Entropy Analysis in Interacting Diffusion Systems on Complex Networks

Shaoting Tang, Xin Jiang, Zhicong Liu, Lili Ma, Zhanli Zhang, and Zhiming Zheng

***Abstract*—**Interacting diffusion systems induced by finite-capacity effect is a typical diffusion model on networks. Its complexity influenced by structure and function has hardly been studied. Aimed at filling this gap, we introduce entropy to quantify such interacting process, and exhibit its strong dependence on topology and routing capacity. The analytical expressions are derived and convinced by simulations. Also, Maximum entropy principle provides an effective measure to design an optimal diffusion process. This will play a crucial role for inference problems emerging in the field of interacting dynamics on complex networks.

***Keywords*—**Entropy, Scale-free networks, Zero range process, Routing strategy, Phase transition

I. INTRODUCTION

Entropy was first applied in thermodynamics, then ranging over statistical mechanics, information theory, dynamical system, etc [1-3]. As the uncertain quantity of a system, entropy is a key indicator representing the complexity. For the past few years, entropy was also introduced into the field of complex networks, which has been a hot topic in many fields, e.g., sociology, economics, transportation science, information technology, etc [4-12]. The main theoretical and empirical interest in the study of complex networks lies in understanding the relations between structure and function. Complex networks display complex structure generating positive entropy [3], based on such topology the complexity of dynamics can be observed but structure is not the only reason.

Previous literatures on entropy in the field of complex network mainly focused on the structure characteristics, as well as the influence of the topology and the navigation rule in a diffusion process, etc. Authors in [13] show how to define the Shannon entropy of a network ensemble and how could it relate to Gibbs and Von Neumann entropies of network ensembles. Further, several different entropy measures have been introduced in the context of complex networks [14-17] to quantify the complexity. Note that diffusion is one of the most common dynamic phenomena on complex networks, e.g. the propagation of information signals among various clients on the Internet [18], the congestion of vehicle flow on large traffic networks [19]. Actually, entropy can be used to characterize a diffusion process on a complex network, and successfully relates to the properties of function with the structure of the underlying network [20].

As a mature measure, entropy of a diffusion process is particularly suited to capture the interplay between structure and diffusion dynamics. Previous studies were performed by quantifying the complexity of network topology or an ordinary diffusion process in which the diffusing packets are non-interactive. In fact, the interaction among diffusing packets can have a dramatic effect on the properties of transportation inducing finite-capacity effect. Finite-capacity effect means the interactive influence among packets caused by the variable finite capacity of routers, which is motivated by the fact that real-world routers do not have enough capacity to store all the packets sent to them. Nevertheless, we still lack means to quantify the complexity of such complex diffusion systems. A precursory work is [21]. It is shown that the dynamics of interacting packets in a network system can be characterized by the zero range process (ZRP), and the corresponding stationary state has a factorized form. We note that interacting diffusion process can be optimized under the frame of congestion mechanism. However, according to the maximum entropy principle, it remains less explored on the features of an interacting diffusion process with maximal entropy, and how to design an optimal diffusion process under the frame of maximum entropy.

Entropy rate introduced here is similar to the measure entropy to characterize such an interacting diffusion process, and to unveil the rich interplay between network topology and dynamics under the finite-capacity effect. A finite-capacity model is established by considering that the local topology and the variable capacity, two crucial elements for routers, are naturally coupled to control diffusing packets. In addition, a finite, undirected network with many packets interacting on it produces a transportation system. We reproduce the packet delivery process by making packets perform random hops in the light of local degree structure and finite-capacity effect, which jointly determine the dynamic properties of diffusion process.

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In this context, we propose two rational definition formalisms of entropy rate based on the diffusion mechanism under the finite-capacity effect. The exact expressions of entropy rate versus γ and δ in the former formalism are deduced to reflect the rules between diffusion complexity and topology structure as well as variable capacity. Interestingly, two completely distinct results simulated by two formalisms, respectively, are observed; also, we will give reasonable discussions about this over the condensation mechanism.

II. Finite-Capacity Model

We consider an undirected graph $G=(V,E)$, where $V=\{1,2,\cdots,N\}$ is the set of nodes and $E=\{(i,j)\,|\,i,j\in V\}$ is the set of edges. The adjacent matrix of G is $A=(a_{ij})_{N\times N}$, where a_{ij} takes the value of 1 if there exists an edge between node i and j, else takes the value of 0. We denote k_i as the degree of node i, i.e., the number of links of node i, can be given by $k_i=\sum_j a_{ji}$.

The total M packets are randomly put into the network initially with the density $\rho=M/N$. We denote the mean occupation number of packets at each node as $m_1,m_2,\cdots,m_N$. The hopping rate of node i is defined by

$$q_i(m_i)=m_i^{\delta}, \tag{1}$$

which reflects the transport capacity of node i. Here we set $\delta\in[0,1]$ as the capacity parameter. Each node of the network is girded with heterogeneous capacity. For instance, the larger the value of δ is, the stronger the capacity of node is, which implies an active traffic state on the network. Actually, for a fixed δ, on the one node, transition efficiency decreases while the occupation number of packets increase, that is, the percentage of delivered packets decreases though the number of delivered number goes up.

At one time step, whether one packet on the node i can be transport or not depends on the hopping rate, if it can hop, then moves to one of its neighbors j with a transition probability

$$\omega_{ji}=\frac{a_{ji}}{k_i}. \tag{2}$$

Such a transition probability corresponds to a classical random walk, and each packet chooses one neighbor node to move randomly.

III. Stationary State of a Diffusion Process under the Finite-Capacity Effect

Under the effect of hopping rate and transition probability, M packets will evolve to a stationary state. To deduce the analytical expression of stationary state, we write single-site weight $f_i(m_i)$ for node i if there exist m_i packets in the configuration

$$f_i(m_i)=\prod_{m'=0}^{m_i}\frac{\omega_i^{\infty}}{q_i(m')}, \tag{3}$$

where ω_i^{∞} is the stationary state probability distribution of a single packet moving around the network with the transition probability ω_{ji} without routing capacity-related consideration, and is given by $\omega_i^{\infty}=k_i/\sum_m k_m$ [23].

The mean occupation number of packets of node i at the stationary state can be easily obtained by considering the ZRP dynamics. With respect to generating function

$$F_i(z)=\sum_{m=0}^{\infty}z^m f_i(m),$$

the mean occupation number of packets of node i at the stationary state is given by the following equation [21]:

$$m_i=z\frac{\partial}{\partial z}\ln F_i(z), \tag{4}$$

where z is the fugacity, satisfied by $|z|\le z$ according to the convergence condition of infinite series, and the value is determined by requiring that the average occupation number of packets over the whole network is equal to the prescribed value $M:\sum_i m_i=M$. For a given network with degree distribution $P_{\deg}(k)$, the number of nodes with degree k is $NP_{\deg}(k)$. We rewrite the condition that determines z in terms of a sum of degree (instead of over nodes): $\sum_k z\frac{\partial}{\partial z}\ln F_k(z)P_{\deg}(k)=\rho$, where we indicate that fugacity z depends on the packet density ρ, and is a positive increasing function of ρ.

IV. Entropy Rate of a Diffusion Process under the Finite-Capacity Effect on Scale-Free Networks

In ergodic theory, entropy indicates the minimum information a random event needs to confirm its consequence. People has found that entropy is an effective tool to characterize both the topology structure and dynamics on the complex networks. On the aspect of transportation, especially under the finite-capacity effect, to investigate the interaction between the topology and diffusion process comprehensively, we have set our model within interacting diffusion system, a practical model on the Internet, social networks, traffic, etc., locating on the scale-free network with power-law distribution $P_{\deg}(k)\sim k^{-\gamma}$. Two rational formalisms of entropy rate are proposed to characterize complexity from different aspects.

In some implication fields, for instance, WWW, cond-mat, traffic, etc., information guided by maximum entropy transportation benefits the communication far and widely. In the following, we show the strong effect of topology and routing capacity to the entropy rate. Further, based on maximum entropy principle, various properties of networks are extracted to design optimal transport process.

A. Entropy rate related to a formalism of pure diffusing state under the finite-capacity effect

We define the entropy rate of an interacting diffusion process as

$$h(\gamma,\delta,N) = -\sum_{i,j}\frac{m_i}{M}\omega_{ji}\ln\omega_{ji}\,, \tag{5}$$

where m_i is the mean occupation number of packets of node i at the stationary state, and ω_{ji} is the transition probability defined in (2).

For a diffusion process with interacted behavior on the scale-free networks with degree distribution $P_{\deg}(k)\sim k^{-\gamma}$, the distribution of packets at the stationary state following ZRP has been obtained [22]. A complete condensation occurs on condition that

$$\delta \le \delta_c = 1/(\gamma-1).$$

Based on this critical point δ_c, we discuss three cases as follows:

1. The case $\delta = 0$ **.** For this case, $q_i(m_i)=1$ means each node can transport only one packet at each time steps, implying a very low capability of each node. Since a singular point of z exists on the hubs, it is reasonable to separate entropy rate h into two parts, one is the entropy rate h_n on normal nodes with degree lower than the maximum degree, denoted by $S=\{i \mid i\in V, k_i<k_{\max}\}$, the other is the entropy rate h_s on hubs whose degree is $k_{\max}$. h_n and h_s form a whole entropy rate as $h=h_n+h_s$. By using ZRP, the mean occupation number of normal nodes $m_{i\in S}=k_i/(k_{\max}-k_i)$, while $m_{hub}=\rho N$ on the hubs.

According to (5), h_n can be given by

$$h_n = \frac{1}{\rho N}\sum_{i\in S}\frac{k_i}{k_{\max}-k_i}\ln k_i\,. \tag{6}$$

The sum calculated over the nodes are replaced by that over the degree, then

$$h_n = \frac{1}{\rho}\sum_{k<k_{\max}}\frac{k}{k_{\max}-k}\ln k\cdot P_{\deg}(k)\,.$$

We get in the continuum-degree approximation,

$$\begin{aligned} h_n &= \frac{1}{\rho}\int_1^{k_{\max}}\frac{k^{1-\gamma}\ln k}{k_{\max}-k}dk \\ &= \frac{1}{\rho}\sum_{l=1}^{\infty}\overline{k^l\ln k}k_{\max}^{-l}\,, \end{aligned} \tag{7}$$

where

$$\overline{k^l\ln k} = \int_1^{k_{\max}} k^l\ln k\cdot P_{\deg}(k)\mathrm{dk}.$$

The entropy rate on the normal nodes leads to

$$\begin{aligned} h_n &= \frac{1}{\rho}k_{\max}^{1-\gamma}\ln k_{\max}\sum_{l=1}^{\infty}\frac{1}{l} - \frac{1}{\rho}k_{\max}^{1-\gamma}\sum_{l=1}^{\infty}\frac{1}{l^2} \\ &\quad + \frac{1}{\rho}\sum_{l=1}^{\infty}\frac{1}{l^2}k_{\max}^{-l}\,, \end{aligned} \tag{8}$$

for large l. Here, for a scale-free network, $k_{\max}$ scales as $\sim N^{1/(\gamma-1)}$, which can be easily obtained by $N\int_{k_M}^{\infty}P_{\deg}(k)dk=1$. On the other hand, within three summation of infinite series, we approximate $\sum_{l=1}^{\infty}\frac{1}{l}\sim\ln n$ with $n\to\infty$, $\sum_{l=1}^{\infty}\frac{1}{l^2}=\pi^2/6$ by using flourier series expansion, and $\sum_{l=1}^{\infty}\left|\frac{1}{l^2}k_{\max}^{-l}\right|$ converges by Weierstrass test even can be approximated by termwise differentiation as $(1-N^{1/(1-\gamma)})\ln(1-N^{1/(1-\gamma)})+N^{1/(1-\gamma)}$.

Hence, entropy rate of the normal nodes as a function of γ writes

$$\begin{aligned} h_n(\gamma,N) &= \frac{1}{\rho(\gamma-1)}N^{-1}(\ln N)(\ln n) + \frac{1}{\rho}N^{-\frac{1}{\gamma-1}} \\ &\quad - \frac{\pi^2}{6\rho}N^{-1} + \frac{1}{\rho}(1-N^{-\frac{1}{\gamma-1}})\left|\ln(1-N^{-\frac{1}{\gamma-1}})\right|, \end{aligned} \tag{9}$$

with $n\to\infty$.

On the other hand, h_s can be calculated by (5) and changes by γ as

$$\begin{aligned} h_s(\gamma,N) &= Nk_{\max}^{-\gamma}\ln k_{\max} \\ &= \frac{1}{\gamma-1}N^{-\frac{1}{\gamma-1}}\ln N. \end{aligned} \tag{10}$$

Then the analytical expression of the entropy rate of the interacting diffusion process with $\delta=0$, as a function of γ, combining h_n in (9) and h_s in (10), forms a whole as

$$
\begin{aligned}
h(\gamma,N) &= h_n(\gamma,N)+h_s(\gamma,N) \\
&= \frac{1}{\rho(\gamma-1)}N^{-1}(\ln N)(\ln n)+\frac{1}{\rho}N^{-\frac{1}{\gamma-1}} \\
&\quad -\frac{\pi^2}{6\rho}N^{-1}+\frac{1}{\rho}(1-N^{-\frac{1}{\gamma-1}})\left|\ln(1-N^{-\frac{1}{\gamma-1}})\right| \\
&\quad +\frac{1}{\gamma-1}N^{-\frac{1}{\gamma-1}}\ln N ,
\end{aligned}
\quad (11)
$$

with $n\to\infty$. Further, divergent $\ln n$ in (9) implies that entropy rate diverges for arbitrary γ in the case $\delta=0$.

2. The case $\delta=1$. This case corresponds to systems with noninteracting feature that M packets play random walk independently. Thus the mean occupation number of packets at the stationary distribution simplifies to $m_i = M\omega_i^{\infty} = \rho k_i/\bar{k}$, here $\bar{k}$ is the average degree of the scale-free network. Then entropy rate reads

$$h=\frac{1}{\bar{k}N}\sum_i k_i \ln k_i .$$

Replace the sum over the node with the sum over the degree, and substitute the summation for integration,

$$
\begin{aligned}
h(\gamma,N) &= \frac{1}{\rho\bar{k}(\gamma-1)(2-\gamma)}N^{(2-\gamma)/(\gamma-1)}\ln N \\
&\quad -\frac{1}{\rho(2-\gamma)^2}N^{(2-\gamma)/(\gamma-1)}+\frac{1}{\rho(2-\gamma)^2}.
\end{aligned}
\quad (13)
$$

Note that entropy rate diverges as $O(N^{(2-\gamma)/(\gamma-1)})$ for $1<\gamma<2$, with $N\to\infty$. Conversely, when $\gamma>2$, the entropy rate is finite in the limit $N\to\infty$ and equal to

$$h=\frac{1}{\rho(2-\gamma)^2}. \quad (14)$$

3. The general case: $\delta>0$. For the general case δ, there exists a critical point

$$\delta_c = 1/(\gamma-1)$$

where the distribution of packets at the stationary state undergoes a transition from a relative homogeneous state to an ultra inhomogeneous state. Further, for a given δ, the whole nodes should be divided into two groups according to a crossover degree k_c, which takes the value $k_c = k_{\max}^{1-\delta/\delta_c}$ for $\delta<\delta_c$ and $k_c=[\ln k_{\max}]$ for $\delta=\delta_c$, then the mean occupation number of packets at those nodes locating in the same group get the same expression [22]. Due to the critical point δ_c divides the phase diagram into two distinct regions, in the following part, we discuss three cases according to different values of δ.

When $\delta<\delta_c$, we obtain two groups of nodes according to $k_c = k_{\max}^{1-\delta/\delta_c}$ denoted by sets

$$B=\{i\,|\,i\in V, k_i<k_c\},$$
$$C=\{i\,|\,i\in V, k_i>k_c\},$$
$$m_{i\in B}=k_i/k_{\max}^{1-\delta/\delta_c},$$
$$m_{i\in C}=k_i^{1/\delta}/k_{\max}^{1/\delta-1/\delta_c}.$$

We deduce the expression of entropy rate in these two groups, respectively,

$$h=\frac{1}{Mk_{\max}^{1-\delta/\delta_c}}\sum_{i\in B}k_i\ln k_i+\frac{1}{Mk_{\max}^{1/\delta-1/\delta_c}}\sum_{i\in C}k_i^{1/\delta}\ln k_i. \quad (15)$$

Replace the sum over the nodes with the sum over the degree, and get in the continuum-degree approximation,

$$
\begin{aligned}
h &= \frac{1}{\rho k_{\max}^{1-\delta/\delta_c}}\int_1^{k_c}k^{1-\gamma}\ln k\,dk \\
&\quad +\frac{1}{\rho k_{\max}^{1/\delta-1/\delta_c}}\int_{k_c}^{k_{\max}}k^{1/\delta-\gamma}\ln k\,dk.
\end{aligned}
\quad (16)
$$

Write the analytical expression of entropy rate with $\delta<\delta_c$,

$$
\begin{aligned}
h(\gamma,\delta,N) &= \frac{\delta-1}{\rho(\gamma-1)(\gamma-2)}N^{\delta(\gamma-1)-1}\ln N \\
&\quad +\left(\frac{1}{\rho(1-\gamma+1/\delta)^2}-\frac{1}{\rho(\gamma-2)^2}\right)N^{\delta(\gamma-1)-1} \\
&\quad +\frac{1}{\rho(\gamma-2)^2}N^{\delta-1/(\gamma-1)}-\frac{1}{\rho(1-\gamma+1/\delta)^2} \\
&\quad +\frac{1}{\rho(\gamma-1)(1-\gamma+1/\delta)}\ln N.
\end{aligned}
\quad (17)
$$

When $N\to\infty$, the entropy rate in scale-free network with $\gamma>1/\delta+1$ diverges as $h\sim O(N^{\delta(\gamma-1)-1}\ln N)$. On the other hand, when $\gamma\le 1/\delta+1$, the entropy rate in the limit $N\to\infty$ diverges as $O(\ln N)$.

When $\delta=\delta_c$, i.e., $\delta=1/(\gamma-1)$. $k_c=[\ln k_{\max}]$ divides nodes into two groups B^* and C^*, also,

$$B^*=\{i\,|\,i\in V, k_i<k_c\}$$
$$C^*=\{i\,|\,i\in V, k_i>k_c\},$$
$$m_{i\in B^*}=k_i/[\ln k_{\max}]^{\delta_c},$$
$$m_{i\in C^*}=k_i^{1/\delta_c}/\ln k_{\max},$$

then

$$h=\frac{1}{M}\sum_{i\in B^*}\frac{k_i}{(\ln k_{\max})^{\delta_c}}\ln k_i+\frac{1}{M}\sum_{i\in C^*}\frac{k_i^{1/\delta_c}}{\ln k_{\max}}\ln k_i. \quad (18)$$

By some calculations, we obtain

$$h = \frac{1}{\rho (\ln k_{\max})^{\delta_c}} \int_1^{k_c} k^{1-\gamma} \ln k dk + \frac{1}{\rho \ln k_{\max}} \int_{k_c}^{k_{\max}} k^{-1} \ln k dk. \tag{19}$$

We can write the analytical expression of entropy rate with $\delta = \delta_c$ as

$$\begin{aligned} h(\gamma, N) = & \frac{1}{\rho(2-\gamma)} (\ln N)^{-1} \ln\left(\frac{\ln N}{\gamma - 1}\right) \\ & + \frac{(\gamma-1)^{1/(\gamma-1)}}{\rho(2-\gamma)^2} (\ln N)^{1/(1-\gamma)} + \frac{1}{2\rho(\gamma-1)} \ln N \\ & - \frac{(\gamma-1)}{\rho(2-\gamma)^2} (\ln N)^{-1} - \frac{1}{2\rho}\left(\frac{\ln N}{\gamma-1}\right)^{(3-\gamma)/(\gamma-1)} . \end{aligned} \tag{20}$$

Let $N \to \infty$, entropy rate in the above expression diverges as $h \sim O((\ln N)^{(3-\gamma)/(\gamma-1)})$ for $1 < \gamma < 2$, while $\gamma \geq 2$ entropy rate in scale-free network diverges as $O(\ln N)$.

When $\delta > \delta_c$, the mean occupation number of packets of each node has a same expression $m_i = k_i^{1/\delta}$, then, h can be calculated by (5) as

$$h = \frac{1}{M} \sum_i k_i^{1/\delta} \ln k_i . \tag{21}$$

After some calculations,

$$h = \frac{1}{\rho} \int_1^{k_{\max}} k^{1/\delta - \gamma} \ln k dk. \tag{22}$$

One can obtain that

$$\begin{aligned} h(\gamma, \delta, N) = & \frac{1}{\rho(\gamma-1)(1-\gamma+1/\delta)} N^{1/(\delta(\gamma-1))-1} \ln N \\ & - \frac{1}{\rho(1-\gamma+1/\delta)^2} N^{1/(\delta(\gamma-1))-1} \\ & + \frac{1}{\rho(1-\gamma+1/\delta)^2}. \end{aligned} \tag{23}$$

Entropy rate in the above expression diverges as $h \sim O(N^{1/(\delta(\gamma-1))-1} \ln N)$ in the scale-free network with $\gamma \leq 1/\delta + 1$, in the limit $N \to \infty$. Conversely, when $\gamma > 1/\delta + 1$ entropy rate is a finite value in the thermodynamic limit $N \to \infty$ and equal to

$$h(\gamma, \delta) = \frac{1}{\rho(1-\gamma+1/\delta)^2}. \tag{24}$$

Our definition in (5), actually, is a weighted average over the entropy rate of each node. Specially, this entropy rate of each node is indeed with no relationship with the topology parameter γ or the finite-capacity parameter δ, just related to the degree of its node writes $\ln k_i$. Thus, hubs, those nodes with the maximum degree, have the maximum entropy rate beyond the other nodes. Based on this, the maximum entropy rate of the diffusion process requires that the macroscopic fraction of packets locate at the hubs. Our previous work in [24] showed that the mean occupation number of packets at the hubs

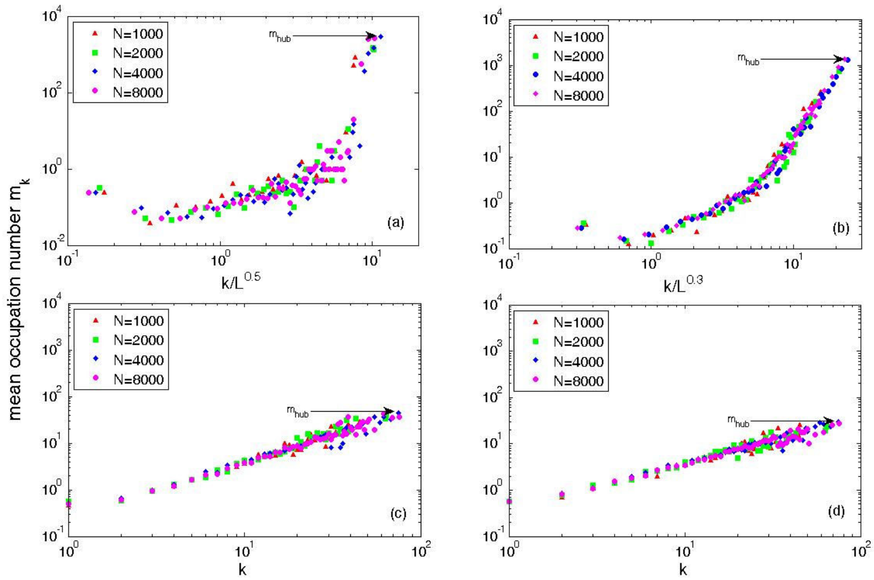

Fig. 1 Mean occupation number m_i versus degree k. The capacity parameter δ takes 0 in (a), 0.2 in (b), 0.8 in (c), 1in (d) respectively. Simulations on different scales of scale-free networks with $\gamma = 3$.

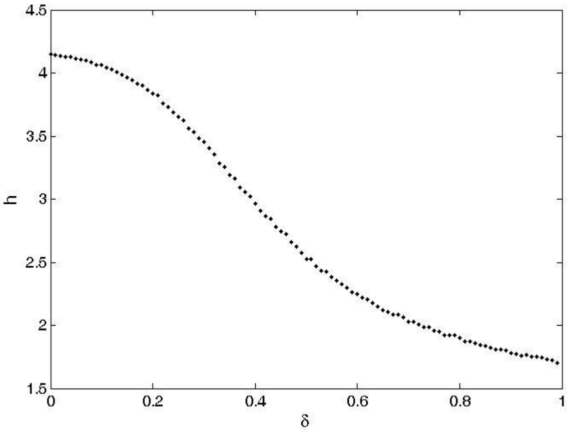

Fig. 2 Entropy rate h of the first formalism in (5) versus δ at $\gamma = 3$, $N = 10^3$ on the scale-free networks constructed by the Molloy-Reed algorithm.

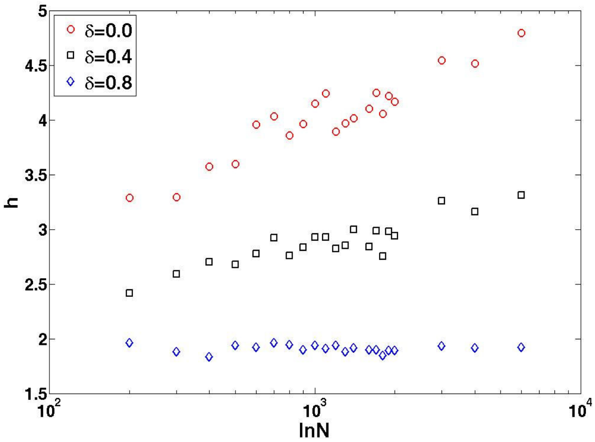

Fig. 3 Entropy rate h of the first formalism in (5) versus N on the scale-free network with exponent $\gamma = 3$ constructed by the Molloy-Reed algorithm.

increases by δ decreasing, which can be found in Fig. 1. In this figure, we focus on the top of the simulated points, the mean occupation number of packets at hub m_{hub} in each subfigure (a), (b), (c), and (d). It is shown that m_{hub} gets the highest value at $\delta = 0$, while $\delta = 1$ induces the lowest m_{hub}. Based on this, we can conclude that entropy rate is a decreasing function of δ.

To confirm the theoretical results, we have illustrated the relationship between the entropy and the routing capacity by Monte Carlo simulations on scale-free networks with power-law degree distribution $P_{\deg}(k)$, exponent $\gamma = 3$, $N = 10^3$, and $\rho = 2$. The simulated results are plotted in Fig. 2. It is shown that entropy rate h is a decreasing function of parameter δ, which confirms to the theoretical results.

Further, to confirm how entropy rate changes by the size of network, we simulated on some scale-free networks with different size but characterized by the same exponent $\gamma = 3$ and constructed by the Molloy-Reed algorithm. As shown in Fig. 3, we select three typical value of parameter δ. Specifically, according to the critical condition $\delta_c = 0.5$ where a complete condensation occurs, $\delta = 0$ corresponds to a system showing an extreme situation that the capacity of router quite low inducing condensation, meanwhile, $\delta = 0.4$ is also lower than δ_c producing condensation, but $\delta = 0.8$ implies a free traffic flow of the delivering packets. From Fig. 3, obviously, a series of simulated points with $\delta = 0$ locate on the top of the graph that coincides with the main result in Fig. 2. Intuitively, entropy rate h increases by N. When systems exhibit condensation, the increasing of N affects entropy rate observably, however, very tiny changes can be found at $\delta = 0.8$. Thus, condensation reinforces the influence of the size of networks on entropy rate.

B. Entropy rate related to a formalism of multi-state under the finite-capacity effect

We have addressed an entropy rate formalism featured that the total probability space of diffusing behavior of one packet on node i was formed by k_i possibilities contributed by its k_i neighbors. Here, we wonder if the staying state is included, that is, the total probability space at node i is formed by $k_i + 1$ possibilities, summing up to 1, the entropy rate will be the same or different from the former formalism. We define the corresponding entropy rate formalism, and it still has relationship with γ and δ, as follows:

$$h(\gamma, \delta, N) = -\sum_{i,j} \frac{m_i}{M} \pi_{ji} \ln \pi_{ji}, \tag{25}$$

where $\pi_{ji} = a_{ji} m_i^{\delta-1} / k_i$ for $i \neq j$, and $\pi_{ji} = 1 - m_i^{\delta-1}$, otherwise.

A number of simulations are used to study how the entropy rate changes by the scale-free topology and finite-capacity effect. To compare with the results in the above section, we simulate under the same conditions in Fig. 2 and Fig. 3, such as both on the scale-free network with $\gamma = 3$ constructed by Molloy-Reed algorithm, the density of packets takes $\rho = 2$, packets distribute randomly on the network and follow the same strategy to diffuse.

Similarly, to investigate how entropy rate changes by N, we pick out the same three value of δ with fig. 3. Differently, as shown in fig. 4, although a complete condensation occurs or not occurs in the interacting diffusion process, N can hardly fluctuate entropy rate apparently.

In Fig. 5, we find a completely different line of simulated results from the results in Fig. 2, although the question is the same. In the latter formalism, since the staying state is considered in the entropy rate of each node, entropy rate

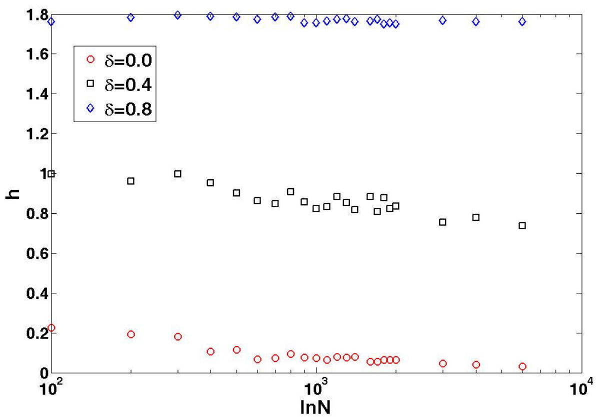

Fig. 4 Entropy rate h of the second formalism in (25) versus N on the scale-free network with exponent $\gamma = 3$ constructed by the Molloy-Reed algorithm.

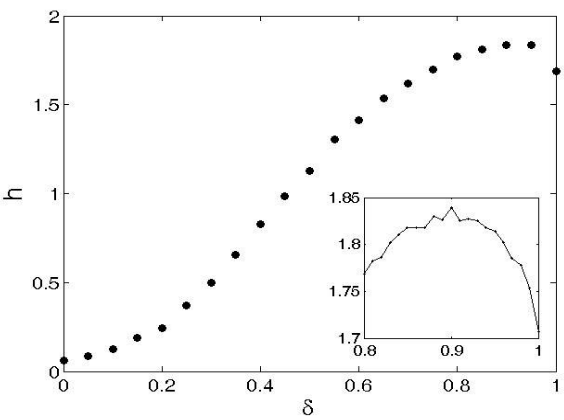

Fig. 5 Entropy rate h of the second formalism in (25) versus δ at $\gamma = 3$, $N = 10^3$ on the scale-free network constructed by the Molloy-Reed algorithm.

changes by δ increasingly for a long range and then decreases. Meanwhile, the value of entropy rate in Fig. 5 is lower than the value in Fig. 2 collectively.

Analytically, when δ is lower, packets tend to condense to hubs leading to an over-loaded situation of hubs [24], consequently, staying state becomes the "most probable" choice, this "most probable" induces quite low entropy rate. Thus, when the value of δ is larger, the effect of condensation becomes weak so that entropy rate increases until a peak value appears. Significantly, this peak is the best point that a balance between staying state and diffusing state is achieved; further, this peak implies an efficient diffusion process characterized by the maximum entropy. During the optimization of a class of interacting diffusion process to be efficient in delivering information packets via changing routing capacity, it is important to get this peak value. And then the entropy rate decreases, since the balance between staying state and diffusing state is broken. Thus, the maximum entropy implies a balance between delay and diffusion state under the finite-capacity effect.

V. Conclusion

We have proposed a class of systems with finite-capacity effects. To reveal its diffusion complexity, entropy rate was introduced and offered a universal formalism for characterizing such diffusion processes. Two crucial elements, the network topology controlled by exponent γ and the finite-capacity effect with parameter δ, decide the entropy together. We have defined two rational formalisms of entropy rate, analyzed and simulated the influence of parameters γ and δ to the entropy rate. Interestingly, there are absolutely distinct results of two formalisms of entropy rate as plotted in Fig. 2 and Fig. 5, implying that finite-capacity effect can have a powerful effect on entropy of such diffusion process. To sum up, in the range of systems with finite-capacity effect, we have proposed a universal and unified framework to analyze and describe an interacting diffusion process, and its complexity can be well characterized.

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