Algebraic construction of exact difference equations from symmetry of functions

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***Abstract*—**Difference equations or exact numerical integration scheme, which have general solutions, are treated algebraically. Eliminating the symmetry in mixed functions, we can construct numerical integration schemes correspond to some ordinary differential equations that have same mixed functions. When arbitrary functions are given, whether we can construct numerical integration schemes that have solution functions equal to given function or not are treated.

***Keywords*—**Exactly integrable numerical scheme, Algebraic treatment, Ordinary difference equation.

I. Introduction

HERE we concentrate the problem what difference equation have solution function, in other words, it is solvable (integrable) or not. This problem seems hard to clear by straight way at present. Then we treat this problem from other side. We change this problem as follows. Firstly, we assume solution function is given. Secondly, we construct difference equation that has such function as the solution function.

We often use difference equations as an approximation of differential equations. However, it is wonderful if the difference equation has same solution function to the one of some differential equation. From this point, correspondence between difference equations and differential equations was investigated [1]. As for extension of the previous study, we treat the case when given solution functions are arbitrary one, not limited to solution functions of some ordinary differential equations (ODE) at the goal.

Here we shortly review the procedure in [1] to construct difference equation that has same solution function to the one of some differential equation.

We assume given solution function as,

$$y = \frac{C_1}{x - C_2}. \tag{1}$$

Then we get differential equation (2), using (1) and derivation of (1), by eliminating integral constant C_1 and C_2 in second derivation of (1).

$$2y_x^2 - yy_{xx} = 0, \quad (y_x = \frac{dy}{dx}). \tag{2}$$

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We can also construct difference equation. Deforming (1) to linear equation respect to integral constant gives,

$$C_1 + C_2 y = xy. \tag{3}$$

Taking appropriate independent point set $\{\{x_{(1)}, y_{(1)}\}, \{x_{(2)}, y_{(2)}\}\}$ and substituting these point to (3) gives two equation. From these equations, we obtain C_1 and C_2. Then substituting them to (3), we get difference equation,

$$xy(y_{(1)} - y_{(2)}) + x_{(1)}y_{(1)}(y_{(2)} - y) + x_{(2)}y_{(2)}(y - y_{(1)}) = 0. \tag{4}$$

It is clear, (2) and (4) have same solution function (1). Here we have simple question, what functions that include integral constants are possible to give difference and differential equations. We treat this problem by concentrate on making difference equations from such given functions. Here we use the word “symmetry” or “invariant” as the same meaning to “integral constant” when it is appropriate for the problem.

II. Procedure For Evolutional Difference Equation With Initial Conditions

We prepare the classification of simultaneous equations respect to initial conditions, which are necessary to construct evolutional difference equations within scope of previous section.

A. *Linear case without derivative (Case 1)*

Let consider linear simultaneous equation which are induced from following given solution function,

$$y = C_1 f_1(x) + C_2 f_2(x) + \cdots + C_n f_n(x) \tag{5}$$

here, C_j are invariant (symmetry or integral constant) of the difference equation which we will construct, and $f_j(x)$ are appropriate function of x. As for elimination of all C_j in (5), we use following simultaneous equations,

$$\begin{cases} y_{(1)} = C_1 f_1(x_{(1)}) + C_2 f_2(x_{(1)}) + \cdots + C_n f_n(x_{(1)}) \\ y_{(2)} = C_1 f_1(x_{(2)}) + C_2 f_2(x_{(2)}) + \cdots + C_n f_n(x_{(2)}) \\ \vdots \\ y_{(n)} = C_1 f_1(x_{(n)}) + C_2 f_2(x_{(n)}) + \cdots + C_n f_n(x_{(n)}) \end{cases} \tag{6}$$

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with well posed initial conditions that are independent respectively,

$$\left\{x_{(1)}, x_{(2)}, \ldots, x_{(n)}\right\}, \left\{y_{(1)}, y_{(2)}, \ldots, y_{(n)}\right\}, \quad y_{(i)} = y(x_{(i)}) \quad (7)$$

here, lower suffix *(j)* for *x* and *y* corresponds to sampling point number and $f_j(x) \in C^0(x)$. We assume (7) are explicitly given.

B. Linear case with derivative (Case 2)

In case 2, we also assume the form of solution function as (5), but use derivatives in simultaneous equations with well posed initial conditions,

$$\begin{cases} y_{(1)} = C_1 f_1(x_{(1)}) + C_2 f_2(x_{(1)}) + \cdots + C_n f_n(x_{(1)}) \\ y_{(2)} = C_1 f_1(x_{(2)}) + C_2 f_2(x_{(2)}) + \cdots + C_n f_n(x_{(2)}) \\ \quad \vdots \\ y_{(s)} = C_1 f_1(x_{(s)}) + C_2 f_2(x_{(s)}) + \cdots + C_n f_n(x_{(s)}) \\ y_{(1)}^{(1)} = C_1 f_1^{(1)}(x_{(1)}) + C_2 f_2^{(1)}(x_{(1)}) + \cdots + C_n f_n^{(1)}(x_{(1)}) \\ y_{(1)}^{(2)} = C_1 f_1^{(2)}(x_{(1)}) + C_2 f_2^{(2)}(x_{(1)}) + \cdots + C_n f_n^{(2)}(x_{(1)}) \\ \quad \vdots \\ y_{(1)}^{(n-s)} = C_1 f_1^{(n-s)}(x_{(1)}) + C_2 f_2^{(n-s)}(x_{(1)}) + \cdots + C_n f_n^{(n-s)}(x_{(1)}) \end{cases} \quad (8)$$

$$\left\{y_{(1)}, y_{(2)}, \ldots, y_{(s)}, y_{(1)}^{(1)}, y_{(1)}^{(2)}, \ldots, y_{(1)}^{(n-s)}\right\}, \quad (9)$$

here, upper suffix *(j)* of *y* and *f* corresponds to j-th derivative $d^j y / dx^j$ and $d^j f / dx^j$. We assume $\{y, f_j(x)\} \in C^{n-s}(x)$ and (9) is given. In this case, we take $s<n$ to treat initial condition same to the one for differential equation. With ease extension, we can consider more general case defined by other initial conditions, for example $\left\{y_{(1)}, y_{(2)}, \ldots, y_{(s)}, y_{(1)}^{(1)}, y_{(2)}^{(1)}, \ldots, y_{(n-s)}^{(1)}\right\}$ and $\left\{y_{(1)}, y_{(2)}, \ldots, y_{(s)}, y_{(t)}^{(r)}, \ldots, y_{(v)}^{(u)}\right\}$, etc. We leave these cases by complicate combination with arbitrary derivatives and sampling points, because of simple description of the problem. Clearly, Case 1 is a part of Case 2.

C. Case 3

Case 3 are defined by generalization of (5),

$$y = g_1(\mathbf{C}) f_1(x) + g_2(\mathbf{C}) f_2(x) + \cdots + g_n(\mathbf{C}) f_n(x), \quad (10)$$

here $\mathbf{C} = \{C_1, \ldots C_n\}$ and g_js are polynomial functions of C_js. Formulation for simultaneous equations and initial conditions are the same to Case 1 or Case 2. We also leave more general cases, for example, generalizing $g_j(\mathbf{C}) f_j(x)$ to $G_j(\mathbf{C}, x)$, and $H(y, x, \mathbf{C}) = 0$, here G_j and H are analytic function of **C**, etc.

III. Elimination And Implicitization Of Seymmetry

We consider elimination of C_js from simultaneous equations in each case. Analytically, local existence of each function $C_j = C_j(\mathbf{y}, \mathbf{x})$ can be verified by implicit function theorem. Here we use notation, $\mathbf{y} = \{y_{(1)}, y_{(2)}, ..., y_{(n)}\}$, $\mathbf{x} = \{x_{(1)}, x_{(2)}, ..., x_{(n)}\}$ and

$$\mathbf{F} = \begin{pmatrix} f_1(x_{(1)}) & f_2(x_{(1)}) & \cdots & f_n(x_{(1)}) \\ f_1(x_{(2)}) & f_2(x_{(2)}) & \cdots & f_n(x_{(2)}) \\ \vdots & \vdots & \cdots & \vdots \\ f_1(x_{(n)}) & f_2(x_{(n)}) & \cdots & f_n(x_{(n)}) \end{pmatrix}. \quad (11)$$

If $\det(\mathbf{F}) \neq 0$, then we can solve C_j explicitly in case 1 and case 2, because (6) and (8) are linear equation by **C**. Substituting C_j to (5), we get explicit evolutional difference equation. Evolution is given by following sequential mapping,

$$\begin{cases} \{x_{(1)}, x_{(2)}, ..., x_{(n)}\} \to \{x_{(2)}, x_{(3)}, ..., x_{(n)}, x\}, x = x_{(n+1)} = x_{(n)} + \delta \\ \{y_{(1)}, y_{(2)}, ..., y_{(n)}\} \to \{y_{(2)}, y_{(3)}, ..., y_{(n)}, y\}, y = y_{(n+1)} \end{cases},$$

here we assume δ is given. These formulations are discrete correspondence to well known Wronskian treatment in differential equation. For the case 3, we use Jacobian matrix instead of (11),

$$\mathbf{F} = \frac{\partial(y_{(1)}, y_{(2)}, ..., y_{(n)})}{\partial(C_{(1)}, C_{(2)}, ..., C_{(n)})} \quad (12)$$

here $y_{(j)}=y(x_{(j)},\mathbf{C})$ by (10). If $\det(\mathbf{F}) \neq 0$, then $C_j = C_j(\mathbf{x}, \mathbf{y})$ exist implicitly and locally by implicit function theorem. However, it is not sufficient to construct difference equation from (10), we must use another approach. Then we change approach from solving C_j to eliminate C_j.

We assumed $g_j(\mathbf{C})$ are polynomial functions in (10), therefore we can use algebraic elimination theorem (or implicitization) using Gröbner basis [2-4]. Here after we treat case 3, but procedures in the following are available to all cases.

We rewrite (10) as,

$$I = g_1(\mathbf{C}) f_1(x) + g_2(\mathbf{C}) f_2(x) + \cdots + g_n(\mathbf{C}) f_n(x) - y. \quad (10)'$$

We define $\mathbf{I} = \{I_{(1)}, I_{(2)}, ..., I_{(n)}\}$ and

$$I_{(j)} = g_1(\mathbf{C}) f_1(x_{(j)}) + g_2(\mathbf{C}) f_2(x_{(j)}) + \cdots + g_n(\mathbf{C}) f_n(x_{(j)}) - y_{(j)}.$$

Then we regard **I** as generating set of ideals in $O(\mathbf{x})[\mathbf{y}, \mathbf{C}]$. $O(\mathbf{x})$ is coefficient function filed for polynomial function of **C** and **y**. We abbreviate $f_j(x_{(i)})$ as f_{ji} and disrespect **x** for a while. It means we neglect the case when $O(\mathbf{x})$ is not algebraically closed field.

We only show difference between solving C_j and eliminating C_j by Mathematica™ example. At the first, we define $\mathbf{I}=\{I_1,I_2,I_3\}$ and $\mathbf{C}=\{c_1,c_2,c_3\}$ as follows,

```
I1 := y1 - (c1 + c2) * f11 - c2^3 * f21;
I2 := y2 - (c1 + c2) * f12 - c2^3 * f22;
I3 := y3 - (c1 + c2) * f13 - c2^3 * f23;
```

Following gives elimination ideal to "temp",

```
temp = GroebnerBasis[{I1, I2, I3}, {y1, y2, y3}, {c1, c2}];
```

We can obtain y_3 explicitly without branch using "Solve",

```
Solve[temp == 0, y3]
```

$$\left\{\left\{\mathrm{y3}\to\frac{-\mathrm{f13\,f22\,y1}+\mathrm{f12\,f23\,y1}+\mathrm{f13\,f21\,y2}-\mathrm{f11\,f23\,y2}}{\mathrm{f12\,f21}-\mathrm{f11\,f22}}\right\}\right\}.$$

On the contrary, we obtain multiple root of C_j using "Solve" directly,

```
Solve[{I1 == 0, I2 == 0}, {c1, c2}]
```

$$\left\{\left\{\mathrm{c1}\to\frac{\frac{\mathrm{f12\,f22\,y1}}{-\mathrm{f12\,f21}+\mathrm{f11\,f22}}+\mathrm{y2}-\frac{\mathrm{f11\,f22\,y2}}{-\mathrm{f12\,f21}+\mathrm{f11\,f22}}-\frac{\mathrm{f12}\,(-\mathrm{f12\,y1}+\mathrm{f11\,y2})^{1/3}}{(-\mathrm{f12\,f21}+\mathrm{f11\,f22})^{1/3}}}{\mathrm{f12}},\ \mathrm{c2}\to\frac{(-\mathrm{f12\,y1}+\mathrm{f11\,y2})^{1/3}}{(-\mathrm{f12\,f21}+\mathrm{f11\,f22})^{1/3}}\right\},\right.$$

$$\left\{\mathrm{c1}\to\frac{\frac{\mathrm{f12\,f22\,y1}}{-\mathrm{f12\,f21}+\mathrm{f11\,f22}}+\mathrm{y2}-\frac{\mathrm{f11\,f22\,y2}}{-\mathrm{f12\,f21}+\mathrm{f11\,f22}}+\frac{(-1)^{1/3}\,\mathrm{f12}\,(-\mathrm{f12\,y1}+\mathrm{f11\,y2})^{1/3}}{(-\mathrm{f12\,f21}+\mathrm{f11\,f22})^{1/3}}}{\mathrm{f12}},\ \mathrm{c2}\to-\frac{(-1)^{1/3}\,(-\mathrm{f12\,y1}+\mathrm{f11\,y2})^{1/3}}{(-\mathrm{f12\,f21}+\mathrm{f11\,f22})^{1/3}}\right\},$$

$$\left.\left\{\mathrm{c1}\to\frac{\frac{\mathrm{f12\,f22\,y1}}{-\mathrm{f12\,f21}+\mathrm{f11\,f22}}+\mathrm{y2}-\frac{\mathrm{f11\,f22\,y2}}{-\mathrm{f12\,f21}+\mathrm{f11\,f22}}-\frac{(-1)^{2/3}\,\mathrm{f12}\,(-\mathrm{f12\,y1}+\mathrm{f11\,y2})^{1/3}}{(-\mathrm{f12\,f21}+\mathrm{f11\,f22})^{1/3}}}{\mathrm{f12}},\ \mathrm{c2}\to\frac{(-1)^{2/3}\,(-\mathrm{f12\,y1}+\mathrm{f11\,y2})^{1/3}}{(-\mathrm{f12\,f21}+\mathrm{f11\,f22})^{1/3}}\right\}\right\}$$

Then we must substitute each root of C_j into I_3 to construct evolutional difference equation. Clearly, we cannot construct single equation. We note that elimination ideal induces Zariski closure of $\mathbf{I}$. From this point, we can regard elimination as $\mathbf{I}\subset O(\mathbf{x})[\mathbf{C},\mathbf{y}]$ and elimination gives $\mathbf{I}\cap O(\mathbf{x})[\mathbf{y}]$.

We found that construction of difference equation from given function with symmetries are appropriately obtained by ideal elimination theorem [3]. It is clear that this result mainly depends on assumption by g_js in (10) are polynomial functions of C_js.

IV. Explicit Evolutional Difference Equation

We found difference equations are obtained by elimination theorem, though we cannot construct explicit evolutional difference equation by the theorem. We need more condition for construction, because difference equations should be evolutional form with appropriate initial conditions. This is difference between implicitization problem and this problem. As a result, we must search suitable simultaneous equations that give evolutional form from considerable combination of variables and initial conditions as in case 2.

A. Algorithm for construction

We shortly summarize procedure to make evolutional difference equation. Here we neglect other factor of construction for simplicity.

Step 1: Construct polynomial from (10)' by expanding and arranging coefficient respect to each $C_i^{\alpha}\cdots C_j^{\beta}$ term.

Step 2: Construct $\mathbf{I}\subset O(\mathbf{x})[\mathbf{C},\mathbf{y}]$ using equation from step1 with (7). Try elimination. If elimination gives single evolutional equation for some $y_{(j)}$, then stop construction. We got suitable evolutional difference equation for $y_{(j)}$ to $y_{(j+1)}$.

Step 3: We try construction of **I** that includes derivatives with initial condition (9), and varying value of s from $s=n-1$ to 1. If elimination gives single evolutional equation by s for some $y_{(j)}$, then stop construction.

We can proceed to construction of evolutional equation when above all step are in fail. Variable transform is one direction, and modification of original function is other direction. For example, in rare occasions, if we consider evolution of $y_{(j)}^{\alpha}\to y_{(j+1)}^{\alpha}$ instead of $y_{(j)}\to y_{(j+1)}$, we can obtain evolutional equation by hiding multiplicity. For other direction, we must back step1. We change subsection for detail treatment by example.

B. Some condition for constructability of unique evolutional equation

Let consider following simultaneous equation with assumption that appropriate initial conditions are given. This example is more general case than (10)'.

$$\begin{cases} I_{(1)}=y_{(1)}-3(C_1+C_2)\,f_{11(1)}+C_1^2C_2\,f_{12(1)}+C_3\,f_{13(1)} \\ I_{(2)}=C_1^2C_2y_{(2)}-(C_1+C_2)\,f_{21(2)}+C_1^2C_2\,f_{22(2)}+C_3\,f_{23(2)} \\ I_{(3)}=y_{(3)}-(C_1+C_2)\,f_{31(3)}+(C_1+C_2)\,f_{32(3)}+C_3\,f_{33(3)} \\ I_{(4)}=y_{(4)}-C_3\,f_{41(4)}+C_1^2C_2\,f_{42(4)}+(C_1+C_2)\,f_{43(4)} \end{cases} \tag{13}$$

Following step 1 in previous subsection, we obtain

$$\begin{cases} I_{(1)}=y_{(1)}-3C_1\,f_{11(1)}-3C_2\,f_{11(1)}+C_3\,f_{13(1)}+C_1^2C_2\,f_{12(1)} \\ I_{(2)}=-C_1\,f_{21(2)}-C_2\,f_{21(2)}+C_3\,f_{23(2)}+C_1^2C_2(y_{(2)}+f_{22(2)}) \\ I_{(3)}=y_{(3)}+C_1(f_{32(3)}-f_{31(3)})+C_2(f_{32(3)}+f_{31(3)})+C_3\,f_{33(3)} \\ I_{(4)}=y_{(4)}+C_1\,f_{43(4)}+C_2\,f_{43(4)}-C_3\,f_{41(4)}+C_1^2C_2\,f_{42(4)} \end{cases} \tag{14}$$

It can be rewrite to matrix form

$$\begin{pmatrix} I_{(1)} \\ I_{(2)} \\ I_{(3)} \\ I_{(4)} \end{pmatrix} = \begin{pmatrix} y_{(1)} & -3f_{11(1)} & -3f_{11(1)} & f_{13(1)} & f_{12(1)} \\ 0 & -f_{21(2)} & -f_{21(2)} & f_{23(2)} & y_{(2)} + f_{22(2)} \\ y_{(3)} & f_{32(3)} - f_{31(3)} & f_{32(3)} + f_{31(3)} & f_{33(3)} & 0 \\ y_{(4)} & f_{43(4)} & f_{43(4)} & f_{41(4)} & f_{42(4)} \end{pmatrix} \begin{pmatrix} 1 \\ C_1 \\ C_2 \\ C_3 \\ C_1^2 C_2 \end{pmatrix} \tag{15}$$

From (15) we found it is wrong construction, because (15) have too many base variables $\{C_1, C_2, C_3, C_1^2 C_2\}$. Clearly it is better to rewrite (13) as,

$$\begin{pmatrix} I_{(1)} \\ I_{(2)} \\ I_{(3)} \\ I_{(4)} \end{pmatrix} = \begin{pmatrix} y_{(1)} & f_{13(1)} & -3f_{11(1)} & f_{12(1)} \\ 0 & f_{23(2)} & -f_{21(2)} & y_{(2)} + f_{212(2)} \\ y_{(3)} & f_{33(3)} & f_{32(3)} - f_{31(3)} & 0 \\ y_{(4)} & -f_{41(4)} & f_{43(4)} & f_{42(4)} \end{pmatrix} \begin{pmatrix} 1 \\ C_3 \\ C_1 + C_2 \\ C_1^2 C_2 \end{pmatrix} \tag{16}$$

We can easily found that (16) gives difference equations by eliminating base variables $\{C_3, C_1 + C_2, C_1^2 C_2\}$. We also require condition $\det(\mathbf{F}) = 0$, here

$$\mathbf{F} = \begin{pmatrix} y_{(1)} & f_{13(1)} & -3f_{11(1)} & f_{12(1)} \\ 0 & f_{23(2)} & -f_{21(2)} & y_{(2)} + f_{212(2)} \\ y_{(3)} & f_{33(3)} & f_{32(3)} - f_{31(3)} & 0 \\ y_{(4)} & -f_{41(4)} & f_{43(4)} & f_{42(4)} \end{pmatrix}. \tag{17}$$

As a result we found following. If we can rewrite simultaneous equations for constructing difference equations with appropriate polynomial base by **C** in $O(\mathbf{x})[\mathbf{y}, \mathbf{C}]$ and its coefficient matrix **F** satisfies $\det(\mathbf{F}) = 0$, then we can construct evolutional differential equations. Here, appropriate polynomial base means that dimension of base by **C** equal to order of difference equation respect to initial condition. Since, the order of difference equation equal to number of simultaneous equations minus one that are generator of elimination ideals for **C**. We assume that the condition $\det(\mathbf{F}) = 0$ should be satisfied by initial conditions or selection of functions and their derivatives. Example (13) is a simple case, because we can easily obtain evolutional difference equation by expanding condition $\det(\mathbf{F}) = 0$ itself.

Algebraic treatment of implicit function theorem relate to regular (Cohen-Macaulay) property. Therefore, if we treat condition $\det(\mathbf{F}) = 0$ more exactly, we have to notice regularity. We will return this topic later section. At the end of this section, we show Mathematica result for this sample.

We define ideal generator for elimination by following input,

```
I1 := y1 - 3 * (c1 + c2) * f11 - c1^2 * c2 * f12 + c3 * f13;
I2 := c1^2 * c2 * y2 - (c1 + c2) * f21 - c1^2 * c2 * f22 + c3 * f23;
I3 := y3 - (c1 + c2) * f31 - (c1 + c2) * f32 + c3 * f33;
I4 := y4 - c3 * f41 - c1^2 * c2 * f42 + (c1 + c2) * f43;
```

Then, we get elimination ideal,

```
temp = GroebnerBasis[{I1, I2, I3, I4}, {y1, y2, y3, y4},
  {c1, c2, c3}]
```

We can confirm the uniqueness of y_4 by elimination with “Solve” from calculated result “temp”,

```
Solve[temp == 0, y4] .
{{y4 → (-f13 f14 f22 y1 - f14 f22 f23 y1 -
      f13 f24 f32 y1 - f23 f24 f32 y1 + f12 f24 f33 y1 +
      f22 f33 f34 y1 + f13 f14 y1 y2 + f14 f23 y1 y2 -
      f33 f34 y1 y2 - f12 f14 f21 y3 + 3 f11 f14 f22 y3 -
      f12 f24 f31 y3 + 3 f11 f24 f32 y3 - f22 f31 f34 y3 +
      f21 f32 f34 y3 - 3 f11 f14 y2 y3 + f31 f34 y2 y3) /
    (f13 f22 f31 + f22 f23 f31 - f13 f21 f32 -
      f21 f23 f32 + f12 f21 f33 - 3 f11 f22 f33 -
      f13 f31 y2 - f23 f31 y2 + 3 f11 f33 y2)}}
```

We found y_4 are unique, that is $\{y_1, y_2, y_3\} \to y_4$. We can easily confirm base variable are appropriate or not by Gröbner basis also,

```
GroebnerBasis[{c1, c2, c3, C1^2 c2}] ,
{c3, c2, c1} .
```

It shows {c1, c2, c3, c1^2c2} is not appropriate bases. On the contrary,

```
GroebnerBasis[{c3, c1 + c2, C1^2 c2}] ,
{C1^2 c2, c1 + c2, c3} .
```

The result shows dimension of these bases are unchanged and equal to order of difference equation. It means these are appropriate bases for the example.

V. Syzygy And Free Resolution In C

We introduce more algebraic treatment in this section. In the previous section, we found that bases of polynomial function by C_js are induced by Gröbner bases. These bases correspond to difference operation or differentiation (We regard C_js to integral constant at the start point of this study). Therefore if we put some base equal to zero, it corresponds to introduction of special solution to the system. For example, if we put $C_n=0$ for (5),

$$y = C_1 f_1(x) + C_2 f_2(x) + \cdots + C_{n-1} f_{n-1}(x), \;\; C_n = 0. \tag{18}$$

Clearly, (18) become general solution function of (n-1)th-order difference equation. In other word, (n)th-order difference equation is integrated, or reduced its order by special condition.

If we call solution of difference equation as kernel of following mapping ***DCE***,

$$\boldsymbol{Sol}: \text{ Solution function}(y) \text{ that include polynimial basis of } C_j$$

$$\underset{\text{Intgeration}}{\overset{\text{Elimination of } C_j \text{ with initial conditions}}{\rightleftarrows}} \boldsymbol{DCE}: \text{Diffrence Equation},$$

$$\boldsymbol{DCE}: y \xrightarrow[\text{Substitution of } y \text{ into } \boldsymbol{DCE} \text{ with initial conditions}]{} \boldsymbol{DCE}(y),\ y \in \boldsymbol{Sol}.$$

Then, putting each base polynomial function by C_js equal to zero makes a element of kernel of ***DCE*** (special solution). In addition, it is clear that syzygies (perpendicular space to the one defined by original base) by these base polynomial functions also generate the part of kernel of ***DCE*** (Free resolution). Syzygies are also invariants of ***DCE*** because they are consist from only C_js (integral constants).

We consider the form (10) with abbreviations,

$$\mathbf{F}\cdot\ \underline{\mathbf{C}}=\mathbf{y} \tag{19}$$

here, **F** is matrix consist from $f_j(x)$s, $\underline{\mathbf{C}}$ is a row vector each element from appropriate $g_j(\mathbf{C})$ base, and **y** is a row vector from initial value of $y_i^{(j)}$. Matrix notation of (6) and (8) are those of simple examples. By assumption, we can use $\mathbf{F}^{-1}$ for solving $\underline{\mathbf{C}}$,

$$\underline{\mathbf{C}}= \mathbf{F}^{-1}\mathbf{y} \tag{20}$$

Using syzygy matrix $\mathbf{S_{yz}}$, we get

$$\mathbf{S_{yz}}\cdot\ \underline{\mathbf{C}}= \mathbf{S_{yz}}\cdot\ \mathbf{F}^{-1}\mathbf{y}=0, \tag{21}$$

since we define $\mathbf{S_{yz}}$ as, $\mathbf{S_{yz}}\cdot\ \underline{\mathbf{C}}=0$. Then $\mathbf{S_{yz}}$ is matrix that each element is polynomial by C_js, we can get additional other descriptions of solution function defined from $\mathbf{S_{yz}}\cdot\ \mathbf{F}^{-1}\mathbf{y}=0$.

VI. General Form For Ordinary Differencing Respect To ODE

A. *Parametric form for ordinary differencing respect to ODE*

We treat following form as an extension of previous section, because general solution function of ODEs are obtained as parametric forms in many cases. Here we put t is parameter and F_j, u_j, v_j, f_j and g_j are appropriate function (:= $O_{(*)}$) of variables *.

$$\text{Solution: } \begin{cases} F_1(y,\ t,\ \mathbf{C})=0 \\ F_2(x,\ t,\ \mathbf{C})=0 \end{cases}, \text{ here } \begin{cases} F_1(y,\ t,\ \mathbf{C})\in O_{(y,t)}[\mathbf{C}] \\ F_2(x,\ t,\ \mathbf{C})\in O_{(x,t)}[\mathbf{C}] \end{cases}. \tag{22}$$

Example,

$$I=\begin{cases} u_1(\mathbf{C})f_1(t)+u_2(\mathbf{C})f_2(t)+\cdots+u_n(\mathbf{C})f_n(t)-y \\ v_1(\mathbf{C})g_1(t)+v_2(\mathbf{C})g_2(t)+\cdots+v_n(\mathbf{C})g_n(t)-x \end{cases}.$$

We can also treat this form with previous procedure by discretizing both x and y in the same system algebraically. For example, we consider 1st order ODE

$$(y^2+2xy+x^2+ay-2ax)\frac{dy}{dx}=-y^2-2xy-x^2-2ay+ax, \tag{23}$$

Its solution function is

$$x=C^2(t^3+4t^2/a)+Ct,\ y=-C^2(t^3+4t^2/a)+Ct. \tag{24}$$

In this special example, we can eliminate t from (24) directly using Gröbner basis putting elimination order of t is highest. Then, it is another form of solution function (24) without t,

$$4Cx-\frac{8Cx^2}{a}+x^2-4Cy-\frac{16Cxy}{a}+3x^2y-\frac{8Cy^2}{a}+3xy^2+y^3=0. \tag{24'}$$

Generally, we have to eliminate **C** from parametric form solution function as (22).

B. *Projective treatment of C respect to particular solution of ODE and rationality*

We notice that example (13) gives intimation that projective treatment for the problem is more appropriate and consistent. Moreover following example forces us to the treatment. Consider Riccati equation,

$$g(x)\frac{dy}{dx}=f_2(x)y^2+f_1(x)y+f_0(x). \tag{25}$$

Using given particular solution $y_0=y_0(x)$ of (25), the general solution can be written as,

$$\begin{aligned} y&=y_0(x)+\Phi(x)\left[C-\int\Phi(x)\frac{f_2(x)}{g(x)}dx\right]^{-1}, \\ \Phi(x)&=\exp\left\{\int[2f_2(x)y_0(x)+f_1(x)]\frac{dx}{g(x)}\right\}, \end{aligned} \tag{26}$$

here particular solution $y_0(x)$ corresponds $C=\infty$. This sample implies projective treatment of C_j. In addition differential equations that have rational functions with moving singularity as its solution functions, give this form [7].

$$y=\frac{a(x-C_1)(x-C_2)\cdots(x-C_n)}{(x-C_{n+1})(x-C_{n+2})\cdots(x-C_m)}. \tag{27}$$

Therefore, we treat solution function in projective space and homogeneous polynomials as for **C**. For example, if solution function is given by affine from as for **C**,

$$y = C_1 f_1(x) + C_2 f_2(x) + \cdots + C_n f_n(x) + r(x), \quad (28)$$

we change variables $\{C_1, C_2, ..., C_n\}$ to $\{C_1/C_0, C_2/C_0, ..., C_n/C_0\}$ and multiply C_0 to both side of (28). It gives,

$$C_0(r(x) - y) + C_1 f_1(x) + C_2 f_2(x) + \cdots + C_n f_n(x) = 0. \quad (29)$$

At a glance, we can obtain particular solution putting $\{C_0, C_1, C_2, ..., C_n\} = \{1, 0, 0, ..., 0\}$. From the example, we can found that particular solutions, moving singularities and symmetry of equation can be treated properly by introducing projective space for **C**.

VII. Regularity Of C and Uniqueness Of Ordinary Differencing

We really want to know is whether C_j's become regular coordinate system or not. If they are regular coordinate system, we obtain uniqueness of solution function by elimination of C_j's with appropriate initial conditions. Algebraically following relations are known, Regular coordinate system → Complete intersection → Gorenstein → Cohen-Macaulay. From this chain, we can introduce many procedures for proving regularity of C, however we return analytic treatment of the problem.

Local regularity of C_j's is easily confirmed by Jacobian with condition $\det(\mathbf{F}) \neq 0$ in a simple case (12). We generalize this approach. We treat solution function as $y_{(j)}=y(x_{(j)}, C_G)$, here C_G means that C_j's are reconstructed from Gröbner (standard) bases. As an example in (16), we find C_G by GröbnerBases(C) $\to C_G$, then we change variables from $\{C_3, C_1+C_2, C_1\text{^}2C_2\}$ to $\{C_{G1}, C_{G2}, C_{G3}\}$ for simplicity. We rewrite $y_{(j)}=y(x_{(j)}, C)$ to $y_{(j)}=y(x_{(j)}, C_G)$ and Jacobian matrix as

$$\mathbf{F} = \frac{\partial(I_{(1)}, I_{(2)}, ..., I_{(n)})}{\partial(C_{G1}, C_{G2}, ..., C_{Gn})}. \quad (30)$$

The condition $\det(\mathbf{F}) \neq 0$ gives regularity condition for $\{C_{G1}, C_{G2}, C_{G3}\}$ on affine formulation for C. In case of projective formulation, we use

$$\mathbf{F} = \frac{\partial(I_{(0)}, I_{(1)}, I_{(2)}, ..., I_{(n)})}{\partial(C_{G0}, C_{G1}, C_{G2}, ..., C_{Gn})}. \quad (31)$$

We can regards that the condition det(**F**) equals to zero or not gives additional integrable condition.

VIII. Conclusion And Discussions

In this study, we treated integrable difference equation algebraically from unusual side. We found that regular property of **C** in solution function is important, and conditions for the property are confirmed properly using theorems around Gröbner base theory. Obtained results relate to application, for example, integrability of finite difference schemes, limiting treatment of function and interpolation theorem, etc. As for constructible condition for the unique evolutional difference and differential equation using more general function including some analytic function should be given more explicitly. Stability problem for obtained difference equation is ignored. However, it seems to have good properties since it has invariants that are defined from constant polynomial bases by C_js [6]. We simply treated rational form of functions and singularities in projective space. More details for this treatment should be study. In this study, we assumed evolutional rule of x is given. This condition should be treated more exactly respect to general case. We can also treat solution function that contains C_i and E_j. In other word, we consider partial difference equation regarding each C_i corresponds to taking difference for variable u, and E_j corresponds to taking difference for variable v. Then solution functions are defined by generally,

$$F(y,\ u,\ v,\ \mathbf{C}, \mathbf{E}) = 0 \quad (32)$$

If we can eliminate all **C** and **E** in (32) with appropriate initial conditions, we get partial difference equation that have (32) as general solution. We leave this natural extension for next study. These left problems should be cleared soon.

Difference equational version of D-module theory [8] may be obtained by hard study for difference operator from algebraic point and contribution by elimination of symmetry from function space like this study, and many other related works, for example algebraic function theory.

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