On a General Efficient Class of Four-Step Root-Finding Methods

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Abstract—In this paper, a general class of four-step iterative methods with four points per iteration is investigated for solving one-variable nonlinear equations. The introduced approximation for the first derivative of the function in the fourth step can be applied on any optimal derivative-involved eighth-order method to attain a new fourteenth-order without memory method with 1.6952 as its efficiency index. The produced methods have better order of convergence and efficiency index in comparison with optimal eighth-order methods and in light of these strong points; they can be observed as robust and efficient multi-point iterative methods. Per cycle, they consist of four evaluations of the function and one evaluation of the first derivative. The error equation for one method of this class is obtained theoretically. And subsequently its efficacy is tested on a series of relevant numerical problems to reveal that the presented methods from the class are efficient and accurate.

Keywords—Error equation, four-step methods, multi-point methods, nonlinear equations, iterative schemes, simple root, efficiency index, optimality, order of convergence.

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I. History and Introduction

LET $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a scalar function which is sufficiently smooth in the real open domain $D = (a, b)$ and has a simple root in this neighborhood, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. There is a vast literature on finding the simple roots of nonlinear equations by iterative methods [1, 18]. Normally, the improvements have been constructed to increase up the rate of convergence and efficiency index of the existing methods (occasionally Newton's iteration).

Newton's method which is defined as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

was described by Isaac Newton in De analysi per aequationes numero terminorum infinitas (written in 1669, published in 1711 by William Jones) and in De metodis fluxionum et serierum infinitarum (written in 1671, translated and published as Method of Fluxions in 1736 by John Colson) [21]. However, his description differs substantially from the modern description given above: Newton applies the method only to polynomials. He does not compute the successive approximations x_n, but computes a sequence of polynomials and only at the end, he arrives at an approximation for the root x. Finally, Newton viewed the iteration as purely algebraic and fails to notice the connection with calculus.

Newton probably derived his iteration from a similar but less precise method by Vieta. The essence of Vieta's method can be found in the work of the Persian mathematician, Sharaf al-Din al-Tusi, while his successor Jamshid al-Kashi used a form of Newton's method for solving

$$x^P - N = 0,$$

to find roots of N.

A special case of Newton's method for calculating square roots was known much earlier and is often called the Babylonian method. Newton's method was used by 17th century Japanese mathematician Seki Kowa to solve single-variable equations, though the connection with calculus was missing as mentioned above.

Newton's scheme was first published in 1685 in A Treatise of Algebra both Historical and Practical by John Wallis. In 1690, Joseph Raphson published a simplified description in Analysis aequationum universalis. Raphson again viewed Newton's method purely as an algebraic method and restricted its use to polynomials, but he describes the method in terms of the successive approximations x_n instead of the more complicated sequence of polynomials used by Newton.

Finally, in 1740, Thomas Simpson described Newton's method as an iterative method for solving general nonlinear equations using fluxional calculus, essentially giving the description above [21]. In the same publication, Simpson also gives the generalization to systems of two equations and notes that Newton's method can be used for solving optimization problems by setting the gradient to zero.

The main goal and motivation in the construction of new methods should be as high as possible computational efficiency for approximating the simple roots; in other words, it is desirable to attain as high as possible the convergence order with fixed number of evaluations per iteration.

To construct optimal fourth-order schemes, based on the still-unproved hypothesis of Kung and Traub in [7] with optimal efficiency index which is defined by

$$p^{(\theta-1)/\theta},$$

wherein p is the rate of convergence and θ is the whole number of evaluations per cycle, many two-step cycles have been considered. For example, the following method had been suggested in [2] with optimal fourth-order convergence

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \frac{f(x_n)+f(y_n)}{f'(x_n)} - \frac{2f(x_n)+f(y_n)}{f'(x_n)}\left(\frac{f(y_n)}{f(x_n)}\right)^2. \end{cases} \quad (1)$$

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This scheme was built through the one-point method of Newton. We know about Newton's scheme that its success depends on the regularity of the function near the solution: the derivative of the function should be non-zero and finite at the solution for the method to converge. Moreover, if the derivative is Lipschitz continuous (does not show sharp needle-like spikes), the method converges quadratically near the solution (error after each successive approximation is some constant times the square of the error in the previous approximation) [5, 14].

Three-step schemes with three points in which we have three evaluations of the function and one evaluation of the first derivative are considered to set up optimal eighth-order methods for solving single variable nonlinear equations.

The idea of such developments is the usage of optimal two-point methods in the first and second steps and the Newton's method in the third step. Subsequently, to reach high efficiency index with eighth-order convergence; a powerful approximation of the new-appeared first derivative of the function in the added iteration is used or the technique of weight function is investigated to increase the rate of convergence. In between, sixth-order [6, 16, 17, 19] or seventh-order [3] methods can be obtained as well. As an illustration, a sixth-order method (in the follow-up form)

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \frac{f(x_n)+f(y_n)}{f'(x_n)}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}, \end{cases}$$

and a seventh-order scheme were obtained (in [3] in the following form)

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n + \frac{f(x_n)+f(y_n)}{f'(x_n)} - 2\frac{f(x_n)}{f(x_n)-f(y_n)}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}, \end{cases} \tag{2}$$

where $f[z_n, y_n]$, $f[z_n, x_n, x_n]$ are divided difference of the function $f(x)$ and could be defined as follows

$$f[z_n, y_n] = \frac{f[z_n] - f[y_n]}{z_n - y_n},$$

and

$$f[z_n, x_n, x_n] = \frac{f[z_n, x_n] - f'(x_n)}{z_n - x_n}.$$

Note that we use the similar notations throughout. Now, let us review some optimal eighth-order methods.

An eighth-order method was provided in [15] in the following form by taking into consideration weight function and an estimation of the new-appeared first derivative of the function in the third step

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)}\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - [1 + \frac{f(z_n)}{f(x_n)} + (\frac{f(z_n)}{f(x_n)})^2]\frac{f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]}. \end{cases} \tag{3}$$

Petkovic in [13] suggested the following optimal eighth-order method by considering the (quasi) Hermite interpolation for estimating the new-appeared first derivative of the function in the third step

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)+tf(y_n)}{f(x_n)+(t-2)f(y_n)}\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \\ \frac{f(z_n)}{2(f[x_n,z_n]-f[x_n,y_n])+f[y_n,z_n]+(\frac{y_n-z_n}{y_n-x_n})(f[x_n,y_n]-f'(x_n))}, \end{cases} \tag{4}$$

wherein $t \in \mathbb{R}$. He further has claimed a new development of a general class of optimal n-point methods with convergence order of 2^n. His efficiency index, unfortunately, has turned out to be far from being optimal due to the some unexpected logical errors appearing in (3.9) of page 4406 in [13], which does not yield $(n + 1)$ evaluations for any $n = 4$. Hence, the iterative methods developed by him will not be of further interest to us, being excluded from our discussion here.

In fact, in case of a four-step method, his technique reaches the sixteenth-order of convergence by using 6 evaluations (with efficiency index 1.587) and in case of a five-step method, the resulted method of his class is of order 24 with seven evaluations per full cycle (with the efficiency index 1.574), and so on. As can be seen this index of efficiency is going down and down by considering more steps. Accordingly, the results given by him are not optimal for the case of four-step or higher step methods.

Petkovic et al. in [12] investigated a new three-step scheme by using a nonlinear fraction for approximating the first derivative of the function in the third step as follows

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)+tf(y_n)}{f(x_n)+(t-2)f(y_n)}\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{\frac{a_2-a_1a_4+a_3(z_n-x_n)(2+a_4(z_n-x_n))}{(1+a_4(z_n-x_n))^2}}, \end{cases} \tag{5}$$

where

$$\begin{cases} a_1 = f(x_n), \\ a_3 = \frac{f'(x_n)f[y_n,z_n]-f[x_n,y_n]f[x_n,z_n]}{x_n f[y_n,z_n]+\frac{y_n f(z_n)-z_n f(y_n)}{y_n-z_n}-f(x_n)}, \\ a_4 = \frac{a_3}{f[x_n,y_n]} + \frac{f'(x_n)-f[x_n,y_n]}{(y_n-x_n)f[x_n,y_n]}, \\ a_2 = f'(x_n) + a_4 a_1. \end{cases} \tag{6}$$

In 2010, Wang and Liu proposed a robust optimal eighth-order method [23] by using weight functions as follows

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \frac{f(x_n)}{f'(x_n)}\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}[\frac{1}{2} + \frac{5f(x_n)^2+8f(x_n)f(y_n)+2f(y_n)^2}{5f(x_n)^2-12f(x_n)f(y_n)} \\ \times(\frac{1}{2} + \frac{f(z_n)}{f(y_n)})]. \end{cases} \tag{7}$$

Recently, Neta and Petkovic in [10] re-presented (methods previously given by Neta in 1981) an eighth-order method by approximating the new-appeared first derivative of the function in the third step using inverse interpolation. Their scheme is as comes next

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)+tf(y_n)}{f(x_n)+(t-2)f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = y_n + c[f(x_n)]^2 - d[f(x_n)]^3, \end{cases} \tag{8}$$

wherein $t \in \mathbb{R}$, and c, d are defined by

$$\begin{cases} d = \frac{1}{(f(y_n)-f(x_n))(f(y_n)-f(z_n))f[y_n,x_n]} - \\ \frac{1}{(f(z_n)-f(x_n))(f(y_n)-f(z_n))f[z_n,x_n]} \\ + \frac{1}{f'(x_n)(f(z_n)-f(x_n))(f(y_n)-f(z_n))} - \\ \frac{1}{f'(x_n)(f(y_n)-f(x_n))(f(y_n)-f(z_n))}, \\ c = \frac{1}{(f(y_n)-f(x_n))f[y_n,x_n]} - \frac{1}{f'(x_n)(f(y_n)-f(x_n))} - \\ d(f(y_n)-f(x_n)). \end{cases} \tag{9}$$

We here remark that, Neta in [11] suggested a four-step method in the following form but did not demonstrate its order of convergence

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)+Af(y_n)}{f(x_n)+(A-2)f(y_n)} \frac{f(y_n)}{f'(x_n)}, \quad A \in \mathbb{R} \\ w_n = z_n - \frac{f(x_n)-f(y_n)}{f(x_n)-3f(y_n)} \frac{f(z_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + \theta_1 f^2(x_n) + \theta_2 f^3(x_n) + \theta_3 f^4(x_n), \end{cases} \tag{10}$$

wherein

$$\theta_3 = \frac{\Delta_1 - \Delta_2}{F_w - F_y},$$

$$\theta_2 = -\Delta_1 + \theta_3(F_w + F_z),$$

$$\theta_1 = \varphi_w + \theta_2 F_w - \theta_3 F_w^2$$

with

$$\Delta_1 = \frac{\varphi_w - \varphi_z}{F_w - F_z}, \Delta_2 = \frac{\varphi_y - \varphi_z}{F_y - F_z},$$

and

$$\begin{cases} \varphi_w = \frac{1}{F_w}\left(\frac{w_n - x_n}{F_w} - \frac{1}{f'(x_n)}\right), \\ \varphi_y = \frac{1}{F_y}\left(\frac{y_n - x_n}{F_y} - \frac{1}{f'(x_n)}\right), \\ \varphi_z = \frac{1}{F_z}\left(\frac{z_n - x_n}{F_z} - \frac{1}{f'(x_n)}\right), \end{cases} \tag{11}$$

with

$$F_w = f(w_n) - f(x_n), F_y = f(y_n) - f(x_n),$$

and

$$F_z = f(z_n) - f(x_n).$$

Newly, Geum and Kim in [4] proved that the order of convergence of (10) is fourteen.

To find another way for solving nonlinear equations, such as Homotopy methods, kindly refer to [24]. For further reading, we refer the readers to [8, 9, 20, 22, 26, 27].

Now after furnishing the outlines of the present work and a short study on the available high order developments of the classical Newton's method, we will provide our contribution in the next section. Section II gives a general class of efficient four-step four-point fourteenth-order methods including four evaluations of the function and one of its first derivative per cycle. This section is followed by Section III where the numerical comparisons are made to manifest the accuracy of the new methods from our class. Finally, the conclusion of the paper will be drawn in Section IV.

II. Construction of a Class of Fourteenth-Order Iterative Methods

In this section, in order to obtain novel methods with better order of convergence and efficiency index, we take account of a Newton's method in the fourth-step of a cycle in which the first three steps are calculated by any optimal derivative-involved eighth-order method, such as (3), (4), (5), (7) and (8). Subsequently, the considered method has sixteenth-order with 1.5874 as its efficiency index which is lower than 1.6817 of optimal eighth-order methods' efficiency index. As a matter of fact, we consider

$$\begin{cases} w_n = \text{optimal eighth} - \text{order method}, \\ x_{n+1} = w_n - \frac{f(w_n)}{f'(w_n)}. \end{cases} \tag{12}$$

In other words, a prominent acceleration technique consists of composing two iterative methods of orders p and q, respectively, to obtain a method of order pq. Anyhow, this process increases the number of evaluations per cycle; and thus, the efficiency index is dropped heavily. At this time, this question is raised: "is there any way to keep the convergence rate and efficiency index up as much as possible?".

Hence, in order to improve the efficiency index of the composed method, the existing idea is to introduce approximations that reduce the number of evaluations, maintaining the convergence order as high as possible. We increase this efficiency index by estimating $f'(w_n)$ with taking into consideration a combination of already computed function values. Let us assume the following nonlinear fraction as the approximation of the function $f(x)$ in the domain D

$$m(t) = \frac{b_1 + b_2(t-x) + b_3(t-x)^2}{1 + b_4(t-x)}, \tag{13}$$

where

$$b_2 - b_1 b_4 \neq 0.$$

As we can see, there are four unknown parameters in the nonlinear fraction (13) which are about to attain with known data. Although we have five known values from the past steps, we have selected this four-parameter nonlinear fraction intentionally. First of all, most of the authors have tried to use all of the past data in order to approximate the new first derivative of the function in a new step, and this increases the computational complexity of the obtained method and in some cases, it just gets bigger the CPU run time of such obtained methods.

Accordingly, our methods are going to have less computational complexity and their operational index are about to be better than the other very high order methods. Second, the obtained approximation can be applied on any optimal eighth-order derivative-involved method to produce a new fourteenth-order method with better efficiency index.

The unknown parameters b_1, b_2, b_3 and b_4 will be determined from the conditions

$$m(x_n) = f(x_n), m'(x_n) = f'(x_n), m(z_n) = f(z_n),$$

and

$$m(w_n) = f(w_n).$$

That is, the known data in the first, third and fourth steps are used. Hence, by solving a system of linear equations, we obtain the four unknowns $(b_1, b_2, b_3$ and $b_4)$ in the following way

$$\begin{cases} b_1 = f(x_n), \\ b_3 = \frac{f'(x_n) f[z_n,w_n] - f[x_n,z_n] f[x_n,w_n]}{x_n f[z_n,w_n] + \frac{z_n f(w_n) - w_n f(z_n)}{z_n - w_n} - f(x_n)}, \\ b_4 = \frac{b_3}{f[x_n,z_n]} + \frac{f'(x_n) - f[x_n,z_n]}{(z_n - x_n) f[x_n,z_n]}, \\ b_2 = f'(x_n) + b_4 b_1, \end{cases} \tag{14}$$

and we attain

$$m'(w_n) \approx f'(w_n).$$

Thus, we generally obtain the following efficient four-step without memory iteration

$$\begin{cases} w_n = \text{optimal 8th} - \text{order derivative} - \text{involved method}, \\ x_{n+1} = w_n - \frac{f(w_n)}{m'(w_n)}. \end{cases}$$

Accordingly, by considering (7) we have

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \frac{f(x_n)}{f'(x_n)} \frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)}, \\ w_n = z_n - \frac{f(z_n)}{f'(x_n)} [\frac{1}{2} + \frac{5f(x_n)^2 + 8f(x_n)f(y_n) + 2f(y_n)^2}{5f(x_n)^2 - 12f(x_n)f(y_n)} \\ \times (\frac{1}{2} + \frac{f(z_n)}{f(y_n)})], \\ x_{n+1} = w_n - \frac{(1 + b_4(w_n - x_n))^2}{f'(x_n) + b_3(w_n - x_n)(2 + b_4(w_n - x_n))} f(w_n). \end{cases} \tag{15}$$

The mathematical proof of this scheme is provided in Theorem 1.

Now it should be mentioned that this simple but efficient approximation of $f'(w_n)$ can be implemented on any optimal eighth-order method in which we have three evaluations of the function and one evaluation of the first derivative to obtain a novel and accurate fourteenth-order scheme.

As an another instance, we provide another scheme of our class by writing the scheme (5) in the first three steps and our approximation for the first derivative of the function in the fourth step. Therefore, we attain

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n) + t f(y_n)}{f(x_n) + (t-2) f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ w_n = z_n - \frac{f(z_n)}{\frac{f'(x_n) + a_3(z_n - x_n)(2 + a_4(z_n - x_n))}{(1 + a_4(z_n - x_n))^2}}, \\ x_{n+1} = w_n - \frac{(1 + b_4(w_n - x_n))^2}{f'(x_n) + b_3(w_n - x_n)(2 + b_4(w_n - x_n))} f(w_n). \end{cases} \tag{16}$$

wherein its error equation is provided in Theorem 2.

Theorem 1. *If an initial guess x_0 is sufficiently close to the simple root α of the function f, then the convergence order of the four-step scheme (15) is equal to fourteen.*

Proof. We demonstrate the order of (15) by providing its Taylor expansion in the last step. Hence, we start by writing the Taylor expansion of $f(x_n)$ and $f'(x_n)$ about the simple root. Let us consider

$$e_n = x_n - \alpha,$$

and

$$c_j = \frac{1}{j!} \frac{f^{(j)}(\alpha)}{f'(\alpha)}, j > 2,$$

thus we have

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + \cdots + O(e_n^{15})]. \tag{17}$$

Furthermore, for the first derivative, we get

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + 7c_7 e_n^6 + 8c_8 e_n^7 + \cdots + O(e_n^{14})]. \tag{18}$$

Dividing (17) by (18), gives us

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + (2c_2^2 - 2c_3) e_n^3 + (7c_2 c_3 - 4c_2^3 - 3c_4) e_n^4 + 2(4c_2^4 - 10c_2^2 c_3 + 3c_3^2 + 5c_2 c_4 - 2c_5) e_n^5 + (-16c_2^5 + 52c_2^3 c_3 - 33c_2 c_3^2 - 28c_2^2 c_4 + 17c_3 c_4 + 13c_2 c_5 - 5c_6) e_n^6 - s_7 e_n^7 - s_8 e_n^8 + \cdots + O(e_n^{15}), \tag{19}$$

where
$s_7 = -2(16c_2^6 - 64c_2^4 c_3 + 63c_2^2 c_3^2 - 9c_3^3 + 36c_2^3 c_4 - 46c_2 c_3 c_4 + 6c_4^2 - 18c_2^2 c_5 + 11c_3 c_5 + 8c_2 c_6 - 3c_7)$,

and
$s_8 = 64c_2^7 - 304c_2^5 c_3 + 408c_2^3 c_3^2 - 135c_2 c_3^3 + 176c_2^4 c_4 - 348c_2^2 c_3 c_4 + 75c_3^2 c_4 + 64c_2 c_4^2 - 92c_2^3 c_5 + 118c_2 c_3 c_5 - 31c_4 c_5 + 44c_2^2 c_6 - 27c_3 c_6 - 19c_2 c_7 + 7c_8$.

Accordingly, for the first step we have

$$x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2 e_n^2 + (-2c_2^2 + 2c_3) e_n^3 + (4c_2^3 - 7c_2 c_3 + 3c_4) e_n^4 + (-8c_2^4 + 20c_2^2 c_3 - 6c_3^2 - 10c_2 c_4 + 4c_5) e_n^5 + \ldots + O(e_n^{15}).$$

By providing the Taylor expansion of $f(y_n)$, we can write the Taylor expansion of (15) at the end of the second step as comes next

$$x_n - \frac{f(x_n)}{f'(x_n)} \frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)} = \alpha + (c_2^3 - c_2 c_3) e_n^4 - 2(2c_2^4 - 4c_2^2 c_3 + c_3^2 + c_2 c_4) e_n^5 + (10c_2^5 - 30c_2^3 c_3 + 18c_2 c_3^2 + 12c_2^2 c_4 - 7c_3 c_4 - 3c_2 c_5) e_n^6 - 2(10c_2^6 - 40c_2^4 c_3 - 6c_3^3 + 20c_2^3 c_4 + 3c_4^2 + 8c_2^2(5c_3^2 - c_5) + 5c_3 c_5 + 2c_2(-13c_3 c_4 + c_6)) e_n^7 + \ldots + O(e_n^{15}).$$

In the same vein, for the third step we have

$z_n - \frac{f(z_n)}{f'(x_n)}[\frac{1}{2} + \frac{5f(x_n)^2+8f(x_n)f(y_n)+2f(y_n)^2}{5f(x_n)^2-12f(x_n)f(y_n)}(\frac{1}{2} + \frac{f(z_n)}{f(y_n)})] = \alpha + \frac{1}{5}c_2(c_2^2-c_3)(11c_2^4-25c_2^2c_3+5c_3^2+5c_2c_4)e_n^8 + (-\frac{268c_2^8}{25} + \frac{1478c_2^6c_3}{25} - 2c_3^4 - \frac{102c_2^5c_4}{5} + 36c_2^3c_3c_4 - 10c_2c_3^2c_4 + 2c_2^4(-41c_3^2 + c_5) + c_2^2(32c_3^3 - 2c_4^2 - 2c_3c_5))e_n^9 + (\frac{4c_2^9}{125} - \frac{19614c_2^7c_3}{125} + \frac{2941c_2^6c_4}{25} - 19c_3^3c_4 + \frac{1}{25}c_2^5(10596c_3^2 - 815c_5) + c_2c_3(52c_3^3 - 26c_4^2 - 17c_3c_5) + c_2^3(-308c_3^3 + 43c_4^2 + 59c_3c_5) + 3c_2^4(-115c_3c_4 + c_6) + c_2^2(216c_3^2c_4 - 7c_4c_5 - 3c_3c_6))e_n^{10} + \ldots + O(e_n^{15})$.

In the fourth step, we use a new evaluation of the function which has the following error equation

$f(w_n) = \frac{1}{5}c_2(c_2^2-c_3)(11c_2^4-25c_2^2c_3+5c_3^2+5c_2c_4)f'(\alpha)e_n^8 - \frac{2}{25}((134c_2^8 - 739c_2^6c_3 + 25c_3^4 + 255c_2^5c_4 - 450c_2^3c_3c_4 + 125c_2c_3^2c_4 + 25c_2^4(41c_3^2 - c_5) + 25c_2^2(-16c_3^3 + c_4^2 + c_3c_5))f'(\alpha))e_n^9 + \ldots + O(e_n^{15})$.

Additionally, we have

$$\frac{(1+b_4(w_n-x_n))^2 f(w_n)}{b_2 - b_1b_4 + b_3(w_n-x_n)(2+b_4(w_n-x_n))} =$$
$$1/5c_2(c_2^2-c_3)(11c_2^4-25c_2^2c_3+5c_3^2+5c_2c_4)e_n^8$$
$$+(-((268c_2^8)/25) + (1478c_2^6c_3)/25 - 2c_3^4 - (102c_2^5c_4)/5$$
$$+36c_2^3c_3c_4 - 10c_2c_3^2c_4 + 2c_2^4(-41c_3^2 + c_5) + c_2^2(32c_3^3$$
$$-2c_4^2 - 2c_3c_5))e_n^9 + ((4c_2^9)/125 - (19614c_2^7c_3)/125$$
$$+((2941c_2^6c_4)/25) - 19c_3^3c_4 + 1/25c_2^5(10596c_3^2 - 815c_5)$$
$$+c_2c_3(52c_3^3 - 26c_4^2 - 17c_3c_5) + c_2^3(-308c_3^3 + 43c_4^2$$
$$+59c_3c_5) + 3c_2^4(-115c_3c_4 + c_6) + c_2^2(216c_3^2c_4$$
$$-7c_4c_5 - 3c_3c_6))e_n^{10} + ((133688c_2^{10})/625 - (237968c_2^8c_3)/625$$
$$-(31168c_2^7c_4)/125 + (4/125)c_2^6(-18019c_3^2 + 5255c_5)$$
$$+2c_3^2(12c_3^3 - 31c_4^2 - 15c_3c_5) + 56/25c_2^5(651c_3c_4$$
$$-20c_6) + 2c_2^3(-848c_3^2c_4 + 67c_4c_5 + 41c_3c_6)+$$
$$4c_2(102c_3^3c_4 - 5c_4^3 - 21c_3c_4c_5 - 6c_3^2c_6)+$$
$$c_2^4((6256c_3^3)/5 - 326c_4^2 - 510c_3c_5 + 4c_7) - 2c_2^2(244c_3^4$$
$$-219c_3c_4^2 - 165c_3^2c_5 + 3c_5^2 + 5c_4c_6 + 2c_3c_7))e_n^{11}$$
$$+(-((4040914c_2^{11})/3125) + (16579584c_2^9c_3)/3125$$
$$-((432011c_2^8c_4)/625) - 1/625c_2^7(3748353c_3^2 + 193735c_5)$$
$$+(2/125)c_2^6(-90523c_3c_4 + 13730c_6) + c_3(212c_3^3c_4 - 85c_4^3$$
$$-191c_3c_4c_5 - 41c_3^2c_6) + c_2^5((70544c_3^3)/125 + (29063c_4^2)/25$$
$$+((9834c_3c_5)/5) - 57c_7) + c_2^3((7786c_3^4)/5 - 2906c_3c_4^2$$
$$-2386c_3^2c_5 + 103c_5^2 + 182c_4c_6 + 105c_3c_7)-$$
$$c_2(314c_3^5 - 596c_3^3c_5 + 94c_4^2c_5 + c_3(67c_5^2 + 116c_4c_6)$$
$$+c_3^2(-1122c_4^2 + 31c_7)) + c_2^4(5814c_3^2c_4 - 934c_4c_5$$
$$-676c_3c_6 + 5c_8) + c_2^2(-3262c_3^3c_4 + 277c_4^3 1299c_3c_4c_5$$
$$+444c_3^2c_6 - 17c_5c_6 - 13c_4c_7 - 5c_3c_8))e_n^{12} + \ldots + O(e_n^{15}).$$

Finally, for the last step by considering the above relations, we obtain

$$e_{n+1} = w_n - \frac{(1+b_4(w_n-x_n))^2 f(w_n)}{b_2 - b_1b_4 + b_3(w_n-x_n)(2+b_4(w_n-x_n))}$$
$$-\alpha = \frac{1}{5}c_2(c_2^2-c_3)^2(-c_3^2+c_2c_4)$$
$$(11c_2^4 - 25c_2^2c_3 + 5c_3^2 + 5c_2c_4)e_n^{14} + O(e_n^{15}). \qquad (20)$$

This completes the proof. □

Theorem 2. *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ has a simple root α in D and x_0 is sufficiently close to α, then the sequence $\{x_n\}$ generated by method (16) converges to α with the convergence order fourteen. And its error equation is as follows*

$$e_{n+1} = x_{n+1} - \alpha =$$
$$c_2c_3^2(c_3^2 - c_2c_4)(c_3(c_2^2+c_3) - c_2c_4)e_n^{14} + O(e_n^{15}). \qquad (21)$$

Proof. The proof of this Theorem is completely similar to the proof of Theorem 1, but however, we give its MATHEMATICA Code for obtaining its order of convergence as comes next.

```
(*e=x-a;*)fx[e_]=dfa*(e+c2*e^2+c3*e^3
+c4*e^4+c5*e^5+c6*e^6+c7*e^7+c8*e^8
+c9*e^9+c10*e^10+c11*e^11+c12*e^12
+c13*e^13+c14*e^14);dfx[e_]=D[fx[e],e];
(*u=y-a;*)u=e-Series[fx[e]/dfx[e],{e,0,14}];
fy[u_]=dfa*(u+c2*u^2+c3*u^3+c4*u^4
+c5*u^5+c6*u^6+c7*u^7+c8*u^8);(*v=z-a;*)
v=u-((2*fx[e]-fy[u])/(2*fx[e]-5*fy[u]))
*(fy[u]/dfx[e]);fz[v_]=dfa*(v+c2*v^2
+c3*v^3+c4*v^4+c5*v^5+c6*v^6+c7*v^7
+c8*v^8);a1=fx[e];a3=(dfx[e]*((fy[u]
-fz[v])/(u-v))-((fx[e]-fy[u])/(e-u))
*((fx[e]-fz[v])/(e-v)))/(e*((fy[u]
-fz[v])/(u-v))+((u*fz[v]-v*fy[u])/(u-v))
-fx[e]);a4=(a3/((fx[e]-fy[u])/(e-u)))
+((dfx[e]-(fx[e]-fy[u])/(e-u))/((u-e)
*((fx[e]-fy[u])/(e-u))));a2=dfx[e]+a4*fx[e];
m=(a2-a1*a4+a3*(v-e)*(2+a4*(v-e)))/((1
+a4*(v-e))^2);(*g=w-a;*)g=v-fz[v]/m;fw[g_]
=dfa*(g+c2*g^2+c3*g^3+c4*g^4);b1=fx[e];
b3=(dfx[e]*((fz[v]-fw[g])/(v-g))-((fx[e]
-fz[v])/(e-v))*((fx[e]-fw[g])/(e-g)))/(e
*((fz[v]-fw[g])/(v-g))+((v*fw[g]-g*fz[v])
/(v-g))-fx[e]);b4=(b3/((fx[e]-fz[v])/(e-v)))
+((dfx[e]-(fx[e]-fz[v])/(e-v))/((v-e)
*((fx[e]-fz[v])/(e-v))));b2=dfx[e]+b4*fx[e];
n=(b2-b1*b4+b3*(g-e)*(2+b4*(g-e)))/((1
+b4*(g-e))^2);e[n+1]=g-fw[g]/n//FullSimplify
```

Thus, we have

$$e_{n+1} = c_2c_3^2(c_3^2 - c_2c_4)(c_3(c_2^2+c_3) - c_2c_4)e_n^{14} + O(e_n^{15}).$$

This ends the proof. □

Remark 1. *The proposed approximation can be applied on any optimal eighth-order derivative-involved method in which we have three evaluations of the function and one of its first derivative to produce fourteenth-order methods with five evaluations per iteration. As a consequence, any of the other methods, such as (3), (4) and (8) can be considered in* *this class as well.*

Hence, by considering (3) we have the follow-up four-point efficient method

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ w_n = z_n - [1 + \frac{f(z_n)}{f(x_n)} + (\frac{f(z_n)}{f(x_n)})^2] \frac{f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]}, \\ x_{n+1} = w_n - \frac{(1+b_4(w_n-x_n))^2}{f'(x_n)+b_3(w_n-x_n)(2+b_4(w_n-x_n))} f(w_n), \end{cases} \tag{22}$$

and subsequently by taking into consideration (4), we attain the following four-step iterative family with fourteenth-order of convergence where $t \in \mathbb{R}$

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)+tf(y_n)}{f(x_n)+(t-2)f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ w_n = z_n - \\ \frac{f(z_n)}{2(f[x_n,z_n]-f[x_n,y_n])+f[y_n,z_n]+(\frac{y_n-z_n}{y_n-x_n})(f[x_n,y_n]-f'(x_n))}, \\ x_{n+1} = w_n - \frac{(1+b_4(w_n-x_n))^2}{f'(x_n)+b_3(w_n-x_n)(2+b_4(w_n-x_n))} f(w_n). \end{cases} \tag{23}$$

As an another result by considering (8) we obtain another new scheme as comes next

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)+tf(y_n)}{f(x_n)+(t-2)f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ w_n = y_n + c[f(x_n)]^2 - d[f(x_n)]^3, \\ x_{n+1} = w_n - \frac{(1+b_4(w_n-x_n))^2}{f'(x_n)+b_3(w_n-x_n)(2+b_4(w_n-x_n))} f(w_n), \end{cases} \tag{24}$$

where c and d are available by (9).

Theorem 3. *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ has a simple root α in D and x_0 is sufficiently close to α, then the sequences $\{x_n\}$ generated by methods (22), (23) and (24) converge to α with the convergence order fourteen.*

Proof. The proof of this Theorem is completely similar to the proof of Theorems 1 and 2, hence it is omitted.

In case of optimality with five evaluations per iteration, we should remark that our proposed schemes with fourteenth-order of convergence do not reach the optimality and have lower convergence rate in contrast by the four-step methods given in [7] and [10], but we build our methods in this way intentionally. As discussed at the beginning of Section II, we did not consider the value of the function in the second step just to reduce the computational effort, i.e. the computational complexity of the presented schemes are really better than the computational complexities of the four-step methods in [7, 10]. If one adds the value of $f(y_n)$ in the calculations then, a method of order sixteen will be obtained which its computational burden is really high.

Remark 2. *The efficiency index of our class is $14^{1/5} \approx 1.6952$, which is bigger than $4^{1/3} \approx 1.5784$ of the optimal fourth-order methods and $8^{1/4} \approx 1.6817$ of the optimal eighth-order methods.*

As we mentioned above, the obtained methods from our class of four-point four-step derivative-involved methods are not optimal according to the Kung-Traub conjecture on the optimality of multi-point iterations without memory. To make such methods optimal without using the value of $f(y_n)$, we must take advantage of weight function approach, i.e. at the end of the fourth step, a weight function should be constructed in order to boost up the order of convergence from 14 to 16. This can be done as future works in this field of study.

III. Numerical Experiments

Experiment is the step in the scientific method that arbitrates between competing models or hypotheses. Experimentation is also used to test existing or new theories in order to support them or disprove them. An experiment or test can be carried out using the scientific method to answer a question or investigate a problem. First an observation is made. Then a question is asked, or a problem arises. Then experiment is used to test that theory. The results are analyzed, a conclusion is drawn, sometimes a theory is formed, and results are communicated through research papers.

An experiment may also test a question or test previous results. It is important that one knows all factors in an experiment. It is also important that the results are as accurate as possible. If an experiment is carefully conducted, the results usually either support or disprove the theory. An experiment can never ”prove” a hypothesis, it can only add support. Thus in this paper, we add the numerical experiments to compare different methods and also support the underlying theory developed in this contribution

Among all the efficient fourteenth-order methods (15), (16), (22), (23) and (24) and due to similarity, we employ only the scheme (15) to solve some nonlinear scalar equations and compare the results with the the fourth-order method of Cordero et al. (1), the seventh-order method of Cordero et al. (2), the eighth-order method of Sharma and Sharma (3), eighth-order method of Petkovic (4) with $t = 3$, eighth-order method Petkovic et al. (5) with $t = -0.5$, eighth-order method of Wang and Liu (7), eighth-order method of Neta and Petkovic (8) with $t = 0$ and the fourteenth-order method of Neta (10) with $A = 0$. All numerical tests were performed on a personal computer with Intel (R) Pentium 4 while the operating system was Windows XP (Vista). We have used the following stopping criterion for computer programs

$$|f(x_n)| < 1.E - 1850.$$

Numerical results are in harmony with the theory developed in this article. Note that if one cannot guess an initial approximation of the root, then the Yun's non-iterative method can be implemented to find a proper guess in the neighborhood. [25] suggested a non-iterative procedure for finding a proper initial guess for starting the iteration process by solving an integral numerically. We should note that again, if one chooses another optimal eighth-order method which is available in literature in the first three steps, then another novel fourteenth-order method will be obtained. The test functions and their simple roots are displayed in the next column of this page.

The computational order of convergence (namely, COC) which can be defined by

$$COC \approx \frac{\ln(\mid x_{n+1} - \alpha \mid / \mid x_n - \alpha \mid)}{\ln(\mid x_n - \alpha \mid / \mid x_{n-1} - \alpha \mid)}, \tag{25}$$

is very close to 14 (to at least the fifth decimal place) for (15).

Note that iteration is the repetition of a particular process like a generalized rule that we adopt in the first step and later implement it to the succeeding steps. The number of iterations used in obtaining the result of a particular problem is an important factor that decides the length of the solution of a problem. Hence, it is preferable to have a process that requires lesser number of iteration to reach its final solution, like (15).

Nowadays, high-order iterative methods are important because numerical applications use high precision in their computations, accordingly in this study, numerical computations have been carried out using variable precision arithmetic in MATLAB 7.6.

The results of comparisons for different methods are given in Tables I, II and III in terms of accuracy to obtain the root. In Tables $Div.$ represents that the considered iterative scheme is divergence for the initial guess and e.g. $0.2e-488$ shows that the absolute value of the test function for the corespondent method is exact (zero) up to 488 decimal places.

In general, computational accuracy strongly depends on the structures of the iterative methods, the sought zeros and the test functions as well as good initial approximations. One should be aware that no iterative method always shows best accuracy for all the test functions. However, a natural question of practical interest arises: does the construction of faster and faster multi-point methods always have a justification? Certainly not if initial approximations are not sufficiently close to the sought zeros.

In those cases it is not possible, in practice, to attain the expected convergence speed (determined in a theoretical analysis). Practical experiments showed that multi-point methods can converge very slowly at the beginning of iterative process for not *so close initial guesses*. It is often reasonable to put an effort into a localization procedure, including the determination of a good initial approximation, instead of using a very fast algorithm with poor starting guesses. It is important to review the proof of convergence for our proposed class of methods (or the compared methods in Tables I, II and III) before implementing it. Specifically, one should review the assumptions made in the proofs when the result of the iterations become divergence. For situations where the method fails to converge, it is because the assumptions made in the proofs are not met. For instance, if the first derivative is not well behaved in the neighborhood of the root, the method may overshoot, and diverge from the desired root as well as a large error in the initial estimate can contribute to non-convergence of the algorithm, which also shows the importance of the basin of attraction for each iteration.

We here remark that, it is widely known that quadratically iterative methods such as Newton's iterative scheme, double the number of correct digits in the convergence phase for the simple roots. As a matter of fact, if an iterative method converges with order p, then after each iteration the number of correct significant digits in the approximation increases by a factor of approximately p. That is why the high-order methods converge faster. Accordingly, our methods from the class of fourteenth-order increases the number of correct significant digits by a factor of approximately fourteen per full iteration.

$f_1(x) = \sqrt{x^4+8}\sin(\frac{\pi}{x^2+2}) + \frac{x^3}{x^4+1} - \sqrt{6} + \frac{8}{17}$,
$\alpha = -2$,
$x_0 = -1.6$,

$f_2(x) = e^{x^2+7x-30} - 1$,
$\alpha = 3$,
$x_0 = 3.1$,

$f_3(x) = \sin^{-1}(x^2-1) - \frac{x}{2} + 1$,
$\alpha \approx 0.594810968398369$,
$x_0 = 1.2$,

$f_4(x) = x^5 - 8x^4$,
$\alpha = 8$,
$x_0 = 9.5$,

$f_5(x) = xe^x$,
$\alpha = 0$,
$x_0 = 1$,

$f_6(x) = (x-2)(x^{10}+x+1)e^{-x-1}$,
$\alpha = 2$,
$x_0 = 2.55$,

$f_7(x) = \sqrt{x^4+8}\sin(\frac{\pi}{x^2+2}) + \frac{x^3}{x^4+1} - \sqrt{6} + \frac{8}{17}$,
$\alpha = -2$,
$x_0 = -1.8$.

Table I. Comparison of different methods after one full iteration.

Test Functions	(1)	(2)	(3)	(4)	(5)	(7)	(8)	(10)	(15)
f_1	Div.	0.7e-1	0.9e-2	Div.	0.2e-2	0.2e-1	0.3e-1	0.1e-1	0.1e-2
f_2	0.2	0.2e-1	0.4e-2	0.4e-1	0.6e-2	0.2e-2	0.7e-2	0.2e-3	0.2e-5
f_3	0.2e-1	0.3e-2	0.2e-3	0.4e-2	0.2e-2	0.1e-2	0.2e-3	0.3e-5	0.1e-5
f_4	856	7.2	28.8	113	0.3e-1	29.6	38	0.1	0.5e-1
f_5	0.2	0.2e-1	0.1e-1	0.1e-1	0.2e-2	0.1e-1	0.2e-1	0.2e-2	0.1e-3
f_6	24	0.6	0.9	0.7	0.3	4.2	4.9	2.2	0.4
f_7	0.1e-2	0.1e-4	0.2e-5	Div.	0.7e-7	0.1e-5	0.3e-5	0.2e-9	0.5e-10

Table II. Comparison of different methods after two full iterations.

Test Functions	(1)	(2)	(3)	(4)	(5)	(7)	(8)	(10)	(15)
f_1	Div.	0.1e-6	0.1e-13	Div.	0.3e-18	0.2e-9	0.4e-9	0.6e-22	0.5e-35
f_2	0.4e-3	0.7e-12	0.2e-21	0.2e-11	0.8e-20	0.8e-24	0.5e-19	0.5e-55	0.4e-85
f_3	0.1e-8	0.3e-20	0.3e-32	0.2e-21	0.1e-23	0.4e-25	0.1e-31	0.2e-82	0.2e-89
f_4	1.2	0.7e-17	0.1e-15	0.3e-9	0.2e-39	0.7e-16	0.3e-14	0.2e-61	0.1e-69
f_5	0.3e-2	0.9e-11	0.5e-14	0.7e-8	0.1e-20	0.3e-15	0.1e-12	0.2e-36	0.4e-57
f_6	2.0	0.1e-7	0.7e-3	0.1e-1	0.4e-12	0.5e-4	0.2e-3	0.6e-7	0.1e-20
f_7	0.1e-10	0.2e-32	0.3e-42	Div.	0.6e-55	0.9e-43	0.1e-40	0.2e-130	0.5e-139

Table III. Comparison of different methods after three full iterations.

Test Functions	(1)	(2)	(3)	(4)	(5)	(7)	(8)	(10)	(15)
f_1	Div.	0.3e-46	0.9e-107	Div.	0.2e-145	0.2e-73	0.3e-72	0.2e-305	0.2e-488
f_2	0.1e-13	0.9e-86	0.1e-175	0.2e-93	0.1e-162	0.2e-195	0.2e-156	0.1e-777	0.6e-1201
f_3	0.1e-36	0.3e-146	0.1e-262	0.9e-176	0.5e-194	0.1e-205	0.4e-257	0.2e-1163	0.9e-1261
f_4	0.1e-10	0.6e-143	0.3e-153	0.4e-101	0.1e-344	0.1e-156	0.4e-142	0.2e-913	0.6e-1030
f_5	0.3e-9	0.1e-76	0.6e-114	0.4e-63	0.4e-167	0.1e-124	0.7e-103	0.9e-512	0.9e-805
f_6	0.7e-2	0.2e-61	0.7e-25	0.9e-20	0.3e-107	0.1e-42	0.5e-36	0.1e-114	0.9e-308
f_7	0.1e-41	0.8e-226	0.1e-336	Div.	0.1e-439	0.3e-341	0.9e-324	0.1e-1822	0

IV. Concluding Remarks

Many mathematical applications involve the solution of a nonlinear equation $f(x) = 0$. For example, from position and velocity coordinates for several given instants, it is possible to determine orbital elements for the preliminary orbit. This theoretical trajectory, also known as Keplerian orbit, is defined taking only into account mutual gravitational attraction forces between both bodies, the Earth and the satellite. Nevertheless it should be refined with later observations from ground stations, whose geographic coordinates are previously known. Different methods have been developed for this purpose, constituting a fundamental element in navigation control, tracking and supervision of artificial satellites. Most of these methods need, in their process, to find a solution of a nonlinear function. There are many methods developed on the improvement of quadratically convergent Newton's method so as to get a superior convergence order than Newton using multi-step (multi-point methods). Multi-step iterative methods have multiple step process to follow the computation route of each step which is generally cumbersome to deal with.

Hence, here we have proposed a new class of methods with fourteenth-order convergence which obtained by taking into consideration optimal eighth-order methods. We also have approximated adequately the last derivative of the function in the fourth step involved by divided differences. The obtained methods of this class have some strong points, such as they could have been written simply and explicitly in four steps wherein no usage of the function value in the second step was used (in the introduced approximation) to reduce the computational complexity. The classical efficiency index of our contributed class is 1.6952 which makes it efficient. The numerical and theoretical results have led us to believe that the new methods from the suggested class have definite practical utility in contrast with the other existing well-known derivative-involved methods.

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