Stabilization of Non-necessarily Inversely Stable First-order Adaptive Systems Under Saturated Input

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Abstract:- This paper is concerned with an indirect adaptive stabilization scheme for first-order continuous-time systems under saturated input which is described by a sigmoidal function. The control singularities are avoided through a modification estimation scheme for the estimated plant parameter vector so that its associated Sylvester matrix is guaranteed to be non-singular and then the estimated plant model is controllable. This strategy implies at the same time the controllability through time of the modified estimation scheme. The estimation modification mechanism involves the use of a hysteresis switching function. An alternative hybrid scheme, whose estimated parameters are updated at sampling instants is also given to solve a similar adaptive stabilization problem. Such a scheme also uses hysteresis switching for modification of the parameter estimates so as to ensure the controllability of the estimated plant model.

Key-Words:- hybrid dynamic systems, discrete systems, saturated input, control, stabilization

I. INTRODUCTION

THE inputs to physical systems usually present saturation phenomena which limit the amplitudes which excite the linear dynamics, [1-2]. Also, the adaptive stabilization and control of linear continuous and discrete systems has been successfully investigated in the last decades, [3-19]. Classically, the plant is assumed to be inversely stable and its relative degree and its high-frequency gain sign are assumed to be known together with an absolute upper-bound for that gain in the discrete case. Attempts of relaxing such assumptions have been made for continuous systems, [5-7]. The assumption on the knowledge of the order can be relaxed by assuming a known nominal order and considering the exceeding modes and unmodelled dynamics, [13-16], [19-20].Similar issues appear in different problems including control theory issues as sliding model designs, adaptive control, adaptive sampling and biological applications, [21-33] The assumption on the knowledge of the high frequency gain has been removed in [6] and [17] and the assumption of the plant being inversely stable has been successfully removed in the discrete case and more recently in the continuous one, [10-16]. The problem has been solved by using either excitation of the plant signals or by exploiting the properties of the standard least-squares covariance matrix combined with an estimation modification rule based upon the use of a hysteresis switching function, [12-16], [18]. Such an estimates modification technique guarantees that the modified estimated plant model is controllable for all time provided that the plant is controllable. *This paper presents an adaptive stabilization algorithm for first -order continuous - time systems with a zero which can be either stable or unstable under saturated input. The saturating device is modelled by a sigmoidal function* . Such an approach is a very good approximation to the common saturations usually modelled as piecewise-continuous functions. Also, it is an exact model for saturations inherent to practical MOS-type amplifiers. The adaptive scheme uses a parameter modification rule which guarantees that the absolute value of the determinant of the Sylvester matrix associated with the modified parameter estimates is bounded from below by a positive threshold and, thus, the estimated model is guaranteed to be controllable. That feature is the main contribution of this manuscript. A simple extension to an hybrid version of the system is also pointed out for the case that the dynamics involves previous samples of the output, input and its derivative at the previous sampling point ran by any predesigned sampling period. The results are then extended to the case when an adaptive stabilizer, which re-updates at sampling instants the plant estimates, modified estimates and controller parameters, is used for the above continuous - time plant. The above strategy results in a hybrid closed-loop system because of the discrete nature of the updating procedure of the parametrical estimation/ modification algorithm.

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II. ADAPTIVE STABILIZATION

A) Plant, Estimation / Estimated- Modification Scheme and Adaptive Stabilization Law

Consider the following continuous-time first-order controllable system under saturated input:

$$\dot{y}+a^{*}y=b_0^{*}\dot{u}'+b_1^{*}u' \quad (1.a)$$

$$u'=sat_{\nu^{*}}(u)\overset{\triangleright}{=}\tanh(\nu^{*}u)=\frac{1-e^{-2\nu^{*}u}}{1+e^{-2\nu^{*}u}} \quad (1.b)$$

where the saturated input u ' to the plant (1.a) is modelled by a sigmoidal function (1.b), [2]. To simplify the writing , the argument (t) is omitted and all the constants are denoted by superscripts by ' * '. Eqn. 1.a can be rewritten as

$$\dot{y}=-a^{*}y+b_0^{*}\dot{u}+b_1^{*}u+b_0^{*}(\dot{u}'-\dot{u})+b_1^{*}(u'-u) \quad (2)$$

Note that the equivalence between (1.a) and (2) is an identity where positive and negative terms concerned with the unsaturated input and its time-derivative are cancelled in the right- hand-side of (2). Define filtered signals

$$\dot{u}_f=-d^{*}u_f+u \ ; \quad \dot{u}'_f=-d^{*}u'_f+u' \quad ; \quad \dot{y}_f=-d^{*}y_f+y \quad (3)$$

for some scalar $d^{*}>0$ so that one gets from (2) for the subsequent filtered signals

$$\dot{y}_f=\theta^{*T}\varphi=-a^{*}y_f+b_0^{*}\dot{u}'_f+b_1^{*}u'_f+\varepsilon_0^{*}e^{-d^{*}t} \quad (4.a)$$

$$=$$

$$\dot{y}_f=-a^{*}y_f+b_0^{*}\dot{u}_f+b_1^{*}u_f+b_0^{*}(\dot{u}'_f-\dot{u}_f)+b_1^{*}(u'_f-u_f)+\varepsilon_0^{*}e^{-d^{*}t} \quad (4.b)$$

where

$$\theta^{*}=[b_0^{*},b_1^{*},a^{*},b_0^{*},b_1^{*},\varepsilon_0^{*}]^T \quad (5.a)$$

$$\varphi=[\dot{u}_f,u_f,-y_f,\dot{u}'_f-\dot{u}_f,u'_f-u_f,e^{-d^{*}t}]^T \quad (5.b)$$

where $\varepsilon_0^{*}=y_f(0)-u'_f(0)$ has been included in θ^{*T} to obtain (4) without neglecting the exponentially decaying term due to initial conditions of the filters $1/(s+d^{*})$ used in (4) as proposed in [13], [15] and[16]. Also, the over-parametrization of (5.a)-(5.b), in the sense that the coefficients of the numerator polynomial are estimated twice with different regressors, allows describing (4.a) as driven by u_f and $u'_f - u_f$. This idea will be then exploited for the stability analysis of the adaptive stabilizer. The parameter vector θ^{*T} can now be estimated by using the least-squares algorithm:

$$e=\dot{y}_f-\theta^T\varphi \quad (6)$$

$$\dot{\theta}=P\varphi e \quad (7)$$

$$\dot{P}=-P\varphi\varphi^TP; P(0)=P^T(0)>0 \quad (8)$$

where e is the prediction error , $\theta=(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)^T$ is the estimate of θ^{*}, defined in (5.a), and P is the covariance matrix . The use of (4.b) into (6) yields

$$\dot{y}_f=\theta_1\dot{u}_f+\theta_2u_f-\theta_3y_f+\theta_4(\dot{u}'_f-\dot{u}_f)+\theta_5(u'_f-u_f)+\theta_6e^{-d^{*}t}+e \quad (9)$$

The following modification rule of the parameter estimates is used to guarantee the controllability of the estimated plant model

$$\bar{\theta}=\theta+P\beta \quad (10)$$

with β being a vector which can be chosen to be equal to one of the following vectors

$$\beta_1=[\overbrace{0,0,\cdots\cdots,0}^{6}]^T \ ; \quad \beta_2=v \ ; \quad \beta_3=-\beta_2 \quad (11.a)$$

$$\beta_4=p_1-p_4+p_3 \ ; \quad \beta_5=-\beta_4 \ ; \quad \beta_6=p_1-p_4-p_3 \quad (11.b)$$

$$\beta_7=-(p_1-p_4)+p_3 \ ; \quad v=(\theta_1-\theta_4)p_3+\theta_3(p_1-p_4)-(p_2-p_5) \quad (11.c)$$

and whose current value is selected from a hysteresis switching function which is defined by the following rule. Define

$$c(\beta)=\left|(\bar{\theta}_1-\bar{\theta}_4)\bar{\theta}_3-(\bar{\theta}_2-\bar{\theta}_5)\right|$$

$$=\left|\operatorname{Det}\begin{bmatrix}1 & 0 & 0\\ \bar{\theta}_3 & 1 & \bar{\theta}_1-\bar{\theta}_4\\ 0 & \bar{\theta}_3 & \bar{\theta}_2-\bar{\theta}_5\end{bmatrix}\right|$$

which is the absolute value of the Sylvester matrix of the modified parameter estimates associated with the estimation of the plant numerator and denominator polynomials obtained from (8)-(9) and (10)-(12) . Assume that $\beta(t^-)=\beta_i(t^-)$ and $c(\beta_j(t^+))\geq c(\beta_m(t^+))$ for some j = 1 , 2 , ..., 7 with $j \neq i$ and all m = 1 , 2 , ... , 7. Thus, for some prefixed design scalar $\alpha^{*}\in(0,1]$:

$$\beta(t^+)=\begin{cases}\beta_j(t^+) \text{ if } c(\beta_j(t^+))\geq(1+\alpha^{*})c(\beta_i(t^+))\\ \beta_i(t^+) \text{ otherwise}\end{cases} \quad (12)$$

where p_i denotes the i-th column of P. This modification strategy, first proposed in [13] for the linear continuous-time case and then extended in [15-16] to linear hybrid systems , guarantees that the parametrical error lies in the image of the of P (see [13]), while allowing that the diophantine equation , which will be then used for the synthesis of the adaptive stabilizer, will have no cancellations at any time. It will be then shown that the two following conditions are satisfied:

C1) β converges
C2) $c(\beta) \geq \delta^* > 0$.

which will be then required in the proofs of convergence and stability. Eqn. 9 can be rewritten as dependent of the modified estimates (10)-(12) as follows :

$$\dot y_f = \bar\theta_1 \dot u_f + \bar\theta_2 u_f - \bar\theta_3 y_f + \bar\theta_4(\dot u'_f - \dot u_f) + \bar\theta_5(u'_f - u_f) + \bar\theta_6 e^{-d^* t} + e - \beta^T P\varphi$$

$$+\bar\theta_5(u'_f - u_f) + \bar\theta_6 e^{-d^* t} + e - \beta^T P\varphi \qquad (13)$$

The filtered control input u_f to the saturating device and its unfiltered version u are generated as follows:

$$\dot u_f = -s_1 u_f - r_0 y_f \qquad (14.a)$$

$$u = d^* u_f + \dot u_f = (d^* - s_1) u_f - r_0 y_f \qquad (14.b)$$

with the parameters r_0 and s_1 of the adaptive stabilizer being calculated for all time from the diophantine polynomial equation:

$$(D + \bar\theta_3)(D + s_1) + [(\bar\theta_1 - \bar\theta_4) D + (\bar\theta_2 - \bar\theta_5)] r_0 = C^*(D)$$
$$\underset{def}{=} D^2 + c_1^* D + c_2^* \qquad (15)$$

with $D = d/dt$ in (15) and $C^*(D)$ being a strictly Hurwitz polynomial that defines the suited nominal closed-loop dynamics.

B) Stability and Convergence Results
They are summarized in the following main result:

Theorem 1. Consider the plant (1) subject to the estimation scheme (6) -(8), the modification scheme (10)-(12) and the control law (14)-(15). Assume that either $a^* \geq 0$ (i. e. , the open- loop plant is stable) or $|y(0)| \leq \left| \frac{b_1^* - a^* b_0^*}{a^*} \right|$ if $a^* < 0$ (i. e. , the initial condition is sufficiently small if the plant is unstable). □

Thus, the resulting closed-loop scheme has the following properties:

(i) The modified estimated plant model is controllable for all time for the chosen β in such a way that $c(\beta) \geq \delta^* > 0$.
(ii) $\tilde\theta = \theta - \theta^* \in L_\infty$ and e and $P\varphi$ are in $L_\infty \cap L_2$.
(iii) θ, P, β, $\bar\theta$, s_1 and r_0 are uniformly bounded and converge asymptotically to finite limits. Also, the number of switches in β is finite. Also, $\dot\theta \in L_2 \cap L_\infty$.
(iv) The signals u, u' and y and their corresponding filtered signals are in $L_\infty \cap L_2$. The signals u, u', u_f, u'_f, y and y_f converge to zero and their time-derivatives are in $L_\infty \cap L_2$ so that they converge to zero asymptotically. □

An outline proof of Theorem 1 is given in Appendix A. Note that the requirement of the initial conditions being sufficiently small when the plant is unstable is a usual requirement for stabilization in the presence of input saturation since it is impossible to globally stabilize an open-loop unstable system with saturated input. This avoids the closed- loop system trajectory to explode. Such a phenomenon occurs when the initial time- derivative of the state vector is positive and continues to be positive for all time because its sign cannot be modified for any input value within the allowable input range. Note also that Theorem 1 (i) -(iii) imply that Conditions C1-C2 for the β - functions of the modification scheme are fulfilled. Finally, note that the controllability of the modified estimation scheme allows to keep coprime the modified estimates of the polynomials for zeros and poles. Thus, the diophantine equation (15) associated with the controller synthesis is solvable for all time without any singularities. The mechanism which is used to ensure local stability for unstable plants and global one for stable ones is to guarantee the boundedness of all the unsaturated filtered and unfiltered signals from the regressor bondedness while the saturated ones are bounded by construction. This also ensures the identification (or adaptation) error to be bounded for all sampling time since the unmodified and modified plant parameter estimates as well as those of the adaptive controller are all bounded. The fact that the control signal is bounded is ensured since it is saturated. In the unsaturated control case, the control boundedness has to be proven explicitly (see, for instance, [21-24]) irrespective of the particular theoretical design or application. On the other hand, it turns out the main future interest of appliying saturating controls to otherwise positive systems in the presence of delays or under hybrid controls (see, [25-27]). Related research would be an interesting future investigation field.

III. ADAPTIVE STABILIZATION AND CONTROLLABILITY OF THE ESTIMATED-MODIFIED MODEL

Now, the continuous-time plant (1) is subject to the control law (14)-(15) under the saturating sigmoidal function (1.b) but the estimation algorithm (6)-(8) only updates parameters at the sampling instants $t_{k+1} = t_k + h = (k+1)h$ of the sampling period h while the regressor is evaluated at all time for re-updating the various estimates at sampling instants only. The estimation modification and calculation of the controller parameters is also updated at sampling instants. The discrete-time parameter estimation and inverse of the covariance matrix adaptation laws are:

$$\theta_k = \theta_{k-1} + \Delta\theta_{k-1} = \theta_{k-1}$$

$$-P_k \frac{\int_0^h \|\varphi[(k-1)h+\tau]\|^2 \varphi[(k-1)h+\tau]\varphi^T[(k-1)h+\tau]\tilde\theta_{k-1} d\tau}{c_k(1 + \int_0^h \varphi^T[(k-1)h+\tau]\varphi[(k-1)h+\tau] d\tau)} \qquad (16.a)$$

$$P_{k+1}^{-1} = P_k^{-1} + \Delta\tilde P_k^{-1} = P_k^{-1}$$

$$+\frac{\int_0^h \|\varphi[(k-1)h+\tau]\|^2 \varphi[(k-1)h+\tau]\varphi^T[(k-1)h+\tau]d\tau}{c_k(1+\int_0^h \varphi^T[(k-1)h+\tau]\varphi[(k-1)h+\tau]d\tau)} \quad (16.b)$$

$$c_k \geq c_{k0} \underset{def}{=} \lambda_{max}^2(P_k) \frac{\int_0^h \|\varphi[(k-1)h+\tau]\|^4 d\tau}{1+\int_0^h \|\varphi[(k-1)h+\tau]\|^4 d\tau} \quad (16.c)$$

with $P(0)=P^T(0) > 0$ and $\tilde{\theta}_k = \theta_k - \theta^*$ for all integer k ≥ 0 .The main result of this section is stated as follows:

Theorem 2. Consider the plant (1) subject to the estimation scheme (6) and (16), i.e., the parameter estimates are only updated at sampling instants, the modification scheme (10)-(12), with (12) being updated only at $t = k\,h$, and the stabilizing control law (14)-(15). Thus, the resulting closed-loop scheme fulfils the same properties of Theorem 1 under the same assumptions. □

The proof of Theorem 2 is outlined in Appendix B.

IV. AN ELEMENTARY EXTENSION TO A CLASS OF HTBRID DYNAMICS

Assume that the system (1.a) is an hybrid one where discrete sample measures influence the dynamics as follows:

$$\dot{y}(t)+a^* y(t)=b_0^* \dot{u}'(t)+b_1^* u'(t)+\left(h_1^* y_k + h_2^* \dot{u}_k' + h_3^* u_k'\right)+u_h(t)$$

where any signal f_k denotes the sampled value $f(kT)$ of f(t) where $k = (\max \text{ integer } z : zT \leq t)$ at any time t with $T \geq 0$ being the sampling period and $u_h(t)$ is the compensating control of the hybrid dynamics. The fact that the discrete-time argument k is related to the continuous one k by $k = (\max \text{ integer } z : zT \leq t)$ is not reflected directly in the notation but interpretable directly from the fact that both arguments appear simultaneously in the same equation. Now, define the output error with respect to its previously sampled value as $\tilde{y}(t) = y(t) - y_k$ and generate the compensating control as:

$$u_h(t) = \left(a^* - h_1^*\right) y_k - h_2^* \dot{u}_k' - h_3^* u_k'$$

then the output error evolves through time subject to $\tilde{y}(0) = 0$ according to the similar equation to (1.a), with the replacement $y(t) \to \tilde{y}(t)$, which follows below:

$$\dot{\tilde{y}}(t)+a^* \tilde{y}(t)=b_0^* \dot{u}'(t)+b_1^* u'(t) \quad \bullet$$

This error equation may be dealt with exactly in the same manner as (1.a) is treated with the saturating control (1.b). Then, the ouput of this class of hybrid system satisfies

$$y(t) = y_k + \tilde{y}(t)$$

Further extensions to the case of hybrid schemes from the point of view of estimation/modification of Section III are direct by combining the results of Sections III and IV. Further extensions to the case that the hybrid part of the dynamics involves sampled output derivatives is also direct by considering extra terms in the whole hybrid dynamics.

V. CONTROL OF A CLASS OF STURATION – FREE HYBRID SYSTEMS WITH INDEPRNDENT CONTINUOUS-TIME AND DISCRETE- TIME OBJECTIVES

Hybrid systems have received important attention in the last years In particular , the optimization of inputs and the fundamental properties of such systems have received attention and the multirate sampling of such systems has been studied. The importance of those systems arises from the fact that continuous and digital subsystems usually operate in a combined and integrated fashion. Another important reason to deal with such systems is that it becomes sometimes suitable the use of either discrete-time or digital controllers for continuous plants by technological implementability reasons. A wide class of linear hybrid systems consists of the couplings between purely continuous- time systems and digital ones. An important issue is the controller synthesis for model matching design with separate or combined continuous-time and discrete time goals.. Such a class of hybrid systems is characterized by the continuous substate being forced by both the current input in continuous time and its sampled value at the last preceding sampling instant as well. The objective of this paper is the design of an hybrid controller that allows the hybrid plant to achieve , in genera l, separate continuous - time and discrete -time model- following objectives in the perfectly modelled situation . In this way, the continuous-time and discrete-time closed-loop dynamics can be separately designed through the synthesis of two subcontrollers which give together the overall , in general, hybrid controller. The subcontroller designed for accomplishing with the discrete-time control objective has a discrete-time nature while that designed to accomplish with the continuous-time objective is of a mixed continuous-time and discrete-time nature. Several particular cases which are included in the general framework are for instance:

- The choice of only a continuous-time reference model . Thus , its digital transfer function is used as discrete model for controller synthesis at sampling instants
- The use of only a discrete-time reference model under a piecewise constant plant input in-between sampling instants. In such a case, the overall scheme becomes a discrete-time one.

The use of the discrete-time reference model for periodic testing of the current closed-loop performance designed for a continuous-time reference dynamics . If the test fails then the continuous-time objective can be on - line modified in terms of re-adjustment of the input to the (continuous-time) reference model or high-frequency gain re-adjustment to modify either the transient reference signal or the steady-state reference set point.

Each subcontroller is designed for the achievement of the corresponding model-following objective in the absence of plant unmodelled dynamics. Also, as a part of the design , each subcontroller generates a compensating signal to annihilate the coupling signals generated from the continuous signals to the discretized output , for the discrete-time control objective , or viceversa. when dealing with the continuous control objective. Such coupling signals are inherent to the structure of the open-loop hybrid plant . Finally, the overall controller is robust against a class of unmodelled dynamics and uniformly bounded state and measurement noises.

A) *Hybrid plant description. The hybrid plant*

Consider the next single-input single-output hybrid linear system ,[15-16]:

$$\dot{x}_c(t)=A_c x_c(t)+A_{cs} x_c[k]+A_{cd} x_d[k] + b_c u(t)+b_{cs} u[k]$$

$$x_d[k+1]=A_d x_d[k]+A_{ds} x_c[k]+b_c u(t)+b_{cs} u[k]$$

$$y(t)=c_c^T x_c(t)+c_{cs}^T x_c[k]+c_d^T x_d[k]+d_c u(t)+d_d u[k] \qquad (16)$$

for $t \in [kT, (k+1)T)$; all nonnegative integer k , with T being the sampling period , where $x_c(.)$ and $x_d[.]$ are, respectively, the n_c and n_d continuous and digital subvectors and u (.) a nd y (.) are the scalar input and output. The continuous time argument is denoted by '(t) ' while the discrete time argument is denoted by ' [k] ' and the associated continuous and digital variables are denoted correspondingly. Thus, a continuous variable at sampling instants is denoted in the same way as a digital variable so that $x_c[k]=x_c(kT)$ and $u[k]=u(kT)$ in (1). In that way , there is no distinction in the treatment of digital and time-discretized variables. The orders of all the real constant matrices in (16) agree with the dimensions of the substates and scalar input and output.

B) Descriptioopn at samoling instants

The input / output solution of (1)at sampling instants is given by the ARMA- model :

$$Q_d(q)\,y[k]=P_d(q)\,u[k]+Q_d(q)\,(c^T\omega[k]) \qquad (17)$$

for all nonnegative ionteger k , where $c^T=c_c^T+c_{cs}^T$, $Q_d(q)$ and $P_d(q)$ are polynomials of real coefficients which might be easility calculaterd of degree $n=n_c+n_d$ and q is the one-step -ahead shift operator. The ARMA - model (17) is obtained from the extended discrete-time system of state $x[k]=[x_c^T[k], x_d^T[k]]^T$ obtained from (16).

C)Description of in-betwwen sampling instants

The input / output differential-differencerelationship for (1) inbetween sampling instants is given by

$$Q_c(D)\,Q_d(q)\,y(t)=P_c(D)\,Q_d(q)\,u(t)+Q_c(D)\left\{ N_{cd}^u(D,q)u[k]+N_{cd}^{\omega^T}(D,q)w[k] \right\} \qquad (18)$$

for $t \in [kT, (k+1)T$ with q and D being the one-step ahead time -shift and time-derivative defined by $q\,v(t)=v(t+T)$ and $\dot{v}(t)=Dv(t)$, respectively , for any differentiable signal v (t) in the continuous-time argument t , where $Q_d(q)$ and $P_d(q)$ are the polynomials in (17) while $Q_c(q)$ and $P_c(q)$ are polynomials of degree n_c and N_{cd}^u and N_{cd}^{ω} are a scalar polynomial and a two-variable n_c - polynomial matrix which have been obtained from the above parameters but the parametrical definition and its development are omitted by space reasons. Note that the term in brackets in the right - hand- side of (18) is a coupling signal from the digital substate and discretized input to the continuous subsystem of (16). The description (18) is obtained from an extended hybrid system of continuous- time substate $x_c(t)$ and the discrete- time substate $x[k]=[x_c^T[k], x_d^T[k]]^T$ used for obtaining (2) at sampling instants . The next simple descriptive example illustrates the decomposition in continuous / discrete (or digital) state variables of an input / output linear mapping involving the operators D and q as it occurs in the general description of (18).

Example

Consider the input/ output linear mapping $v(t)=H_1(D)\,H_3(q)\,\delta[k]+H_2(D)\,\upsilon(t)$ driven by the discrete input $\delta[k]$ and the continuous one $\upsilon(t)$ where $H_1(D)=\frac{D+a}{D+b}$; $H_2(D)=\frac{1}{D+c}$; $H_3(q)=\frac{q+1}{q+2}$ with a , b and c being real constants. Define now two continuous-time variables $v_1(t)$ and $v_2(t)$ and a digital variable $\delta_1[k]$ given by the dynamics $v_1(t)=\frac{D+a}{D+b}\,\delta_1[k]$, $v_2(t)=\frac{1}{D+c}\,\upsilon(t)$ and $v_3[k]=\frac{q+1}{q+2}\,\delta[k]$. Thus , the overall state - space representation is described by

$$v(t)=v_1(t)+v_2(t)+v_3[k] \quad \dot{v}_1'(t)=-bv_1'(t)+(a-b)v_3[k]$$

$$v_1(t)=v_1'(t)+v_3[k]$$

$$\dot{v}_2(t)=-cv_2(t)+\upsilon(t);$$

$$v_3[k]=-2\,v_3[k-1]+\delta[k]+\delta[k-1]$$

subject to initial conditions $v_i(0)=v_{i0}$ (i = 1, 2) and $v_3[0]=v_{30}$. . □

Remark 1

The description of (18) also describes eqns. 16 at sampling instants and results to be

$$Q_c(D)Q_d(q)\,y[k]=[P_c(D)Q_d(D)+Q_c(D)N_{cd}^u(D,q)]u[k] \;+Q_c(D)N_{cd}^{\omega^T}(D,q)\omega[k]$$

whose discrete-time solution is (17). Note through a comparison with (18) that the parametrization of the differential- difference solution to (16) becomes modified at sampling instants with respect to the intersample parametrization since additive terms involving the sampled continuous substate and sampled input result from the plant parametrization given by (16) at sampling instants. □

D) Some Mathematical Results an d Controller synthesis

Global exponential stability conditions for the open-loop plant

The global exponential stability of the unforced system (16) is only dependent on the stability of the *A* -matrix defined by

$$\begin{bmatrix} e^{A_c T}[I+(\int_0^T e^{-A_c\tau}d\tau)A_{cd}] & e^{A_c T}(\int_0^T e^{-A_c\tau}d\tau)A_d \\ A_{ds} & A_d \end{bmatrix} \quad (19)$$

obtained after omitted calculations . This follows from building the extended unforced discrete dynamics x [k +1] = A x[k] with $x[k] = (x_c^T[k], x_d^T[k])^T$. Thus, the continuous - time solution of the continuous substate in (16) satisfies :

$$x_c(kT+\tau)=[e^{A_c\tau}(I+\int_0^\tau e^{-A_c\tau'}d\tau')A_{cs}) \\ , \quad e^{A_c\tau}(\int_0^\tau e^{-A_c\tau'}d\tau')A_{cd}]x[k]$$

Thus , if A is strictly Hurwitzian , then $x_d[k]$, $x_c[k]$ and $x_c(t)$ converge to zero exponentially fast exponentially fast for any bounded rinitial conditions. The next result , whose proof is omitted, is concerned with the stability of the A - matrix under that of A_c and A_d provided that the coupling signals between continuous and discretized (or, indistinctly, digital) variables are sufficiently small.

Proposition 1

Assume that A_c and A_d are strictly Hurwitzian with their maximum eigenvalues satisfying :

$$e^{-\rho' T} \le \left|\lambda_{max}(e^{A_c T})\right| \\ \le e^{-\rho T}(\text{i.e.}, -\rho \le \lambda_{max}(A_c) \le -\rho')$$

and $|\lambda_{max}(A_d)| \le e^{-\rho T}$.Thus, the open-loop unforced plant is globally exponentially stable if $\left|\lambda_{max}(\begin{bmatrix} A_{cs} & A_d \\ A_{ds} & I \end{bmatrix})\right| < \text{Min}(e^{\rho T}-1, \frac{\rho'(e^{\rho T}-1)}{e^{\rho' T}-1})$

Controller synthesis

General design philosophy and Assumptions. The controller to be synthesized will consist of two subcontrollers each one being designed to satisfy a different (respectively, continuous-time or discrete - time) control objective , namely

Objective 1

u [k] = u [k T] is generated in such a way that a prescribed stable discrete reference model of transfer function $W_{md}(q)$ is matched at sampling instants. A discrete subcontroller (Subcontroller 1) which will be then synthesized accomplishes with this control objective. As a part of the design , the coupling signal in (17) from the continuous-time subsystem to the discrete - time subsystem , caused by the signal $\omega[k] = [e^{A_c T}(\int_0^T e^{-A_c t}u(kT+t)dt)b_c$,that includes the contribution of the continuous-time input over one sampling period to the output at sampling instants , is annihilated by synthesizing the appropriate compensator as addressed below.

Objective 2

u (t) ($t \neq kT$) is generated in such a way that the closed-loop system matches a prescribed stable continuous-time reference model of transfer function $W_{mc}(D)$ inbetween sampling instants. A mixed continuous / discrete subcontroller (Subcontroller 2) is synthesized to accomplish with such a control objective. As a part of the design , the couplings between the discretetized signals u [k] and $\omega[k]$ and the continuous subsystem are cancelled by synthesizing the appropriate compensator as addressed below.

Since u [k] and u (t) , $t \in [kT, (k+1)T]$, all nonnegative integer k are , in general , synthesized to satisfy two different control objectives , discontinuities of the control input at sampling instants occur in general. Also , there are input discontinuities caused by the influence in the feedback signals of the modification of the digital substate at sampling instants while it is kept constant inbetween sampling instants. When suitable, the two reference models can be appropriately related to each other in order to state the problem with a unique control objective as discussed later. Those input discontinuities translate in output discontinuities at sampling instants in the more general case when $W_{mc}(D)$ and $W_{md}(q)$ are chosen independently. The combined objective can be intuitively figured as of the actions of Subcontrollers 1- 2 synthesized to satisfy the Control Objectives 1 - 2 . There are two control channels integrated in the actuator that generate the input ' at ' and ' inbetween' sampling instants as u (t) = u ' [k] (t = k T) ; u (t) = u ' (t) ($t \neq kT$) Channel 1 is used to generate (inbetween sampling instants)the input for model -matching of $W_{md}(q)$ while Channel 2 is used to match $W_{mc}(D)$. Note that once Channel 1modifies its state, it supplies u [k] at sampling instants .

Assumptions

1. $P_d(q)$ and $P_c(D)$ have all their zeros in $|q|<1$ and $\text{Re}(D)<0$ **.**

2. All common zeros of $P_d(q)$ and $Q_d(q)$ (of $P_c(D)$ and $Q_c(D)$) , if any , are strictly Hurwitzian and closed - loop zeros and poles of the discrete-time (continuous- time) dynamics , i. e . , they are zero- pole cancellations of $W_{md}(q)$ in $|q|<1$ (of $W_{mc}(D)$ in $\text{Re}(D)<0$). Also, the zeros of $P_c(D)$ and $P_d(q)$ which are cancelled by the controller, if any, are closed- loop poles and thus poles of $W_{mc}(D)$ and $W_{md}(q)$, respectively .

3. $W_{mc}(D)$ and $W_{md}(q)$ are proper , strictly Hurwtzian and of relative orders non less than those of $P_c(D)/Q_c(D)$ and $P_d(q)/Q_d(q)$, respectively. □

Note that Assumption 1 means that both (open-loop) discrete and continuous-time descriptions eqns. 2 and 3 are inversely stable. Assumption 2 means that if any of the discrete or continuous plant dynamics is uncontrollable (i.e., there are zero-pole cancellations) then the associated uncontrollable modes have to be stable and closed-loop poles of the corresponding dynamics. The need for such an assumption will then arise from the solvability of the diophantine equations associated with the pole-placement problems of Objectives 1-2. Note also that if $d = d_c + d_d$ (d_c) is nonzero in (1) then $P_d(q) / Q_d(q)$ ($P_c(D) / Q_c(D)$) is nonstrictly proper and then the realizability of Subcontroller 1 (Subcontroller 2) is realizable for any realizable $W_{md}(q)$ ($W_{mc}(D)$). Thus, the relative order constraint of Assumption 3 holds automatically under the realizability of the discrete-time (continuous- time) reference model guaranteed by its properness of the first partof the assumption.

Objective 1: Synthesis of Subcontroller 1 and Generation of u [k] = u (kT)

The next discrete control law is designed to achieve Objective 1 when the plant (16) is perfectly known and noisy-free:

$$u[k]=\frac{G_{1d}(q)}{L_d(q)}u[k]+\frac{G_{2d}(q)}{L_d(q)}y[k] + \frac{G_{3d}^T(q)}{L_d(q)}\omega[k]+\frac{R_{1d}(q)}{L_d(q)}r_{1d}[k] \qquad (20)$$

The compensating signal $r_{1d}[\,.\,]$ is forwarded to the plant input from the reference model input $r_d[k]$ and $\omega[k] = (\int_0^T e^{A_c(T-\tau)}u(kT+\tau)d\tau)b_c$ according to generation laws given below . All the transfer functions in the above control law are expressed as quotients of polynomials and realizable. The above law is explicited as follows:

$$u[k]= C_{yu}^d(q)y[k]+C_{\omega u}^{d^T}(q)\,\omega[k] + C_{r_1u}^d(q)\, r_{1d}[k] \qquad (21.a)$$

where the compensator transfer functions are

$$C_{yu}^d(q)=\frac{G_{2d}(q)}{L_d(q)-G_{1d}(q)}$$
$$C_{\omega u}^d(q)=\frac{G_{3d}(q)}{L_d(q)-G_{1d}(q)}$$
$$C_{r_1u}^d(q)=\frac{R_{1d}(q)}{L_d(q)-G_{1d}(q)L_d(q)} \qquad (21.b)$$

The problem of accomplishing with Objective 1 consists of designing the polynomials $G_{id}(q)$ (i=1,2) and $R_{1d}(q)$, the polynomial vector $G_{3d}(q)$ as well as the compensating signal $r_{1d}[.]$, for a given stable $L_d(q)$ so that $W_{md}(q)$ is matched if the plant is perfectly known and free of unmodelled dynamics and noise. The next result addresses the controller design :

Theorem 1

Suppose that the control law (21) is applied , $r_d[k]$ ($k \ge 0$)is the uniformly bounded reference input sequence to $W_{md}(q)$ and that the next assumptions hold :

4. Assumptions 1 - 3 hold for $P_d(q)$, $Q_d(q)$ and the poles of $W_{md}(q)$, and that all the roots of $R_{1d}(q)$ and $L_d(q)$ are in $|q|<1$. Assume also that $d_c = -d_d$ in (1) and $\deg(R_{1d}) \le \deg(L_d - G_{1d})$.

5. $P_d(q) = \overline{Q}_d(q)P'_d(q)$ and $Q_d(q) = \overline{Q}_d(q)Q'_d(q)$ where $\overline{Q}_d(q)$ is the strictly Hurwitzian (from Assumption 2) ma ximum common factor of $P_d(q)$ and $Q_d(q)$. Also , $P'_d(q)=P_{1d}(q)P_{2d}(q)$ with $P_{1d}(q)$ being defined by the discrete strictly Hurwitzian plant zeros (from Assumption 2) which are not plant poles and they are transmitted to the reference model $W_{md}(q) = B_{md}(q) / A_{md}(q)$.

6. $L_d(q)$ is factorized as $L_d(q) = P_{2d}(q)L'_d(q)$ in (20) .

Thus , the discrete closed - loop transfer function equalizes that of $W_{md}(q)$ provided that Subcontroller 1 and its associated compensating signal $r_{1d}[.]$ are synthesized as follows :

$$r_{1d}[k]=\frac{B_{md}(q)}{R_{1d}(q)P_d(q)}r_d[k] + \frac{L_d(q)-G_{1d}(q)}{R_{1d}(q)}M_d c^T\omega[k] \qquad (22)$$

where $G_{1d}(q)=G'_{1d}(q)P_{2d}(q)$, $d = d_c + d_d$, $c = c_c + c_{cs}$ with $M_d(q)$ being an arbitrary polynomial satisfying $\deg(M_d(q)) < \deg(L_d(q) - G_{1d}(q)) - \deg(R_{1d})$, $G_{3d}(q) = -(1+R_{1d}M_d c)$, and $G'_{1d}(q)$, $G_{2d}(q)$ being polynomials which are the unique solution to the diophantine equation :

$$Q'_d(q)\,G'_{1d}(q)+P_{1d}(q)G_{2d}(q) = Q'_d(q)L'_d(q)-A''_{md}(q) \qquad (23)$$

subject to the degree constraints $\deg(G_{2d}(q)) < \deg(Q'_d(q))$ or $\deg(G'_{1d}(q)) < \deg(P_{1d}(q))$ for $A''_{md}(q)$ being a polynomial satisfying the factorizations

$$A_{md}(q)=\overline{Q}_d(q)A'_{md}(q) =\overline{Q}_d(q)P_{2d}(q)A''_{md}(q) \qquad (24)$$

which exist from Assumption 2 . □

Corollaries

1 . Theorem 1 also holds under the same assumptions if G_{3d} is a rational function and the compensating signal in the controller satisfy :

$$G_{3d}(q)=\frac{Q'_d(q)(G'_{1d}(q)-L'_d(q))c}{P_{1d}(q)} \qquad (25)$$

$$r_{1d}[k] = \frac{1}{R_{1d}(q)}\frac{B'_{md}(q)}{P_{2d}(q)}r_d[k] \qquad (26)$$

and all the remaining compensators of the control law remaining identical as in Theorem 1 .

2 . Theorem 1 and Corollary 1 also apply directly to the regulation case with $r_d[k] = r_{1d}[k] = 0$ with the closed-loop dynamics resulting to be $A_{md}(q)\, y[k] = 0$. □

The proof of Corollary 1 becomes direct from the application of Assumptions 3 - 6 of Theorem 1 and the use of the cancelled factors $Q_d P_{2d}$ and $G_{1d}=G'_{1d}P_{2d}$ to yield :

$$G_{3d}=\frac{\overline{Q}_d Q'_d(G_{1d}-L_d)c}{P_d} = \frac{Q'_d(G'_{1d}-L'_{1d})c}{P_{1d}}$$

$$\Rightarrow C^d_{r_1 u}=\frac{G_{3d}}{L'_d-G'_{1d}}=-\frac{Q'_d}{P'_d}=-\frac{Q_d}{P_d}$$

which is nonstrictly proper and stable since P_d is strictly Hurwitzian and $d_c \neq d_d$. The use of the above relationships leads to

$$[(L_d-G_{1d})Q_d-P_d G_{2d}]y[k] = Q_d(q)R_{1d}(q)r_{1d}[k] \qquad (27)$$

from (22)-(26) and Corollaries 1-2 follow as Theorem 1. Corollary 2 follows when $r_d[k]\equiv 0$. . □

Note that the main difference between the design of Theorem 1 and Corollary 1 is the choice of the compensator $C^d_{r_1 d}(q)$ in (21). In Theorem 1 , this proper compensator of high - frequency being $-d^{-1}c = -(d_c+d_d)^{-1}(c_c+c_{cs})$ which cancels the high- frequency gain of $\frac{G'_{1d}-L'_d}{P_{1d}}Q'_d c$. Thus , the closed-loop dynamics depends on $\omega[k-1]$ but not on $\omega[k]$ and $M_d(q)$ is kept arbitrary. However, the decomposition of all the transfer functions from the components of $\omega[k]$ to u [k] in Corollary 1 with their high- frequency gains being cancelled is not used. The synthesis mechanism in that case is the choice of $G_{3d}(q)$ such that the transfer function from $\omega[k]$ to u [k] is cancelled.

Objective 2: Synthesis of Subcontroller 2 and generation of $u(t)$ $(t \neq kT)$

The next control law is designed for the achievement of Objective 2 when the known plant is perfectly modelled and free- noise and has the following implicit structure :

$$u(t)=\frac{G_{1c}(D,q)}{L_c(D,q)}u(t)+\frac{G_{2c}(D,q)}{L_c(D,q)}y(t) + \frac{G_{3c}(D,q)}{L_c(D,q)}u[k]+\frac{G^T_{4c}(D,q)}{L_c(D,q)}\omega[k]+\frac{R_{1c}(D)}{L_c(D,q)}r_{1c}(t) \quad (28)$$

for all $t \in (kT, (k+1)T)$ and all nonnegative integer k , with $r_{1c}(t)$ being a compensating signal to be generated as a part of the controller design and $L_c(D,q)$ being a strictlty Hurwitzian two-variable polynomial. The various filters are formed by two variable polynomials and the associated hybrid realizations can be obtained as addressed in the given example. The above control law becomes explicited as follows:

$$u(t)=C^c_{yu}(D,q)y(t)+C^c_{uu}(D,q)u[k] + C^{c\,T}_{\omega u}(D,q)\omega[k]+C^c_{r_1 u}(D,q)r_{1c}[k] \qquad (29)$$

for all $t \in (kT, (k+1)T)$ with

$$C^c_{yu}(D,q)=\frac{G_{2c}(D,q)}{L_c(D,q)-G_{1c}(D,q)} \qquad (30)$$

$$C^c_{uu}(D,q)=\frac{G_{3c}(D,q)}{L_c(D,q)-G_{1c}(D,q)} \qquad (31)$$

$$C^c_{\omega u}(D,q)=\frac{G_{4c}(D,q)}{L_c(D,q)-G_{1c}(D,q)} \qquad (32)$$

$$C^c_{r_1 u}(D,q)=\frac{R_{1c}(D)}{L_c(D,q)-G_{1c}(D,q)} \qquad (33)$$

Note that the compensators of (39)-(33) are dependent on D and q because of structure of (3) . The problem of fulfilling Objective 2 consists of synthesizing (14), subject to (30)-(33), as well as the compensating signal $r_{1c}(.)$ as addressed in the next result which applies the philosophy of Theorem 1 and Corollary 1 to the problem of model-matching of the continuous reference model . In the following, the degree of two-variable polynomials with respect to one of the variables is denoted with the corresponding subscript.

Theorem 2

Suppose that $r_c(t)$ is the uniformly bounded reference input to W_{mc} D) and that the next assumptions hold

7. Assumptions 1 - 3 hold for $P_d(q)$, $Q_d(q)$, $P_c(D)$ and $Q_c(D)$ and that the poles of $W_{mc}(D)$ and all the roots of $R_{1c}(D)$ and $L_c(D)$ are in $\mathrm{Re}(D)<0$. Assume also that $d_c \neq -d_d$.

8. $P_c(D)$ admits the polynomial factorization $\overline{Q}_c(D)\, P_{1c}(D)\, P_{2c}(D)$ where $\overline{Q}_c(D)$ includes the (stable) common roots of $P_c(D)$ and $Q_c(D)$, $P_{1c}(D)$ contains eventual zeros of $P_c(D)$ transmitted from the plant to the reference model and $P_{2c}(D)$ includes the (stable) plant zeros which are closed-loop poles and controller poles .

Thus, the closed -loop dynamics is globally exponentially stable and defined by

$$A_{mc}(D)y(t)=B_{mc}(D)r_c(t) \qquad (34)$$

if the compensators in (30)-(33) and compensating signal $r_{1c}(t)$ are chosen to satisfy $G_{1c}(D,q) = P_2(D,q)\,G'_{1c}(D,q)$ where $(G'_{1c}(D,q)\,,\ G_{2c}(D,q))$ is a polynomial pair being a unique solution to the two- variable diophantine equation :

$$Q'_c(D)G'_{1c}(D,q)+P'_c(D)G_{2c}(D,q) = L'_c(D,q)-A''_{mc}(D,q) \qquad (35)$$

with $L_c(D,q) = P_{2c}(D,q)\,L'_c(D,q)$ and $A_{mc}(D,q) = \overline{Q}_c(D,q)P_{2c}(D,q)A''_{mc}(D,q)$ subject to any of the two the next degree constraints

$$\deg_D(L_c(D,q)-A_{mc}(D)) \le \deg_D(G_{1c}(D,q))$$
$$\deg_D(G_{2c}(D,q)) < \deg(Q_c(D)) = \deg(P_c(D)) \qquad (36.a)$$

$$\deg_D(L_c(D,q)-A_{mc}(D)) \le \deg_D(G_{2c}(D,q))$$
$$\deg_D(G_{1c}(D,q)) < \deg(Q_c(D)) = \deg(P_c(D)) \qquad (36.b)$$

and

$$G_{3c}(D,q)=\frac{G'_{1c}(D,q)-L'_c(D,q)}{\overline{Q}_c(D)P_{1c}(D)Q_d(q)}N^{u}_{cd}(D,q) \qquad (37.a)$$

$$G_{4c}(D,q)=\frac{G'_{1c}(D,q)-L'_c(D,q)}{\overline{Q}_c(D)P_{1c}(D)Q_d(q)}N^{\omega}_{cd}(D,q) \qquad (37.b)$$

$$r_{1c}(t)=\frac{B'_{mc}(D)}{P_{2c}(D)R_{1c}(D)}r_c(t) \qquad (37.c)$$

with $B'_{mc}(D)$ being the free- design zeros of $W_{mc}(D)$ (i. e. , those of W_{mc} D) excluding the factor $\overline{Q}_c(D)P_{1c}(D)$). □

The proof is omitted by space reasons. Note that Theorem 2 applies the same philosophy for pole-placement for the continuous reference model as the previously used for the discrete one in Corollary 1 since the coupling signals from the discrete subsystems to the continuous one are cancelled by the controller (29)-(32) with the compensators and compensating signal fulfilling (35)-(37) while the compensating signal in (37.c) is used to cancel the unsuitable plant zeros . A more general choice of $r_{1c}(t)$ based on an arbitrary design of $G_{ic}(D,q)$ $(i = 3, 4)$ could be established without difficulty in the same way as addressed in Theorem 1 for the discrete model , although at the expense of more involved calculations .

E) Summary of the controller synthesis method and guidelines for particular designs of interest

The synthesis of the hybrid controller for the hybrid plant (1) consists of firstly defining the discrete and continuous reference models $W_{md}(q) = B_{md}(q) / A_{md}(q)$ and $W_{mc}(q) = B_{mc}(q) / A_{mc}(q)$ for uniformly bounded reference inputs $r_d[k]$ and $r_c(t)$, $t \in [kT, (k+1)T)$. Then, $u[k]$ and $u(t)$, $t \in [kT, (k+1)T)$ are generated from (6) , with the compensators designed according to Theorem 1 or Corollary 1 , and (30)-(33) with the compensators designed according to Theorem 2 , respectively . There are several particular designs of practical interest, within the above general framework, which are now described concerning the use of a unique reference model or the way of combining the dynamics of two separate discrete and continuous reference models to improve the performances of the closed-loop system .

Design 1 (<u>Continuous - time reference mod</u>*el*). The reference input to the continuous-time reference model$W_{mc}(D)$is piecewise continuous with discontinuities at sampling instants and being constant inbetween sampling instants and the discrete-time reference model $W_{md}(q)$ is the z - transform of $W_{mc}(D)$. Choose the reference signal as $r(t) = r_c(t) = r_c[k] = r_d[k] = r[k]$, $t \in [kT, (k+1)T)$. Thus, the reference output is generated by a unique reference model for all $t \geq 0$. The , in general discontinuous , plant input is generated from (6) and Theorem 1 , or Corollary 1 , for $t = kT$ and from (30) -(33)and Theorem 2 for $t \neq kT$. The main difference of Design 1 with respect to Design 2 below is that the plant input is generated at sampling instants from a discrete-time model - following philosophy while it is generated from a continuous-time model- matching philosophy inbetween sampling instants despite that a unique continuous - time model is available together with its discretization at sampling instantse. In other words, the diophantine equation solving the pole-placement problem at sampling instants is of a discrete nature and it is related to the q-operator while that used for the continuous dynamics pole-placement is of a continuous nature and it is related to the D-operator.

Design 2 (<u>Continuous- time reference model with the controller using periodic plant reparametrization</u>). $W_{mc}(D)$ is used as the unique reference model at all time. The use of a discrete-time reference model $W_{md}(q)$ is omitted in this design . At each new sampling instant $t = kT$, the continuous-time description of the plant is reparametrized with the replacement $P_c(D)\,Q_d(q)$ with $P_c(D)\,Q_d(q) + Q_c(D)N^{u}_{cd}(D,q)$ in (3), according to Remark 1, since the right - hand -side terms of (1.a) and (1.c) that involve to $u(t)$ and $u[k]$ have to be summed up when $t = kT$. Thus , (30) -(18)and Theorem 2 are used to generate the control signal for each $t = kT$ with $r(t)=r[k]= r_c[k]$. Subsequently, $r_{1c}(t) = r_1(t)$ and $r(t) = r_c(t)$, $t \in (kT, (k+1)T)$ and the plant input $u(t)$ is generated from (32) and Theorem 2 for $t \neq kT$. The main difference of Design 2 with respect to

Design 1 is that now the plant input is always generated from Theorem 2 (i. e., from the continuous-time dynamics) with the plant involving a reparametrization at sampling instants (see Remark 1). In other words, the associated pole-placement problem problem is given by two diophantine equations at and inbetween sampling instants. Those equations are associated to , in general , different plant parametrizations which arise fro the fact that the input , state and output signals become additive at sampling instants in the rigth-hand-sides of eqns. 1 .

Design 3 (Discrete - time reference model). The plant input is restricted to be piecewise continuous with discontinuities at sampling instants only while being constant inbetween sampling instants , i. e ., it is generated by a zero-order-hold and $u(t) = u[k] = u(kT)$, $t \in (kT, (k+1)T)$. Thus , only the discrete-time reference model $W_{md}(q)$ is used in this particular design . Thus, $r_1[k] = r_{1d}[k]$ and $r(t) = r[k] = r_d[k]$. Simple calculus yields after substitution in (17) and (21) yields directly :

$$Q_d y[k] = \bar{P}_d(q) u[k] \quad (38.a)$$

$$u(t) = u[k] = C^d_{yu}(q) y[k] + C^d_{r_1 u}(q) r_{1d}[k] \quad (38.b)$$

all $t \in [kT, (k+1)T)$ with

$$\bar{G}_{1d}(q) = G_{1d}(q) + G^T_{3d}(q)\ell = \bar{G}'_{1d}(q) P_{2d}(q) \quad (38.c)$$

$$\bar{P}_d(q) = P_d(q) + Q_d(q) c^T \ell \quad (38.d)$$

$$C^d_{yu}(q) = \frac{G_{2d}(q)}{L_d(q) - \bar{G}_{1d}(q)} \quad (38.e)$$

$$C^d_{r_1 u}(q) = \frac{R_{1d}(q)}{L_d(q) - \bar{G}_{1d}(q)} \quad (38.f)$$

the model- matching problem is solved by applying the controller to the plant while solving the diophantine equation (23) with the replacement $G_{1d} \to \bar{G}_{1d}$ in the solution polynomials $\bar{G}'_{1d}$ and G_{2d} , which are unique if $\deg(\bar{G}'_{1d}) = \deg(P_{1d}) - 1$ and the compensating signal $r_1(t) = r_1[k] = r_{1d}[k] = R^{-1}_{1d} P^{-1}_{2d} B'_{md} r[k]$, all $t \in [kT, (k+1)T]$ and all nonnegative integer k .

Design 4 (<u>General combined continuous - time and discrete-time reference models with large sampling periods</u>). This design keeps both Objectives 1-2. The discrete-time reference model $W_{md}(q)$ is designed with a large sampling period compared to the dominant constant of the continuous- time subsystem while keeping Assumption 3. I this context, Objective 2 over the continuous- time reference model $W_{mc}(D)$ is the basis of the overall design . Objective 1 is used for periodic testing of the closed-loop performance and eventual re- adjustment of the continuous-time model in case of performance' s test failure. If such a test fails in terms of excessive deviations of the sampled output from its neighbouring values generated by Objective 1 then either the high- frequency gain of $W_{mc}(D)$ or its reference input $r_c(t)$ can be re-updated appropriately. This model re-updating procedure makes justifiable the use of two separate continuous-time and discrete- time reference models and two associated control objectives as stated in the general design procedure. An important advantage is that the possible re-updating could be implemented while keeping two independent continuous-time and discrete - time reference dynamics. Another important key issue which can be extended to all the above designs is that the continuous-time equations of both the plant and continuous-time reference model can be implemented in practice through a discretization at very low sampling periods compatred to the sampling period that regulates the discrete dynamic. All the formulation has been extended to the presence of unmodelled dynamics and noise environments and also it has been tested with numerical examples in both of these situations as well as in the ideal deterministic one, but the obtained results have been omitted by space reasons.

VI. CONCLUDING REMARKS

This paper has developed a continuous-time adaptive stabilizer for a continuous-time first-order controllable plants which can have an unstable zero and is subject to an input saturation of sigmoidal function type. The mechanism used to guarantee the scheme' s closed-loop stability is a modification scheme of the parameter estimates which is based on the use of a hysteresis switching function. The switches are built so that the modified plant estimated model is controllable and then it has no pole-zero cancellation. An alternative adaptive stabilizer which only modifies the parameter estimates at sampling instants, but which is based on continuous-time input / output measurements, is also addressed for the same kind of simple plant. The resulting closed-loop system is of a hybrid nature because of the discrete updating of the estimation scheme. A similar hysteresis switching function, which operates at sampling instants, is also used in that case so as to guarantee the controllability of the modified estimated plant model. Some extensions to design control objectives of a class of saturation- free hybrid systems have been developed . Such a class consists of coupled continuous-time and digital dynamics.

APPENDICES

A) Outline of Proof of Theorem 1

Define the Lyapunov-like function candidate $V = 1/2\left(\tilde{\theta}^T P^{-1} \tilde{\theta}\right)$ by using the parametrical error $\tilde{\theta} = \theta - \theta^*$ and the inverse of the covariance matrix. It follows that $P^{-1}\tilde{\theta}$ is constant for all time so that $\theta^* = \theta + P\beta$ .Thus,

$$0 < \delta^*_0 \le c\left(\beta^*\right) = \left|f_1\right|$$
$$\le \left(\left|f + f_1\right| + \|v\| + \|p_3\| \|p_1 - p_2\|\right) \max\left(1, \|\beta^*\|^2\right)$$

where

$$f=\theta_1\theta_4-\theta_4^2+\theta_5-\theta_2$$
$$f_1=b_0^* a^* - b_1^*$$
$$v^T=\left(p_5-p_2\right)^T+\left(p_1-p_4\right)^T\theta_3+p_3^T\left(\theta_1-\theta_4\right)$$

It follows directly that

$$c(\beta)=\left|(\bar\theta_1-\bar\theta_4)\bar\theta_3-(\bar\theta_2-\bar\theta_5)\right|$$
$$=\left|f+\beta^T\left(v+\left(p_1-p_4\right)\beta^T p_3\right)\right|>0$$

since $f+f_1$, v, p_3 and p_1-p_4 cannot be simultaneously zero since $c\left(\beta^*\right)>0$. $f_1=-f\neq 0$ if $f+f_1=0$ so that $c\left(\beta\right)>0$. If $\beta=\pm v\neq 0$ then $c\left(\beta\right)>0$. If $f=v=0$ then β equalizes one of the combinations $\pm\left(p_1-p_4\right)\pm p_3$ and $c\left(\beta\right)>0$. Property (i) has been proven. Property (ii) is proven as follows. First note that $2\dot V=-e^2\leq 0$ what implies that $V\leq V\left(0\right)<\infty$. Then, e(t) is bounded and square-integrable and the parametrical error is also bounded for al time. Finally, $d\left(\mathrm{tr}\,P\right)/dt=-\varphi^T P^2\varphi\leq 0$ what implies that $P\varphi$ is bounded and square-integrable. Properties (iii)-(iv) follow from the fact that P is non-increasing and positive semidefinite from its updating rule so that it converges. Also,

$$\theta\left(t\right)-\theta\left(0\right)\leq\int_0^t\left\|\dot\theta\left(\tau\right)\right\|d\tau=\int_0^t\left\|P(\tau)\varphi\left(\tau\right)e\left(\tau\right)\right\|d\tau$$
$$\leq\frac{1}{2}\int_0^t\left(\left\|P(\tau)\varphi(\tau)\right\|^2+\left\|e(\tau)\right\|^2\right)d\tau<\infty$$

for all time. It follows that the parametrical error converges asymptotically to a finite limit. From this partly result, the remaining of the proof follows by calculating a bounded upper-bound of the norm-square integral of the time derivative of the estimate time-.derivative. It follows that $\dot\theta$ is bounded and square-integrable. Then , using the Diophantine equation for the controller synthesis, it follows that the modified estimated vector $\bar\theta$ also converges asymptotically as well as they converge the various controller parameters. □

B) Outline of Proof of Theorem 2

One gets from (16) that $\Delta\tilde\theta_{k-1}=-P_k\Delta\tilde P_k^{-1}\tilde\theta_{k-1}$ with the one-step incremental error being:

$$\Delta\tilde\theta_{k-1}=\tilde\theta_k-\tilde\theta_{k-1}\quad\text{and}\quad\Delta\tilde P_k^{-1}=P_{k+1}^{-1}-P_k^{-1}$$

Then, for a Lyapunov sequence candidate $V_k=\tilde\theta_k^T\tilde P_k^{-1}\tilde\theta_k$, one gets a one-step increment from (16) :

$$\Delta V_{k-1}=V_k-V_{k-1}$$

$$=-\tilde\theta_{k-1}^T\Delta\tilde P_k^{-1}\left(I-P_k\Delta\tilde P_k^{-1}P_k\right)\Delta\tilde P_k^{-1}\tilde\theta_{k-1}\leq 0$$

if $c_k\geq c_{k0}$. Then, the candidate is a Lyapunov sequence with bounded eigenvalues of the covariance matrix implying strictly positive eigenvalues of its inverse, what leads to the results of Theorem 2. □

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