Fuzzy Multi-objective Linear Plus Linear Fractional Programming Problem: Approximation and Goal Programming Approach

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***Abstract*—In this article, a fuzzy goal programming (GP) and method of approximation is presented for the solution of a multiobjective linear plus linear fractional programming problem. In the proposed approach, membership functions are defined for each fuzzy goal and then a method of variable change on the under- and over- deviational variables of the membership functions associated with the fuzzy goals of the model is introduced. Then the problem is solved efficiently by using goal programming(GP) methodology and method of approximation(MAP). Three numerical examples is given for verification of the method. The examples are solved by optimization software TORA@ 2.0 version, 2006.**
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I. INTRODUCTION

Optimization is the task of finding one or more solutions which corresponds to minimizing (or maximizing one or more specified objectives and which satisfy all constraints (if any). A single objective optimization problem involves a single objective function usually results a single solution, called an optimal solution. On the other hand, a multi-objective optimization task considerers several conflicting objectives simultaneously. In such cases there is usually no single optimal solution, but a set of alternatives with different trade - offs, called Pareto optimal solutions, or non dominated solutions. Despite the existence of multiple Pareto optimal solutions, in practice, usually only one of these solutions is to be chosen. Thus compared to single objective optimization problems, in multiobjective optimization, there at least two equally important task: an optimization task for finding Pareto optimal solution and a decision making task for choosing single most preferred solution.
The main interest in fractional programming was generated by the fact that a lot of optimization problems from engineering, natural resources and economics require the optimization of a ratio between physical and /or economic functions. The problems, where the objective functions appear as a sum of a linear function and a fractional function constitute a linear plus fractional programming problem. If we take more than one objective in this problem then the problem is referred to multiobjective linear plus linear fractional programming problem.

A general linear plus linear fractional programming (LLFP) problem is defined as the following way:

$$\begin{aligned}
&\text{Maximize} \quad F(x) = (p^T x + \theta) + \frac{c^T x + \alpha}{d^T x + \beta} \\
&\text{subject to} \\
&Ax = b \qquad\qquad (1)\\
&x \geq 0, \\
&\text{where} \quad x, c, d, p \in R^n, b \in R^m, \alpha, \beta, \theta \in R.
\end{aligned}$$

For some values of x, $d^T x + \beta$ may be equal to zero but here we take only the case $d^T x + \beta > 0$. If we take more than one objectives in general linear plus fractional programming problem, then the problem is known as multiobjective linear plus linear fractional programming problem, mathematically it can be written as:

$$\begin{aligned}
&\text{Maximize}\, F(x) = [F_1(x), F_2(x), \ldots F_k(x)], \\
&\text{where} \quad F_i(x) = l_i(x) + \frac{f_i(x)}{m_i(x)}, \qquad (2)\\
&\qquad x \in X.
\end{aligned}$$

and, $l_i(x) = p_i^T x + \theta_i$, $f_i(x) = c_i^T x + \alpha_i$, $m_i(x) = d_i^T x + \beta_i$, are real valued function on X, where $X = \{x : Ax\,(\leq, =, \geq)\,b,\, x \geq 0, x \in R^n, b \in R^m, A = (a_{ij})_{m\times n}, \theta_i, \alpha_i, \beta_i \in R\}$, and $d_i^T x + \beta_i > 0 \quad (i = 1, 2, \ldots, k)\,\forall \quad x \in X$.
Here, X is assumed to be non - empty convex bounded set in R^n.

If an uncertain aspiration level is introduced to each of the objectives of MOLLFP, then these fuzzy objectives are called fuzzy goals. The Fuzzy multiobjective linear plus linear fractional programming problem (FMOLLFP) can be defined as

$$\begin{aligned}
&\text{Find } X(x_1, x_2, ...x_n) \quad \text{such that} \\
&F_i(x) \lessapprox g_i \quad \text{or} \quad F_i(x) \ \gtrapprox \ g_i \,\forall (i = 1, 2, \ldots, k) \\
&\text{subject to} \\
&x \in X = \{x \in R^n, \, Ax \leq b, x \geq 0 \,\text{with}\, b \in R^m, \\
&A \in R^{m \times n}\} \qquad (3) \\
&F_i(x) = (p_i^T x + \theta_i) + \frac{c_i^T x + \alpha_i}{d_i^T x + \beta_i}
\end{aligned}$$

where g_i is the aspiration level of the i^{th} objective F_i and $\lessapprox$, $\gtrapprox$ indicate fuzziness of the aspiration level. The membership function $\mu_i(x)$ must be described for each fuzzy goal. A membership function can be explained as given below:

If $\quad F_i(x) \lessapprox g_i,$ then

$$\mu_i(x) = \begin{cases} 1 & if \quad F_i(x) \leq g_i \\ \frac{\overline{t_i} - F_i(x)}{\overline{t_i} - g_i} & if \quad g_i \leq F_i(x) \leq \overline{t_i} \\ 0 & if \quad F_i(x) \geq \overline{t_i} \end{cases} \qquad (4)$$

If $\quad F_i(x) \geq g_i,$ then

$$\mu_i(x) = \begin{cases} 1, & if \quad F_i(x) \geq g_i \\ \frac{F_i(x) - \underline{t_i}}{g_i - \underline{t_i}}, & if \quad \underline{t_i} \leq F_i(x) \leq g_i \\ 0, & if \quad F_i(x) \leq \underline{t_i} \end{cases} \qquad (5)$$

and $\overline{t_i}$ and $\underline{t_i}$ are the upper tolerance limit and lower tolerance limit, respectively, for the i^{th} fuzzy goal.

Multiobjective linear plus linear fractional programming (MOLLFP) are applied to different disciplines such as transportation, problem of optimizing enterprize capital, the production development fund and social, cultural and construction fund [7]. Basically it is used for modeling real life problems with one or more objectives when compromisation situation occurs [10]. Multiobjective linear plus linear fractional programming problem have been extensively studied by authors and the research is based on the theoretical background of fractional programming. As a matter of fact, many ideas and approaches have their foundation in the theory of fractional programming (See [1, 15]).

Teterev [10] pointed out this type of problem and he derived optimality criteria for (LLFP) using simplex type algorithm. Shaible [1, 2] has pointed out for LLFP that for these problems a local maximum is not a global one and an optimal solution is not attained at an extreme point of polyhedron in general. Chaddha [8] presented a dual of a maximization problem for linear plus linear fractional programming problem under linear constraints. His approach is based on assertion of Teterev [10] which have been already proven to be erroneous. In [4], Hirche clarified some deficiencies of Chaddha's [8] proposed duality and illustrated the fact about the behavior of the objective function. Singh, Gupta and Bhatia [5] studied multiparametric sensitivity analysis for LLFP using the concept maximum volume in the tolerance region. They constructed critical regions for simultaneous and independent perturbations in the objective function coefficients and in the right-hand - side vector in the given problem. They derived necessary and sufficient conditions to classify perturbation parameter as focal and non-focal. In [6] , Gupta and Singh studied multiparametric sensitivity analysis under perturbations in multiple rows or columns of the constraint matrix in linear plus linear fractional programming problem.

Recently, Jain and Lachhawani [3] have given the solution procedure for sum of linear plus linear fractional multiobjective programming problem fuzzy under rule constraint. They suggested the use of if - then fuzzy reasoning method to determine the crisp functional relationship between the objective function and decision variables under the assumptions that the denominator of the fractional part of the objective function is non - zero on the constraint set and finally solved the the resulting programming problem to find a pair of optimal solution of original problem. Kheirfam [13] studied classical sensitivity analysis, when the coefficients of the objective function and right hand side are parameterized. Recently Singh [14] derived optimality and duality conditions for transportation with linear plus linear fractional programming problem. He also established weak duality and strong duality theorem for the dual model.

In this article, we propose an algorithm to the solution of multiobjective linear plus linear fractional programming problem (MOLLFP) using goal programming procedure. In the Goal Programming (GP) model formulation of the problem, first the objectives are transformed in to fuzzy goals by means of assigning an aspiration level to each of them. Then achievement of the highest membership value (unity) to the extent possible of each of the fuzzy goals is considered. In the solution process, the under- and over- deviational variables of the membership goals associated with the fuzzy goals are introduced to transform the proposed model in to an equivalent non - linear goal programming (NLGP) model to solve the problem using Wolf - Frank method of approximation programming (MAP). Our attempt is to give simple solution procedure for MOLLFP.

II. GOAL PROGRAMMING

Goal programming is one of the first method expressly created for multiobjective optimization (Charnes et.al. 1955; Charnes and Cooper, 1961[20]). It has been originally developed for MOLP problems (Ignizio, 1985).

In goal programming, the DM is asked to specify aspiration levels $g_i (i = 1, 2, \ldots k)$ for the objective functions. Then

deviation from these aspiration levels are minimized. An objective function jointly with an aspiration level is refereed to as goal. For minimization problems, goals are of the form $F_i(x) \leq g_i$ and aspiration levels are assumed to be selected so that they are not achievable simultaneously. After the goal have been formed, the deviations $d_i^+ - d_i^- = \max[0, F_i(x) - g_i]$ of the objective function values are minimized.
A typical GP is expressed as follows

$$\begin{aligned} &\text{Minimize} \sum_{i=1}^{k} |F_i(x) - g_i| \\ &\text{subject to} \\ &x \in X = \{x \in R^n;\, Ax \leq b,\, x \geq 0\}. \end{aligned} \tag{6}$$

Where F_i is the linear function of the i^{th} goal and g_i is the aspiration level of i^{th} goal.
Let $F_i(x) - g_i = d_i^+ - d_i^-, \quad d_i^-, d_i^+ \geq 0$. Equation (5) can be formulated as follows

$$\begin{aligned} &\text{Minimize} \sum_{i=1}^{k} (d_i^+ + d_i^-) \\ &\text{subject to} \\ &F_i(x) - d_i^+ + d_i^- - g_i = 0,\, i = 1, 2, \dots k \\ &d_i^+,\, d_i^- \geq 0 \\ &x \in X = \{x \in R^n;\, Ax \leq b,\, x \geq 0\}. \end{aligned} \tag{7}$$

Where $d_i^- \geq 0$, $d_i^+ \geq 0$ are, respectively under - and over -deviations of $i^t h$ goal.
Problem (6) has been applied to solve many real world problems.

A. *Weighted Goal Programming*

In the weighted goal programming approach (Charnes and Cooper, 1977), the weighted sum of deviation is minimized. This means that in addition to the aspiration levels. The DM must specify positive weights. Then we solve a problem

$$\begin{aligned} &\text{Minimize} \sum_{i=1}^{k} w_i (d_i^+ + d_i^-) \\ &\text{subject to} \\ &F_i(x) - d_i^+ + d_i^- \leq g_i, \qquad i = 1, 2, \dots k \\ &d_i^+,\, d_i^- \geq 0 \\ &x \in X = \{x \in R^n;\, Ax \leq b,\, x \geq 0\}. \end{aligned} \tag{8}$$

On the other hand, in the lexicographic goal programming approach, the DM must specify a lexicographic order order of the goals in addition to the aspiration levels. After lexicographic ordering, the problem with the deviations as objective functions is solved lexicographically subject to the constraints. It is also possible to use a combination of the weighted and lexicographic approaches. In this case,. several objective functions may belong to the same class of importance in the lexicographic order. In each priority class, a weighted sum of deviations is minimized. Let us also mention a so called min - max goal programming approach where maximum of deviations is minimized and meta - goal programming (Rodriguez Uria et.al., 2002), where different variants of goal programming are incorporated.
Goal programming is a very widely used and popular solution method. Goal setting is an understandable and very easy of making decisions. The specification of the weights or the lexicographic ordering may be more difficult.
Let us finally add that goal programming has been used in a variety of further developments and modifications. Among others, goal programming is related to some fuzzy multiobjective optimization methods where fuzzy sets are used to express degree of satisfaction from the attainments of goals and from satisfaction of soft constraints.

B. *Fuzzy Goal Programming*

In fuzzy goal programming approaches, the highest degree of membership function is 1. So, for the defined membership function in (4) and (5), the flexible membership goals with aspiration levels 1 can be expressed as

$$\begin{aligned} &\frac{F_i(x) - \underline{t_i}}{g_i - \underline{t_i}} + d_i^- - d_i^+ = 1 \\ \text{or} \\ &\frac{\overline{t_i} - F_i(x)}{\overline{t_i} - g_i} + d_i^- - d_i^+ = 1 \end{aligned} \tag{9}$$

Where $d_i^- \geq 0$, $d_i^+ \geq 0$ with $d_i^+ . d_i^- = 0$ are, respectively under - and over -deviations from the aspiration levels.
In conventional GP, the under- and over-deviational variables are included in the achievement function or minimizing them and that depend upon the type of the objective functions to be optimized.
In this approach, only the under - deviational variable d_k^- is required to be achieve the aspired levels of the fuzzy goals. It may be noted that any over - deviation from fuzzy goal indicates the full achievement of the membership value. Recently, B. B. Pal. et.al [17] proposed an efficient goal programming (GP) method for solving Fuzzy multiobjective linear fractional programming problems.In this paper, the idea of B.B.Pal for FMOLFP is extended to FMOLLFP.

III. LINEAR APPROXIMATIONS OF NON - LINEAR PROGRAMS

Algebraic procedures such as pivoting are so powerful for manipulating linear equalities and inequalities that many nonlinear programming algorithms replace the given problem by an approximating linear problems [2]. Separable programming is a prime example and also one of the most useful of these procedures. As in separable programming these non - linear algorithms usually solve several linear

approximations by letting the solution of the last approximation suggest a new one.
By using different approximation schemes, this strategy can be implemented in several ways. we are restricted ourself only Frank - Wolf Algorithm and its extension.

A. *Frank - Wolf Algorithm*

Let $x^{(0)} = (x_1^0, x_2^0, \dots x_n^0)$ be any feasible solution with linear constraints [2]:

$$\begin{aligned} &\text{Max} \quad F(x_1, x_2, \dots x_n) \\ &\text{subject to} \\ &\sum_{j=1}^{n} a_{ij} x_j \le b_i, \qquad (i = 1, 2, \dots m) \\ &x_j \ge 0, \qquad\qquad (j = 1, 2, \dots n). \end{aligned}$$

Here $x^{(0)}$ might be determined by Phase I of the simplex method. This algorithm forms a linear approximation at the point $x^{(0)}$ by replacing the objective function with its current value plus a linear correction term; that is by the linear objective

$$F(x^{(0)}) + \sum_{j=1}^{n} c_j (x_j - x_j^0)$$

where c_j is the slope, or partial derivative of F with respect to x_j, evaluated at the point $x^{(0)}$. Since $f(x^{(0)}), c_j,$ and x_j are fixed, maximizing the objective function is equivalent to maximizing

$$Z = \sum_{j=1}^{n} c_j x_j.$$

This linear approximation problem is solved, giving an optimal solution $y = (y_1, y_2, \dots y_n)$. At this point the algorithm recognizes that, although the linear approximation problem indicates that the objective improves steadily from $x^{(0)}$ to y. Therefor, the algorithm uses a procedure to determine the maximum value for $F(x_1, x_2, \dots x_n)$ along the line - segment joining $x^{(0)}$ to y. Letting $x^1 = (x_1^1, x_2^1, \dots x_n^1)$ denote the optimal solution of the line - segment optimization, we repeat at x^1. Continuing in this way, we determine a sequence of points approach $x^1 = (x_1^1, x_2^1, \dots x_n^1)$ any point $x^* = (x_1^*, x_2^*, \dots x_n^*)$ that these point approach in the limit is an optimal solution to the original problem. The Frank - Wolf algorithm is convergent computationally because it solves linear programs with the same constraints as the original problem.

B. *MAP(Method of Approximation)[2]*

The Frank - Wolf algorithm can be extended to general nonlinear programs by making linear approximations to the constraints as well as the objective function. When the constraints are highly nonlinear, however, the solution to the approximation problem can become far removed from feasible region since the algorithm permits large moves from any candidate solution. The Method of approximation programming (MAP) is a simple modification of this approach that limits the size of any move. As a result, it is sometimes referred to as a small - step procedure. Let $x^{(0)} = (x_1^0, x_2^0, \dots x_n^0)$ be any candidate solution to the optimization problem:

$$\begin{aligned} &\text{Max} \quad F(x_1, x_2, \dots x_n) \\ &\text{subject to,} \\ &g_i(x_1, x_2, \dots x_n) \le 0, (i = 1, 2, \dots m) \end{aligned}$$

Each constraint can be linearized, using its current value $g_i(x^{(0)})$ plus a linear correction term, as:

$$\tilde{g_i}(x) = g_i(x^{(0)}) + \sum_{j=1}^{n} a_{ij} (x_j - x_j^0) \le 0,$$

where a_{ij} is the partial derivative of constraint g_i with respect to variable x_j evaluated at the point $x^{(0)}$. This approximation is a linear inequality, which can be written as

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i^0 \equiv \sum_{j=1}^{n} a_{ij} x_j^0 - g_i(x^{(0)}.$$

since the terms on the right hand side are all constants. The MAP algorithm uses these approximations, together with the linear objective function approximation and solve the linear programming problem:

$$\begin{aligned} &\text{Maximize} \quad Z = \sum_{j=1}^{n} c_j x_j \\ &\text{subject to} \\ &\sum_{j=1}^{n} a_{ij} x_j \le b_i^0, (i = 1, 2, ..m), x_j \ge 0, (j = 1, 2, ..n). \end{aligned}$$

We might expect then that the additional work required by the line - segment optimization of Frank - Wolf algorithm is not worth the slightly improved solution that it provides. MAP operates on this premise, taking the solution to the linear programs as the new x^1. The partial derivative data a_{ij}, b_i, and c_j is recalculated at x^1, and the procedure is repeated. Continuing in this manner determines points $x^1, x^2, \dots, x^k, \dots$ and as in Frank - Wolf procedure any point $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ that these points approach in the limit is considered a solution.

IV. Solution Method

We consider the multiobjective linear plus linear fractional programming problem of the form (2).
Assume fuzzy aspiration level g_i and tolerance limit $(\bar{t}_i, \underline{t}_i)$ for each objective function F_i, then we construct membership function for each objective using Zimmermann Max - Min approach [16], then the problem (2) becomes

FMOLLFP (3). The membership function μ_i must be described for each fuzzy goal as equation (4) and (5).

The proposed algorithm can be explained in three steps:
Step 1: Determine $x_i^* = \{x_1^*, x_2^*, ..., x_n^*\}$ for which the i^{th} membership function μ_i is to be constructed, associated with i^{th} objective function $F_i(x) \quad \forall\ (i = 1, 2, \ldots, k)$, where n is the number of variable.
Step 2: The i^{th} membership goal (8) can be written as

$$H_iF_i - H_i\underline{t}_i + d_i^- - d_i^+ = 1, \qquad H_i = \frac{1}{g_i - \underline{t}_i}. \tag{10}$$

Substituting the expression for F_i

$$\begin{aligned} &H_i\{(p_i^Tx + \theta_i)(d_i^Tx + \beta_i) + (c_i^Tx + \alpha_i)\} \\ &+ d_i^-(d_i^Tx + \beta_i) - d_i^+(d_i^T + \beta_i) \\ &= H_i^{'}(d_i^Tx + \beta_i), \end{aligned} \tag{11}$$

where $H_i^{'} = 1 + H_i\underline{t}_i$.

Similar expression for other membership goal can also be obtained. However, for model simplification, the expression in (10) can be considered as a general form of goal expression for any type of the stated membership goals. Using the the method of variable change as presented by Kornbluth and Steuer [21], the goal expression in (10) can be written as follows: The simplified form of the expression in (10) is obtained as

$$\begin{aligned} &C_ix^2 + F_ix + D_i^- - D_i^+ = G_i \qquad (12)\\ &\text{where,} \quad G_i = H_ip_i^T\beta_i + H_i\alpha_i - H_i^{'}\beta_i \\ &C_i = H_ip_i^Td_i^T \\ &F_i = H_ip_i^Td_i^T + H_ip_i^T\beta_i + c_i^TH_i - d_i^TH_i^{'} \\ &D_i^- = d_i^-(d_i^Tx + \beta_i) \\ &D_i^+ = d_i^+(d_i^Tx + \beta_i) \\ &\text{with } D_i^-,\ D_i^+ \geq 0 \text{ and } D_i^-.D_i^+ = 0, \\ &\text{since } d_i^-,\ d_i^+ \geq 0,\ d_i^-(d_i^Tx + \beta_i) \geq 0. \end{aligned}$$

Step 3: Now in making decision, minimization of d_i^- means $\frac{D_i^-}{(d_i^Tx+\beta_i)}$, which is also a non - linear one.

It may be noted that when a membership goal is fully achieved, $d_i^+ = 0$ and when its achievement is zero, $d_i^- = 1$ are found in the solution.

So involvement of $d_i^- \leq 1$ in the solution leads to impose the following constraint to the model of the problem.

$$\begin{aligned} &\frac{D_i^-}{(d_i^Tx + \beta_i)} \leq 1, \\ &\text{i.e.} \quad -d_i^Tx + D_i^- \leq \beta_i. \end{aligned} \tag{13}$$

It may be pointed out that any such constraint corresponding to d_i^+ does not arise in the formulation and simplest version of GP (i.e. minsum GP)[17] is introduced to formulate the model of the problem under consideration, then the GP model formulation becomes:

$$\begin{aligned} &\text{Minimize} \quad \bar{F} = \sum_{i=1}^{k} w_i^- D_i^- \\ &\text{also satisfy} \quad C_ix^2 + F_ix + D_i^- - D_i^+ = G_i \\ &\text{subject to} \\ &x \in X = \{x \in R^n,\ Ax \leq b, x \geq 0 \qquad (14)\\ &-d_i^Tx + D_i^- \leq \beta_i. \\ &D_i^-,\ D_i^+ \geq 0, \qquad i = 1, 2 \ldots k \\ &w_i^- = \begin{cases} \frac{1}{g_i - \underline{t}_i} & \text{for} \quad \mu_i(x) \text{ in } (5) \\ \\ \frac{1}{\overline{t}_i - g_i} & \text{for} \quad \mu_i(x) \text{ in } (4). \end{cases} \end{aligned}$$

Where $\bar{F}$ represents the fuzzy achievement function consisting of the weighted under - deviational variables, and the numerical weights $w_i^- \geq 0,\ i = 1, 2, \ldots k$ represent the relative importance of achieving the aspired level of the respective fuzzy goals subject to the constraints sets of the decision situation. Now above non-linear G.P can be solve easily using Wolf-Frank method of approximation (MAP) satisfying the nonlinear constrains.

V. NUMERICAL EXAMPLES

Example 1 Consider a MOLLFP with two objective functions:

$$\begin{aligned} &\text{Max}\{F_1(x) = (-x_1 - 1) + \frac{-5x_1 + 4x_2}{2x_1 + x_2 + 5}, \\ &F_2(x) = (x_2 + 1) + \frac{9x_1 + 2x_2}{7x_1 + 3x_2 + 1}\} \\ &\text{subject to} \\ &\qquad x_1 - x_2 \geq 2 \\ &\qquad 4x_1 + 5x_2 \leq 25 \qquad (15)\\ &\qquad x_1 \geq 5 \\ &\qquad x_1,\ x_2 \geq 0. \end{aligned}$$

It is observed that $F_1 < 0,\ F_2 \geq 0,$ for each x in the feasible region.

If the fuzzy aspiration levels of the two objectives are -7.31, and 3.21, find x in order to satisfy the following fuzzy goals.

$$F_1(x) \gtrapprox -7.31, \qquad F_2(x) \gtrapprox 3.21.$$

The tolerance limits for the two fuzzy goals are $(-9.04,\ 2.21)$ respectively. The membership function for the two fuzzy goals are i.e.

$$\mu_1(x) = \begin{cases} 1, & if \quad F_1(x) \geq -7.31 \\ \dfrac{(-x_1 - 1) + \dfrac{-5x_1 + 4x_2}{2x_1 + x_2 + 5} + 9.04}{1.73} \\ if \quad -9.04 \leq F_1(x) \leq -7.31 \\ 0, & if \quad F_1(x) \leq -9.04 \end{cases} \tag{16}$$

$$\mu_2(x) = \begin{cases} 1, & if \quad F_2(x) \geq 3.21 \\ \dfrac{(x_2 + 1) + \dfrac{9x_1 + 2x_2}{7x_1 + 3x_2 + 1} - 2.21}{1}, \\ if \quad 2.21 \leq F_2(x) \leq 3.21 \\ 0, & if \quad F_2(x) \leq 2.21. \end{cases} \tag{17}$$

Then the membership goal can be expressed as

$$\frac{(-x_1 - 1) + \dfrac{-5x_1 + 4x_2}{2x_1 + x_2 + 5} + 9.04}{1.73} + d_1^- - d_1^+ = 1 \tag{18}$$

$$\frac{(x_2 + 1) + \dfrac{9x_1 + 2x_2}{7x_1 + 3x_2 + 1} - 2.21}{1} + d_2^- - d_2^+ = 1 \tag{19}$$

where, $d_i^-,\ d_i^+ \geq 0$, with $d_i^-.d_i^+ = 0$, $i = 1,\ 2,\ \ldots k$. Following the procedure, the membership goals are restated as

$$-2x_1^2 - x_1x_2 + 2.54x_1 + 10.27x_2 + D_1^- - D_2^+ = -31.35, \tag{20}$$

$$3x_2^2 + 7x_1x_2 - 6.41x_1 + 3x_2 + D_2^- - D_2^+ = 8.84 \tag{21}$$

where

$$\begin{aligned} D_1^- &= 1.73(2x_1 + x_2 + 5)d_1^- \\ D_1^+ &= 1.73(2x_1 + x_2 + 5)d_1^+ \\ D_2^- &= (7x_1 + 3x_2 + 1)d_2^- \\ D_2^+ &= (7x_1 + 3x_2 + 1)d_2^+ \end{aligned}$$

Now the restrictions $d_1^- \leq 1$ and $d_2^- \leq 1$ gives

where

$$\begin{aligned} &D_1^- - 3.46x_1 - 1.73x_2 \leq 8.65 \\ &D_2^- - 7x_1 - 3x_2 \leq 1 \end{aligned}$$

Thus the final equivalent GP formulation is obtained as

$$\begin{aligned} &\text{Find } X(x_1,\, x_2) \\ &\text{Min } (\frac{1}{1.73} D_1^- + D_2^-) \\ &\text{ and satisfy} \\ &-2x_1^2 - x_1x_2 + 2.54x_1 + 10.27x_2 + \\ &D_1^- - D_2^+ = -31.35 \\ &3x_2^2 + 7x_1x_2 - 6.41x_1 + 3x_2 + D_2^- - D_2^+ = 8.84 \\ &\text{subject to} \\ &D_1^- - 3.46x_1 - 1.73x_2 \leq 8.65 \\ &D_2^- - 7x_1 - 3x_2 \leq 1 \\ &x_1 - x_2 \geq 2 \\ &4x_1 + 5x_2 \leq 25 \\ &x_1 \geq 5 \\ &x_1,\, x_2 \geq 0. \\ &D_1^-,\ D_1^+, D_2^-,\ D_2^+ \geq 0, \text{with}, D_i^-.D_i^+ = 0 \\ &i = 1,\, 2 \ldots k \end{aligned} \tag{22}$$

Now apply Frank - Wolf method of approximation(MAP). If we assume the initial solution is $x_1 = 5,\ x_2 = 0,\ D_1^- = 0, D_1^+ = 0,\ D_2^- = 0,\ D_2^+ = 0$ from the feasible region, then the non - linear problem transformed in to linear approximation program as follows

$$\begin{aligned} &\text{Find } X(x_1,\, x_2) \\ &\text{Min } (\frac{1}{1.73} D_1^- + D_2^-) \\ &\text{subject to} \\ &-17.46x_1 + 5.27x_2 + D_1^- - D_1^+ = -81.35 \\ &-6.41x_1 + 38x_2 + D_2^- - D_2^+ = -72.9 \\ &D_1^- - 3.46x_1 - 1.73x_2 \leq 8.65 \\ &D_2^- - 7x_1 - 3x_2 \leq 1 \\ &x_1 - x_2 \geq 2 \\ &4x_1 + 5x_2 \leq 25 \\ &x_1 \geq 5 \\ &x_1,\, x_2 \geq 0 \\ &D_1^-,\ D_1^+, D_2^-,\ D_2^+ \geq 0, \text{with}, D_i^-.D_i^+ = 0 \\ &i = 1,\, 2 \ldots k \end{aligned} \tag{23}$$

The optimal solution of the problem (23) is at the point $x_1 = 5,\ x_2 = 1,\ D_1^- = 0.68, D_1^+ = 0,\ D_2^- = 0,\ D_2^+ = 78.85$. and minimum value is 0.39. Now repeat the process for the point $x_1 = 5,\ x_2 = 1,\ D_1^- = 0.68, D_1^+ = 0,\ D_2^- =$

$0,\ D_2^+ = 78.85$. and the new LPP is obtained

$$\begin{array}{ll}
\text{Find } X(x_1, x_2) & \\
\text{Min } (\frac{1}{1.73} D_1^- + D_2^-) & \\
\text{subject to} & (24) \\
-18.46x_1 + 5.27x_2 + D_1^- - D_1^+ = -86.35 & \\
0.59x_1 + 44x_2 + D_2^- - D_2^+ = 46.84 & \\
D_1^- - 3.46x_1 - 1.73x_2 \leq 8.65 & \\
D_2^- - 7x_1 - 3x_2 \leq 1 & \\
x_1 - x_2 \geq 2 & \\
4x_1 + 5x_2 \leq 25 & \\
x_1 \geq 5 & \\
x_1, x_2 \geq 0. & \\
D_1^-,\ D_1^+, D_2^-,\ D_2^+ \geq 0, \text{with}, D_i^-.D_i^+ = 0 & \\
i = 1, 2 \ldots k &
\end{array}$$

The optimal solution of the problem (24) is at the point $x_1 = 5,\ x_2 = 1,\ D_1^- = 0.68, D_1^+ = 0,\ D_2^- = 0,\ D_2^+ = 0.11$. and minimum value is 0.3944. since we obtained the same value for $x_1 = 5,\ x_2 = 1$. So $x_1 = 5,\ x_2 = 1$ is the final solution for the problem.
The solution for the original problem is given by $x_1 = 5, x_2 = 1,\ F_1 = -7.31, F_2 = 3.21$. The membership function values at $(5, 1)$ indicate that goal F_1 and F_2 are satisfied 100% and 100% respectively, for the obtained solution.

Example 2: Let us consider a MOLLFP with three objective functions

$$\begin{aligned}
\text{Max} \quad \{F_1(x) &= (-x_1 - 1) + \frac{-5x_1 + 4x_2}{2x_1 + x_2 + 5}, \\
F_2(x) &= (x_2 + 1) + \frac{9x_1 + 2x_2}{7x_1 + 3x_2 + 1}, \\
F_3(x) &= (x_1 + 1) + \frac{3x_1 + 8x_2}{4x_1 + 5x_2 + 3}\}
\end{aligned}$$

subject to

$$\begin{array}{ll}
x_1 - x_2 \geq 2 & \\
4x_1 + 5x_2 \leq 25 & (25) \\
x_1 + 9x_2 \geq 9 & \\
x_1 \geq 5 & \\
x_1, x_2 \geq 0. &
\end{array}$$

It is observed that $F_1 < 0, F_2 \geq 0, F_3 \geq 0$ for each x in the feasible region.

If the fuzzy aspiration levels of three objectives are $(-7.31, 3.21, 7.54)$ respectively, find x in order to satisfy the following goals:

$$\begin{aligned}
F_1(x) &\gtrapprox -7.31, \\
F_2(x) &\gtrapprox 3.21, \qquad (26) \\
F_3(x) &\gtrapprox 7.54.
\end{aligned}$$

The tolerance limits for the three fuzzy goals are $-8.43, 2.59, 6.72$ respectively. The membership functions for the three fuzzy goals are given by

$$\mu_1(x) = \begin{cases} 1, & if \quad F_1(x) \geq -7.31 \\ \dfrac{(-x_1 - 1) + \dfrac{-5x_1 + 4x_2}{2x_1 + x_2 + 5} + 8.43}{1.12}, & \\ if \quad -8.43 \leq F_1(x) \leq -7.31 & \\ 0, & if \quad F_1(x) \leq -8.43 \end{cases}$$

$$\mu_2(x) = \begin{cases} 1, & if \quad F_2(x) \geq 3.21 \\ \dfrac{(x_2 + 1) + \dfrac{9x_1 + 2x_2}{7x_1 + 3x_2 + 1} - 2.59}{0.62}, & \\ if \quad 2.59 \leq F_2(x) \leq 3.21 & \\ 0, & if \quad F_2(x) \leq 2.59 \end{cases}$$

$$\mu_3(x) = \begin{cases} 1, & if \quad F_3(x) \geq 7.54 \\ \dfrac{(x_1 + 1) + \dfrac{3x_1 + 8x_2}{4x_1 + 5x_2 + 3} - 6.72}{0.82}, & \\ if \quad 6.72 \leq F_3(x) \leq 7.54 & \\ 0, & if \quad F_3(x) \leq 6.72. \end{cases}$$

Then the membership goal can be expressed as

$$\frac{(-x_1 - 1) + \dfrac{-5x_1 + 4x_2}{2x_1 + x_2 + 5} + 9.04}{1.73} + d_1^- - d_1^+ = 1 \qquad (27)$$

$$\frac{(x_2 + 1) + \dfrac{9x_1 + 2x_2}{7x_1 + 3x_2 + 1} - 2.21}{1} + d_2^- - d_2^+ = 1 \qquad (28)$$

where, $d_i^-,\ d_i^+ \geq 0, \quad$ with $d_i^-.d_i^+ = 0, \qquad i = 1, 2, \ldots k$
Following the procedure, the membership goals are restated as

$$-2x_1^2 - x_1x_2 + 2.54x_1 + 10.27x_2 + D_1^- - D_2^+ = -31.35, \qquad (29)$$

$$3x_2^2 + 7x_1x_2 - 6.41x_1 + 3x_2 + D_2^- - D_2^+ = 8.84 \qquad (30)$$

where

$$D_1^- = 1.73(2x_1 + x_2 + 5)d_1^-$$
$$D_1^+ = 1.73(2x_1 + x_2 + 5)d_1^+$$
$$D_2^- = (7x_1 + 3x_2 + 1)d_2^-$$
$$D_2^+ = (7x_1 + 3x_2 + 1)d_2^+$$

Now the restrictions $d_1^- \leq 1$ and $d_2^- \leq 1$ gives

where

$$D_1^- - 3.46x_1 - 1.73x_2 \leq 8.65$$
$$D_2^- - 7x_1 - 3x_2 \leq 1$$

Thus the final equivalent GP formulation is obtained as

$$\begin{aligned}
&\text{Find } X(x_1, x_2)\\
&\text{Min } \left(\frac{1}{1.73} D_1^- + D_2^-\right)\\
&\text{ and satisfy}\\
&-2x_1^2 - x_1x_2 + 2.54x_1 + 10.27x_2\\
&+D_1^- - D_2^+ = -31.35\\
&3x_2^2 + 7x_1x_2 - 6.41x_1 + 3x_2\\
&+D_2^- - D_2^+ = 8.84\\
&\text{subject to}\\
&D_1^- - 3.46x_1 - 1.73x_2 \leq 8.65\\
&D_2^- - 7x_1 - 3x_2 \leq 1\\
&x_1 - x_2 \geq 2\\
&4x_1 + 5x_2 \leq 25\\
&x_1 \geq 5\\
&x_1,\, x_2 \geq 0.
\end{aligned} \tag{31}$$

The solution of the above non - linear programming problem is given by Frank - Wolf method of approximation(MAP)

The optimal solution of the problem (22) is at the point $(5, 1)$ and maximum value is 2. The point $(5, 1)$ is the efficient solution of the given original problem in the feasible region. The solution for the original is given by $x_1 = 5$, $x_2 = 1$, $F_1 = -7.31, F_2 = 3.21$. The membership function values at $(5, 1)$ indicate that goal F_1 and F_2 are satisfied 100% and 100% respectively, for the obtained solution.

Example 3: Let us consider a MOLLFP with three objective functions

$$\begin{aligned}
\text{Max}\quad &\{F_1(x) = (-x_1 - 1) + \frac{-x_1 + 2x_2 - 5}{7x_1 + 3x_2 + 1},\\
&F_2(x) = (-2x_2 - 1) + \frac{2x_1 - 3x_2 - 5}{x_1 + 1},\\
&F_3(x) = (-3x_1 - 1) + \frac{5x_1 + 2x_2 - 19}{-5x_1 + 20}\}
\end{aligned}$$

subject to

$$\begin{aligned}
&x_1 \leq 6\\
&x_2 \leq 6\\
&2x_1 + x_2 \leq 9\\
&-2x_1 + x_2 \leq 5\\
&x_1 - x_2 \leq 5\\
&x_1,\, x_2 \geq 0.
\end{aligned} \tag{32}$$

It is observed that $F_1 < 0$, $F_2 < 0$, $F_3 < 0$ for each x in the feasible region.

If the fuzzy aspiration levels of the three objectives are -1.21, -0.17, -1.95 respectively.

$$\begin{aligned}
F_1(x) &\gtrapprox -1.21,\\
F_2(x) &\gtrapprox -0.17,\\
F_3(x) &\gtrapprox -1.95.
\end{aligned} \tag{33}$$

The tolerance limits for the two fuzzy goals are -6.28, -21, -17.2 respectively. The membership functions for the three fuzzy goal are

$$\mu_1(x) = \begin{cases} 1, & if \quad F_1(x) \geq -1.21 \\ \dfrac{(-x_1 - 1) + \dfrac{-x_1 + 2x_2 - 5}{7x_1 + 3x_2 + 1} + 6.28}{5.07}, \\ if \quad -6.28 \leq F_1(x) \leq -1.21 \\ 0, & if \quad F_1(x) \leq -6.28 \end{cases}$$

$$\mu_2(x) = \begin{cases} 1, & if \quad F_2(x) \geq -0.17 \\ \dfrac{(-2x_2 - 1) + \dfrac{2x_1 - 3x_2 - 5}{x_1 + 1} + 21}{20.83}, \\ if \quad -21 \leq F_2(x) \leq -0.17 \\ 0, & if \quad F_2(x) \leq -21 \end{cases}$$

$$\mu_3(x) = \begin{cases} 1, & if \quad F_3(x) \geq -1.95 \\ \dfrac{(-3x_1 - 1) + \dfrac{5x_1 + 2x_2 - 19}{-5x_1 + 20} + 17.2}{15.25}, \\ if \quad -17.2 \leq F_3(x) \leq -1.95 \\ 0, & if \quad F_3(x) \leq -17.2. \end{cases}$$

Then the membership goal can be expressed as

$$\frac{(-x_1 - 1) + \dfrac{-5x_1 + 4x_2}{2x_1 + x_2 + 5} + 9.04}{1.73} + d_1^- - d_1^+ = 1 \tag{34}$$

$$\frac{(x_2 + 1) + \dfrac{9x_1 + 2x_2}{7x_1 + 3x_2 + 1} - 2.21}{1} + d_2^- - d_2^+ = 1 \tag{35}$$

where, $d_i^-,\ d_i^+ \geq 0, \quad$ with $d_i^-.d_i^+ = 0, \qquad i = 1, 2, \ldots k$
Following the procedure, the membership goals are restated as

$$-2x_1^2 - x_1x_2 + 2.54x_1 + 10.27x_2 + D_1^- - D_2^+ = -31.35, \tag{36}$$

$$3x_2^2 + 7x_1x_2 - 6.41x_1 + 3x_2 + D_2^- - D_2^+ = 8.84 \tag{37}$$

where

$$D_1^- = 1.73(2x_1 + x_2 + 5)d_1^-$$
$$D_1^+ = 1.73(2x_1 + x_2 + 5)d_1^+$$
$$D_2^- = (7x_1 + 3x_2 + 1)d_2^-$$
$$D_2^+ = (7x_1 + 3x_2 + 1)d_2^+$$

Now the restrictions $d_1^- \leq 1$ and $d_2^- \leq 1$ gives

where

$$D_1^- - 3.46x_1 - 1.73x_2 \leq 8.65$$
$$D_2^- - 7x_1 - 3x_2 \leq 1$$

Thus the final equivalent GP formulation is obtained as

$$\begin{aligned} &\text{Find } X(x_1,\, x_2) \\ &\text{Min } \left(\frac{1}{1.73} D_1^- + D_2^-\right) \\ &\text{and satisfy} \\ &-2x_1^2 - x_1x_2 + 2.54x_1 + 10.27x_2 + \\ &7D_1^- - D_2^+ = -31.35 \\ &3x_2^2 + 7x_1x_2 - 6.41x_1 + 3x_2 + \\ &D_2^- - D_2^+ = 8.84 \\ &\text{subject to} \\ &D_1^- - 3.46x_1 - 1.73x_2 \leq 8.65 \\ &D_2^- - 7x_1 - 3x_2 \leq 1 \\ &x_1 - x_2 \geq 2 \\ &4x_1 + 5x_2 \leq 25 \\ &x_1 \geq 5 \\ &x_1,\, x_2 \geq 0. \end{aligned} \tag{38}$$

The solution of the above non - linear programming problem is given by Frank - Wolf method of approximation(MAP)

The optimal solution of the problem (22) is at the point $(5, 1)$ and maximum value is 2. The point $(5,\, 1)$ is the efficient solution of the given original problem in the feasible region. The solution for the original is given by $x_1 = 5,\ x_2 = 1,\ F_1 = -7.31, F_2 = 3.21$. The membership function values at $(5,\, 1)$ indicate that goal F_1 and F_2 are satisfied 100% and 100% respectively, for the obtained solution.

VI. Conclusion

Various fuzzy approaches have been proposed for the solution of multiobjective linear plus linear fractional programming problem and most of the approaches have computational burdensome. Our approach is to give simple procedure for the solution of multiobjective linear plus linear fractional programming problem using fuzzy set theory, goal programming and method of approximation(MAP).

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