
$\lambda_1 = 300$ W.m^{-1}.K^{-1}, $\lambda_2 = 20$ W.m^{-1}.K^{-1}

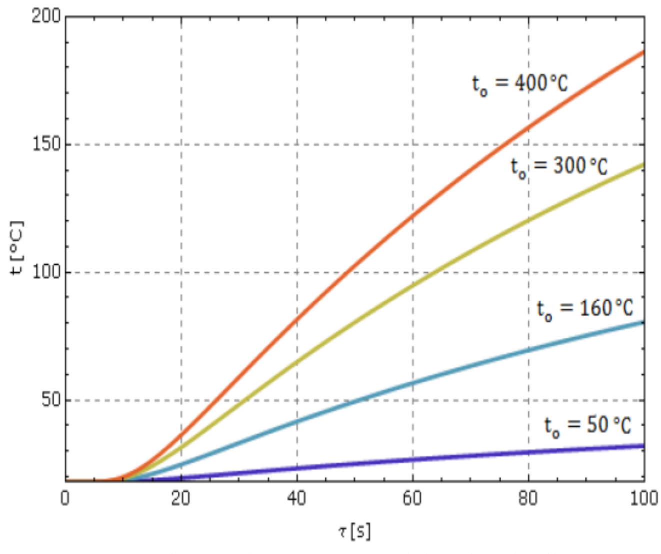

Fig. 16 Dependence of temperature of the plate in distance 10 cm from the left margin on time of the process for various ambient temperatures
$t_p = 0$ °C, t_o =variable,
$b_1 = 0.15$ m, $b_2 = 0.10$ m,
$\tau = 100$ s, $\rho_1 = 7810$ kg.m^{-3}, $\rho_2 = 1900$ kg.m^{-3},
$c_{p1} = 600$ J.kg^{-1}.K^{-1}, $c_{p2} = 740$ J.kg^{-1}.K^{-1}
$\lambda_1 = 300$ W.m^{-1}.K^{-1}, $\lambda_2 = 20$ W.m^{-1}.K^{-1}

Fig. 17 demonstrates dependence of temperature of the heated plate on the thickness of first layer (layer 1) in distance 4 cm from the left margin on time. It is clear, that the process will be faster in the thinner plates than in the thicker plates under the same conditions.

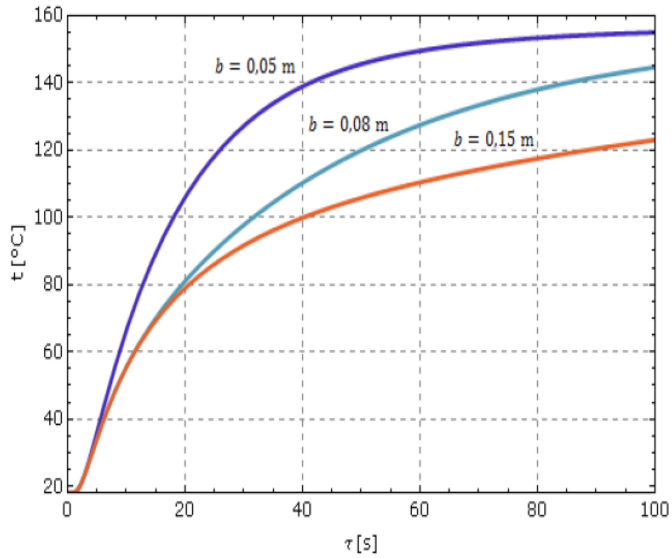

Fig. 17 Dependence of temperature of the first layer (layer 1) of the plate in distance 10 cm from the left margin on thickness of the layer
$t_p = 0$ °C, $t_o = 160$ °C,
b_1 = variable, $b_2 = 0.10$ m,
$\tau = 100$ s, $\rho_1 = 7810$ kg.m^{-3}, $\rho_2 = 1900$ kg.m^{-3},
$c_{p1} = 600$ J.kg^{-1}.K^{-1}, $c_{p2} = 740$ J.kg^{-1}.K^{-1},
$\lambda_1 = 300$ W.m^{-1}.K^{-1}, $\lambda_2 = 20$ W.m^{-1}.K^{-1}

VII. Economical Balance of the Heating (Cooling) Process

The mathematical model and its analytical solution described by equations (1) – (12) enable us to compute economical costs of the real processes that are based on similar mechanism that we described in this paper.

An optimal time to achieve the required temperature in the specific place of first or second layer of the plate can be numerical computed from analytical solutions (8) or (9).

In technical practice, the main economical costs needed for achievement of the required temperature in the plate depend on costs of unit of energy to the drive of machine, power of energy supply machine and on time of the process:

$$N = P \cdot \tau \cdot K_E \qquad (13)$$

where:

N - total costs , €;

τ - time, h;
P - power of energy supply time, kWh;
K_E - costs of energy unit, €/kWh.

VIII. Conclusion

We formulated mathematical model that is suitable for description of heating or cooling of two-layer plane plate in the case of the semi-infinite region.

Analytical solution of the model we used for programming of the software application that can compute temperature fields and enabled to get quick notion about studied process course. The software application can compute data necessary for optimization of many technological operations which are based on the similar mechanism as we described in this paper.

Validity of the formulated model we verified by comparison of the computed data with computer simulation of the process by use of Comsol Multiphysics. The obtained results proved possibility to use our software application for modeling of the real technological processes.

The software application enabled us also determine main parameters that influence heating or cooling process course and compute economical costs of the studied process.

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