Impact response of circular pre-stressed orthotropic and transversely isotropic plates*

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Abstract – The problem on normal low-velocity impact of an elastic falling body upon a pre-stressed orthotropic plate possessing curvilinear anisotropy is studied with consideration for the changes in the geometrical dimensions of the contact domain. At the moment of impact, shock waves (surfaces of strong discontinuity) are generated in the target, which then propagate along the plate during the process of impact. The classification of transient waves propagating in a thin pre-stressed plate possessing curvilinear orthotropy is presented. Behind the wave fronts upto the boundary of the contact domain, the solution is constructed with the help of the theory of discontinuities and one-term ray expansions. Nonlinear Hertz's theory is employed within the contact region. For the analysis of the processes of shock interaction of the elastic sphere with the pre-stressed orthotropic plate, a nonlinear integro-differential equation has been obtained with respect to the value characterizing the local indentation of the impactor into the target, which has been solved analytically in terms of time series with integer and fractional powers. The particular case of a pre-stressed transversely isotropic plate is analyzed in detail. A critical review of the approaches for investigating the transient wave propagation in isotropic and pre-stressed orthotropic plates impacted by falling objects is presented.

Keywords – Wave theory of impact, orthotropic plate possessing curvilinear anisotropy, transversely isotropic plate, ray method, Hertz's contact law, dynamic contact interaction, surface of strong discontinuity

1 Introduction

The problems connected with the analysis of the shock interaction of thin bodies (rods, beams, plates, and shells) with other bodies have widespread application in various fields of science and technology. The physical phenomena involved in the impact event include structural responses, contact effects and wave propagation. These problems are topical ones just as from the viewpoint of fundamental research in applied mechanics, so also with respect to their applications. Since these problems belong to the problems of dynamic contact interaction, their solution is connected with severe mathematical and calculation difficulties. To overcome this impediment, a rich variety of approaches and methods has been suggested, what is embodied in a great quantity of articles and reviews (see e.g. a long list of references in [1] and [2]).

The state-of-the-art article [1] highlights in more detail one of the important but scantily known aspect of the given problem, namely: the influence of the transient waves generated at the moment of impact upon the process of the shock interaction of solids, and the connection of this aspect with other facets of this challenge is shown as well.

An impact response analysis requires a good estimate of contact force throughout the impact duration. Low-velocity impact problems, which also took the local indentation into account, have been solved by many authors. Reference to the state-of-the-art papers [1, 2] shows that in most studies it was assumed that the impacted structure was free of any initial stresses. But this does not adequately reflect the real multidirectional complex loading states that the materials experience during their service life.

In practice, the composite facing of a structure may be under a preload, e.g., a sandwich structure with laminate facing under bending loads, jet engine fan blades subjected to centrifugal forces [3]. Even when stationary on the runway a composite airframe is under pre-stress [4]. The other example of great practical interest is the analysis of impact response of pipes pressurized for hydro-tests subjected to dropped tools [5].

Very few works have reported on the impact response of anisotropic and composite plates and beams subjected to an initial uniaxially tensile preloading [6]-[9], as well as biaxial preloading [4, 5], [10]-[12], in so doing only rectangular plates are considered as the targets.

The impact behaviour under compressive preloads is addressed in even fewer papers [4, 5], [13]-[16]. This preloading condition is yet more complex because plate buckling becomes an issue for relatively thin composite structures [14]. Analytical investigation of the low-

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velocity impact response of circular orthotropic and transversely isotropic plates possessing curvilinear anisotropy under compressive preloding has been carried out recently by Rossikhin and Shitikova in [15] and [16], respectively. The equations of plate motion take the rotary inertia and transverse shear deformations into account. In the case of the orthotropic target [15], the changes in the geometrical dimensions of the contact domain have been ignored and the contact interaction is modeled by a linear spring, and a force arising in it is the linear approximation of Hertz's contact force. Stability or instability of the plate is established by analyzing the behavior of transient waves generating in the plate at the moment of impact, which further propagate along its median surface as 'diverging circles'. In the case of the transversely isotropic target [16], which was impacted by a long elastic thin cylindrical rod, the waves of strong discontinuity are generated in the plate and begin to propagate. Behind the fronts of these waves, the solution is constructed in terms of ray series, the coefficients of which are the different order discontinuities in partial-time derivatives of the desired functions, and a variable is the time elapsed after the wave arrival at the plate's point under consideration. The ray series coefficients are determined from recurrent equations within an accuracy of arbitrary constants, which are then determined from the conditions of dynamic contact interaction of the impactor and the target. The found arbitrary constants allow one to construct the solution both within and out of the contact region.

In the present paper, the problem on normal low-velocity impact of an elastic sphere upon a pre-stressed orthotropic plate possessing curvilinear anisotropy is studied with the consideration of the changes in the geometrical dimensions of the contact domain. Behind the wave fronts upto the boundary of the contact domain, the solution is constructed with the help of the theory of discontinuities and one-term ray expansions, while the nonlinear Hertz's theory is employed within the contact region [17].

It should be noted that after the publication of the review paper [1], the authors are still interested in the field and have made some advanced contributions to the wave theory of impact [16]-[20]. And of course we are still trace new publications in the field trying not to miss important technical papers. Thus, recently our attention was attracted by a paper [21] published online by Springer on July 9, 2011 followed by the hard version in *Acta Mechanica*. This paper is written in the best traditions of rebus compilers, since it is difficult if not impossible for a reader without a strong background in the wave theory of impact to understand anything from what the author is doing in this pape nor and how he has received each of his statements, i.e. each result is to be hardly comprehended. Having got acquainted with the cited article, the authors of this paper, to their great surprise, have found that only one equation was written there without mistakes!

The paper by Loktev [21] is devoted to the analysis of the impact response of a circular simply supported pre-stressed orthotropic plate possessing curvilinear anisotropy. Moreover, the impact occurs not at the center of the curvilinear anisotropy, which coincides with the center of the orthotropic plate, but at its arbitrary point.

In Introduction of [21], a short review of papers dealing with the application of Uflyand–Mindlin type plates, which allow one to consider transient wave propagation in the target after the impact by a falling object, is presented, and in particular one of our papers [15] is cited with the following short comments:

> "In Ref. 7 (paper [15] in our list of references), the external radial force compressing the round plate, the dynamic behaviour of which was described by *simplified equations* (here and below italic is used by these authors) for *a buckling* and *a turning angle* of the normal to the radius, was presented".

And this is despite the fact that hyperbolic equations of motion of a prestressed orthotropic plate possessing curvilinear anisotropy were derived by Rossikhin and Shitikova [15] in the polar coordinate system with due account for transverse shear deformations and rotary inertia starting from the equations describing its statical behavior [22]. The paper [15] pioneers in studying the dynamic stability of a precompressed circular plate with respect to shock loading, and *buckling* was not even mentioned, since buckling problems are usually solved in statical formulations.

However, for some reason, the other paper by Rossikhin and Shitikova [16] devoted to the similar problem as in [21] has not been cited. The equations of motion of the plate are practically the same, although the transversely isotropic plate impacted by an elastic rod with a plane circular end has been considered in [16].

It was a great surprise to these authors to read in the conclusion of Introduction [21] that "in the aforementioned papers, the wave processes taking place in the target after the impact *were not taken into account*". But it is not the case at all! It could be noted that the wave theory of impact was initiated in [23] and [24], and it is developed uninterruptedly by researchers from various countries (see e.g. the review of papers in the field in [1]).

Thus, the second aim of this paper is to show the inconsistency and falseness of the calculating scheme presented in Loktev [21], which further will be referred to as the 'L scheme', and to suggest an alternative effective approach allowing one to consider the impact response of a prestressed orthotropic plate possessing curvilinear anisotropy, as well as to analyze its dynamic stability with respect to transient loading.

2 On a critique of the governing equations and boundary conditions

In Sect. 2 of [21], the problem of impact of a spherical body upon an orthotropic plate is formulated. In what follows, all quotations from [21] used in the given paper, i.e. equations of motion, initial and boundary conditions, the method of solution and calculation scheme and so, on will be referred to as the 'L equations of motion, initial and boundary L conditions, the L method of solution and calculation L scheme', respectively, and will be properly indicated.

Thus, we will proceed from the equations of motion of a target following [21]:

> "the round-shaped simply supported over the contour orthotropic plate, the displacements of whose points are determined from the Uflyand–Mindlin equations [Ref. 4] (paper [25] in our list of references) in a dimensionless form that represent the generalized Hooke's law *considering geometrical nonlinearity of the plate's material*, is studied:

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}+\frac{c_2\sigma_r+c_3}{c_1 r}\frac{\partial^2\psi}{\partial r\partial\theta}$$
$$-\frac{1}{r^2}\frac{c_2}{c_1}\varphi-\frac{c_2+c_3}{c_1 r^2}\frac{\partial\varphi}{\partial\theta}+\frac{12c_4}{c_1}\left(\frac{\partial w}{\partial r}-\varphi\right)$$
$$=-\frac{\partial^2\varphi}{\partial\tau^2}+M+\frac{M_r\triangle_r u}{c_1 h\rho},$$

$$\frac{c_4}{c_1}\left(\frac{\partial^2 w}{\partial r^2}-\frac{\partial\varphi}{\partial r}\right)+\frac{c_4}{c_1}\left(\frac{\partial^2 w}{r^2\partial\theta^2}-\frac{\partial\psi}{r\partial\theta}\right)$$
$$+\frac{c_4}{c_1}\left(\frac{\partial w}{r\partial r}-\frac{\varphi}{r}\right)=\frac{\partial^2 w}{\partial\tau^2}+q_1+\frac{N\triangle_r w}{c_1 h^2\rho},$$

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{c_3}{c_1}\frac{\partial^2 u}{r^2\partial\theta^2}-\frac{c_2}{c_1}\frac{u}{r^2}$$
$$+\frac{c_2\sigma_r+c_3}{c_1 r}\frac{\partial^2 v}{\partial r\partial\theta}-\frac{c_2+c_3}{c_1 r^2}\frac{\partial v}{\partial\theta}$$
$$=\frac{\partial^2 u}{\partial\tau^2}+\frac{N\triangle_r u}{c_1 h^2\rho}+\frac{M_r\triangle_r\varphi}{c_1\rho},\qquad (1L)$$

$$\frac{c_2}{c_1}\frac{\partial^2 v}{r^2\partial\theta^2}+\frac{c_3}{c_1}\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right)$$
$$+\frac{\sigma_\theta+c_3}{c_1 r}\frac{\partial^2 u}{\partial r\partial\theta}+\frac{c_2+c_3}{c_1 r^2}\frac{\partial u}{\partial\theta}$$
$$=\frac{\partial^2 v}{\partial\tau^2}+\frac{M_z\triangle_\theta\psi}{c_1\rho},$$

$$\frac{c_3}{c_1}\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{\psi}{r^2}\right)+\frac{c_2}{c_1}\frac{\partial^2\psi}{r^2\partial\theta^2}$$
$$+\frac{\sigma_\theta+c_3}{c_1 r}\frac{\partial^2\varphi}{\partial r\partial\theta}+\frac{c_2+c_3}{c_1 r^2}\frac{\partial\varphi}{\partial\theta}$$
$$+\frac{12c_5}{c_1}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}-\psi\right)=-\frac{\partial^2\psi}{\partial\tau^2}+\frac{M_z\triangle_\theta v}{c_1 h\rho},$$

> where

$$w=\frac{w}{h},\ u=\frac{u}{h},\ v=\frac{v}{h},\ r=\frac{r}{h},$$
$$\tau=\frac{t\sqrt{c_1}}{h},\ q_1=\frac{qh}{\rho c_1},$$
$$c_1=\frac{E_r}{(1-\sigma_r\sigma_\theta)\rho},\ c_2=\frac{E_\theta}{(1-\sigma_r\sigma_\theta)\rho},$$
$$c_3=\frac{G_{r\theta}}{\rho},\ c_4=\frac{KG_{rz}}{\rho},\ c_5=\frac{G_{\theta z}}{\rho},$$
$$M=\frac{12qR_1\cos\alpha_1}{\rho hc_1}=\frac{12R_1\cos\alpha_1}{h^2}q_1,$$

> $E_r\sigma_r=E_\theta\sigma_\theta$, $K=5/6$, E_r, E_θ and σ_r, σ_θ – the coefficients of elasticity and Poisson's ratios for r and θ directions, respectively; G_{rz}, $G_{\theta z}$ – the moduli of rigidity for rz and θz planes, respectively; $w(r,\theta)$ – the normal displacement of the median plane; $u(r,\theta)$ and $v(r,\theta)$ – the tangential displacements of the medial surface with respect to r and θ coordinates, respectively; $\varphi(r,\theta)$ and $\psi(r,\theta)$ – the arbitrary functions of r and θ coordinates, ρ – the density, h – the plate's thickness, q – the load, R_1 – the radius of the spherical indenter, $\triangle_r=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$, $\triangle_\theta=\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$, N – the external longitudinal force operating in a radial direction, M_r – the external bending moment the vector of which is directed along the radius, M_z – the external rotational moment, the vector of which is directed along the normal to the median plane of the target."

First of all it should be noted that Eqs. (1L) contain so many mistakes, starting from the erroneous record of terms involving the external forces and moments in the dimensionless form (it seems likely that the author of these equations does not know the dimensions of internal as well external forces and moments used in the theories of plates and shells!). Thus, the correct dimensionless values should be the following:

$$q^*=\frac{q}{\rho c_1},\ N^*=\frac{N}{\rho c_1 h},\ M_r^*=\frac{M_r}{\rho c_1 h^2},\ M_z^*=\frac{M_z}{\rho c_1 h^2},\qquad (1)$$

as well as a mystery moment (its physical meaning together with the value α_1 has not been explained)

$$M^*=\frac{12R_1\cos\alpha_1}{h}\,q^*.$$

In the first equation of (1L), the third term should be multiplied by c_3/c_1, while in its sixth term the derivative $\partial\psi/\partial\theta$ should appear instead of $\partial\varphi/\partial\theta$. In the last two equations of (1L), the Poisson's coefficient σ_θ is added to the coefficient c_3, the dimension of which is equal to the squared velocity! The correct coefficient is $(\sigma_\theta c_1 + c_3)/c_1$.

It would be shown later that the negative inertia terms in the first and last equations from (1L) are also incorrect, since such a choice results in erroneous velocities of transient wave propagation.

It is evident from Eqs. (1L) that they do not involve 'the generalized Hooke's law considering geometrical nonlinearity of the plate's material'. It is well known fact that the curvilinear anisotropy does not couple with the effect of geometrical nonlinearity by any means [22].

The statement of the boundary conditions follows Eqs. (1L):

> "the simply supported over the contour round plane is described by the following boundary conditions:
>
> $$w\Big|_{r=R} = 0, \quad \frac{\partial^2 w}{\partial r^2}\Big|_{r=R} = 0, \qquad (2L)$$
>
> where R – the radius of the plate."

It is intriguing fact that only *two* boundary conditions (2L) have been prescribed for a set of *five* governing Eqs. (1L). Moreover, the second condition (the absence of the bending moment on the plate's boundary) is valid only for a classical plate, and it is not satisfied for the plate under consideration.

From the above quotation a reader could recognize that Eqs. (1L) were determined from the Uflyand–Mindlin equations in [25]. However it is not the case. Equations (1L) were derived neither by Uflyand [26] nor by Mindlin [27], and certainly not in [25].

As it has been shown in [15], equations of the 1L-type could be obtained from the five fundamental differential equations suggested in [22], which describe the equilibrium of an orthotropic plate with a cylindrical anisotropy for the case when all radial planes crossing the axis of anisotropy are the planes of elastic symmetry. It was assumed in [22] that the pole of anisotropy, i.e. a point of intersection of the axis of anisotropy and the median plane of the plate, is the origin of the cylindrical set of coordinates $r\theta z$, and the z-axis is directed along the axis of anisotropy. The Ambartsumian equations have the form [22]:

$$C_r\left(r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\right) + C_k\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} - C_\theta\frac{u}{r} + (C_\theta\sigma_r + C_k)\frac{\partial^2 v}{\partial r\partial\theta}$$
$$-(C_\theta + C_k)\frac{1}{r}\frac{\partial v}{\partial\theta} = 0,$$
$$C_\theta\frac{1}{r}\frac{\partial^2 v}{\partial\theta^2} + C_k\left(r\frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r} - \frac{v}{r}\right) + (C_r\sigma_\theta + C_k)\frac{\partial^2 u}{\partial r\partial\theta}$$
$$+(C_\theta + C_k)\frac{1}{r}\frac{\partial u}{\partial\theta} = 0,$$
$$\frac{1}{r}\frac{\partial(r\varphi)}{\partial r} + \frac{\partial\psi}{\partial\theta} = -\frac{12}{h^3}\,q, \qquad (1A)$$
$$D_r\,\frac{\partial}{\partial r}\left(r\frac{\partial^2 w}{\partial r^2}\right) + D_{r\theta}\,\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial^2 w}{\partial\theta^2}\right)$$
$$-\frac{h^2}{10}\left\{a_r\left[D_r\,\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + D_k\,\frac{1}{r}\frac{\partial^2\varphi}{\partial\theta^2} - D_\theta\,\frac{\varphi}{r}\right]\right.$$
$$\left.+a_\theta\left[(D_\theta\sigma_r + D_k)\frac{\partial^2\psi}{\partial r\partial\theta} - (D_\theta + D_k)\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right]\right\}$$
$$-D_\theta\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + \frac{h^3}{12}\,r\varphi = 0,$$
$$D_\theta\left(\frac{1}{r^2}\frac{\partial^3 w}{\partial\theta^3} + \frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta}\right) + D_{r\theta}\,\frac{\partial^3 w}{\partial r^2\partial\theta}$$
$$-\frac{h^2}{10}\left\{a_\theta\left[D_k\left(r\frac{\partial^2\psi}{\partial r^2} + \frac{\partial\psi}{\partial r} - \frac{\psi}{r}\right) + D_\theta\,\frac{1}{r}\frac{\partial^2\psi}{\partial\theta^2}\right]\right.$$
$$\left.+a_r\left[(D_r\sigma_\theta + D_k)\frac{\partial^2\varphi}{\partial r\partial\theta} + (D_\theta + D_k)\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right]\right\}$$
$$+\frac{h^3}{12}\,r\psi = 0,$$

where $E_r\sigma_\theta = E_\theta\sigma_r$ (note that this very important relationship for an orthotropic plate possessing cylindrical anisotropy was written in Eqs.(1L) with a mistake), D_r, D_θ and C_r, C_θ are rigidities due to bending and compression-tension in the $r-$ and θ- directions, respectively, D_k and C_k are rigidities due to torsion and shear, respectively,

$$D_r = \frac{h^3}{12}B_r, \quad D_\theta = \frac{h^3}{12}B_\theta, \quad D_k = \frac{h^3}{12}B_k,$$
$$D_{r\theta} = D_r\sigma_\theta + 2D_k, \quad C_r = hB_r, \quad C_\theta = hB_\theta,$$
$$C_k = hB_k, \quad a_r = \frac{1}{G_{rz}}, \quad a_\theta = \frac{1}{G_{\theta z}},$$
$$B_r = \frac{E_r}{(1-\sigma_r\sigma_\theta)}, \quad B_\theta = \frac{E_\theta}{(1-\sigma_r\sigma_\theta)}, \quad B_k = G_{r\theta}.$$

In Eqs. (1A), $\varphi(r,\theta)$ and $\psi(r,\theta)$ are arbitrary desired functions in the coordinates r,θ (which do not coincide with those in Eqs. (1L) as it will be shown below) are connected with the transverse forces Q_r and Q_θ, respectively, by the following formulae:

$$Q_r = \frac{h^3}{12}\,\varphi, \quad Q_\theta = \frac{h^3}{12}\,\psi. \qquad (2A)$$

The boundary conditions for the case of a simply supported circular plate with R radius are the following [22]:

$$N_r = 0, \;\; M_r = 0, \;\; T_{r\theta} = 0, \;\; H_{r\theta} = 0, \;\; w = 0, \quad (3A)$$

where N_r and $T_{r\theta}$ are internal tangential forces, and M_r and $H_{r\theta}$ are internal bending moment and torque, respectively, per unit length of the plate's middle surface.

For describing the dynamic behavior of the circular orthotropic plate subjected to transient loading, Rossikhin and Shitikova [15] suggested first to replace the functions φ and ψ in Eqs. (1A) with

$$\varphi(r,\theta,t) = \frac{12}{h^2}\, KG_{rz}\left(\frac{\partial w}{\partial r} - \varphi_r\right),$$
$$\psi(r,\theta,t) = \frac{12}{h^2}\, KG_{\theta z}\left(\frac{1}{r}\,\frac{\partial w}{\partial \theta} - \varphi_\theta\right), \qquad (2)$$

resulting in the known relationships for the transverse forces

$$Q_r = hKG_{rz}\left(\frac{\partial w}{\partial r} - \varphi_r\right),$$
$$Q_\theta = hKG_{\theta z}\left(\frac{1}{r}\,\frac{\partial w}{\partial \theta} - \varphi_\theta\right), \qquad (3)$$

where K is the shear coefficient, the magnitude of which is chosen further as $5/6$, $\varphi_r(r,\theta,t)$, and $\varphi_\theta(r,\theta,t)$ are the angles of rotation of the normal to the plate in the $r-$ and $\theta-$axes directions, respectively.

Substituting (2) into Eqs. (1A), in so doing dividing the last two of them by r, and introducing into the right-hand sides of the final equations the following forces of inertia, respectively:

$$\rho\ddot{u}, \quad \rho\ddot{v}, \quad \frac{12}{h^2}\,\rho\ddot{w}, \quad \frac{h^3}{12}\,\rho\ddot{\varphi}_r, \quad \frac{h^3}{12}\,\rho\ddot{\varphi}_\theta, \qquad (4)$$

where an overdot denotes the time-derivative, yields

$$C_r\left(r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\right) + C_k\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} - C_\theta\frac{u}{r}$$
$$+(C_\theta\sigma_r + C_k)\frac{\partial^2 v}{\partial r\partial\theta} - (C_\theta + C_k)\frac{1}{r}\frac{\partial v}{\partial \theta} = \rho\ddot{u}, \qquad (5)$$

$$C_\theta\frac{1}{r}\frac{\partial^2 v}{\partial \theta^2} + C_k\left(r\frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r} - \frac{v}{r}\right)$$
$$+(C_r\sigma_\theta + C_k)\frac{\partial^2 u}{\partial r\partial\theta} + (C_\theta + C_k)\frac{1}{r}\frac{\partial u}{\partial \theta} = \rho\ddot{v}, \qquad (6)$$

$$KG_{rz}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\,\frac{\partial w}{\partial r} - \frac{\partial \varphi_r}{\partial r} - \frac{1}{r}\,\varphi_r\right)$$
$$+\frac{1}{r}\,KG_{\theta z}\left(\frac{1}{r}\,\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial \varphi_\theta}{\partial \theta}\right) = \rho\ddot{w}, \qquad (7)$$

$$D_r\left(\frac{\partial^2 \varphi_r}{\partial r^2} + \frac{1}{r}\,\frac{\partial \varphi_r}{\partial r}\right) + (D_\theta\nu_r + D_k)\,\frac{1}{r}\,\frac{\partial^2 \varphi_\theta}{\partial r\partial\theta}$$
$$+hKG_{rz}\left(\frac{\partial w}{\partial r} - \varphi_r\right) + D_k\,\frac{1}{r^2}\,\frac{\partial^2 \varphi_r}{\partial \theta^2}$$
$$-D_\theta\,\frac{1}{r^2}\,\varphi_r - (D_\theta + D_k)\,\frac{1}{r^2}\,\frac{\partial \varphi_\theta}{\partial \theta} = \frac{\rho h^3}{12}\,\ddot{\varphi}_r, \qquad (8)$$

$$D_k\left(\frac{\partial^2 \varphi_\theta}{\partial r^2} + \frac{1}{r}\,\frac{\partial \varphi_\theta}{\partial r} - \frac{1}{r^2}\,\varphi_\theta\right)$$
$$+hKG_{\theta z}\left(\frac{1}{r}\,\frac{\partial w}{\partial \theta} - \varphi_\theta\right) + (D_r\nu_\theta + D_k)\,\frac{1}{r}\,\frac{\partial^2 \varphi_r}{\partial r\partial\theta}$$
$$+(D_\theta + D_k)\,\frac{1}{r^2}\,\frac{\partial \varphi_r}{\partial \theta} = \frac{\rho h^3}{12}\,\ddot{\varphi}_\theta. \qquad (9)$$

The set of Eqs. (5)-(9) is subjected to the boundary conditions (3A), where the internal forces and moments have the form

$$N_r = C_r\,\frac{\partial u}{\partial r} + C_r\sigma_\theta\,\frac{1}{r}\left(\frac{\partial v}{\partial \theta} + u\right), \qquad (10)$$

$$T_{r\theta} = C_k\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right), \qquad (11)$$

$$M_r = -D_r\left[\frac{\partial \varphi_r}{\partial r} + \sigma_\theta\,\frac{1}{r}\left(\frac{\partial \varphi_\theta}{\partial \theta} + \varphi_r\right)\right], \qquad (12)$$

$$H_{r\theta} = -D_k\left[\frac{\partial \varphi_\theta}{\partial r} + \frac{1}{r}\left(\frac{\partial \varphi_r}{\partial \theta} - \varphi_\theta\right)\right]. \qquad (13)$$

For axially symmetric problems, the functions u, w and φ_r are independent of θ, but $v = 0$ and $\varphi_\theta = 0$, and therefore the set of Eqs. (5)-(9) goes over into one uncoupled equation for u and a set of two coupled equations for w and φ_r, which was used in [15] for the analysis of the dynamic stability of a circular pre-stressed elastic orthotropic plate impacted by a falling flat-end rod.

3 Loktev's classification of transient waves

At the end of Sect. 2 in [21], a reader could find very 'impressive' remarks

> "In Eq. (1L), c_1, c_2, c_3, c_4, c_5 represent the square of the velocities of independent waves, i.e., waves in the non-prestressed target; coefficients 1 and 2 correspond to the longitudinal waves of the tension-compression propagating in r and θ directions, respectively; coefficient 3 corresponds to the shear wave of the longitudinal sections in $r\theta$ plane; coefficients 4 and 5 correspond to the transverse shear waves in rz and θz planes, respectively."

which result further in the classification of transient waves (surfaces of discontinuity) suggested in Fig. 1L in [21]:

1. LWR - the quasilongitudinal wave of tension-compression propagating in r-direction,

2. LWθ - quasilongitudinal wave of tension-compression propagating in θ-direction,

3. TRWRθ - quasitransverse shear wave propagating in $r\theta$ plane,

4. TRWRZ - quasitransverse shear wave propagating in rz plane,

5. TRWθZ - quasitransverse shear wave propagating in θz plane.

This classification is worth of notable attention, since it is based on the utilization of those elastic moduli which, from the Loktev point of view, should enter into the velocities of the corresponding "independent" transient waves propagating in a non-prestressed plate, what points to the complete incompetence of its author in the field.

The waves propagating with the velocities $\sqrt{c_2}$ and $\sqrt{c_5}$ do not exist in nature. These virtual waves, to be named as carousel-type waves, exist only in Loktev's imagination.

However, this fact does not embarrass Loktev at all, and in Sect. 3.1 of [21] (which, by the way, has no any connection with the further part of the paper under consideration), the author describes the method with the help of which he assertedly obtained the force N and moments M_r and M_z dependence of the velocities of the enumerated above five waves based on the condition of compatibility (3L):

> "Applying the procedure described in Ref. 6 (paper [25] in our list of references), from the obtained equations at $k = -1$, one can come to the system written in terms of displacements and waves' velocities. The result of the solution of the following system is represented graphically in Figs. 2,3,4 in the form of dependencies $G^{(x)} = f(N, M_r, M_z)$."

First of all it should be emphasized that this 'procedure' was pioneered in [24], and afterwards it was applied by many researchers (see review papers [28, 1]), and thus it is very shamelessly to assign this procedure to the name of the author of [25].

Secondly, this brings up the question: Why did not he present these dependences $G^{(x)} = f(N, M_r, M_z)$ in [21] (moreover that they have rather simple form and could be found easily)? Instead of this, he drew the curves appeared from no quarter in Figs. 2L–4L (it should be noted that in these figures the dimensions of both abscissa and ordinate are incorrect, while the correct abscissa is $G/\sqrt{c_1}$ and ordinates are N^*, M_r^* and M_z^*), which could not be constructed without the knowledge of these dependences.

Generally speaking, with the help of the compatibility condition (3L)

$$G\left[\frac{\partial x_{,(k)}}{\partial \alpha}\right] = -\left[x_{,(k+1)}\right]\nu_\alpha + \frac{\delta\left[x_{,(k)}\right]}{\delta t}\nu_\alpha, \tag{3L}$$

where G–the normal velocity of the expansion of the wavefront; $[x_{,(k)}] = x_{,(k)}^{+} - x_{,(k)}^{-} = [\partial^k x/\partial t^k]$–the discontinuity in the derivatives of the k-degree by time t from the unknown function x on the wave surface Σ; the upper indices "+" and "-" indicate that the value is found directly in front of and behind the wavefront, respectively; x takes values ϕ, ψ, w, u, v; the magnitude α takes values r, θ; $\nu_\alpha (\nu_r = \cos\phi,\ \nu_\theta = \cos\theta)$–the components of the normal vector to the wave surface; $\delta/\delta t$ – δ-time derivative.

it is impossible to determine the velocities of nonstationary waves from the set of Eqs. (1L) due to the simple reason that such a condition of compatibility does not exist. Moreover, dimensionless Eqs. (1L) are subjected to the boundary conditions (2L) and the compatibility conditions (3L) written in dimension form!

If a point of the contact force application does not coincide with the pole of the curvilinear anisotropy of the plate, as it takes place in the L-problem considered in [21], then the condition of compatibility should take another form, namely [29]:

$$\begin{gathered} G\left[\partial Z_j/\partial x^i\right] = -\left[\partial Z_j/\partial t\right]\nu_i + \nu_i \delta[Z_j]/\delta t \\ -G\Gamma_{jn}^m [Z_m]\nu^n \nu_i + G\Gamma_{ji}^m [Z_m] + G g^{\alpha\beta} g_{ik}[Z_{j,\alpha}] x_\beta^k, \end{gathered} \tag{14}$$

where Z_j are the covariant components of the desired vector, g_{ij} and $g_{\alpha\beta}$ are the covariant components of the metric tensors of the space and the wave surface, respectively, $g^{\alpha\beta} g_{\alpha\gamma} = \delta_\gamma^\beta$, ν^i and ν_i are the contravariant and covariant components of the vector normal to the wave surface, $x_\beta^i = \partial x^i/\partial u^\beta$, x^i are the curvilinear spatial coordinates, u^β are the curvilinear surface coordinates, Γ_{ji}^m are the spatial Christoffel symbols, and an index after a comma denotes the covariant derivative with respect to the corresponding surface coordinate.

This is connected with the fact that a partial derivative with respect to the spatial coordinate should be substituted by the covariant derivative with respect to the same coordinate, while the δ-derivative is substituted with

$$\frac{D[Z_j]}{Dt} = \frac{\delta[Z_j]}{\delta t} - G[Z_m]\Gamma_{ji}^m \nu^i,$$

and further it is needed to take into account that the physical components of the values Z_j in the orthogonal system of coordinates have the following form:

$$(Z_j)_{\text{phys}} = Z_j \sqrt{g^{jj}},$$

where the summation over the j-index is absent.

Moreover, in this case the velocities of waves depend on the direction of their propagation, what results in the distortion of the wave front during its propagation.

Thus, if the wave velocities are considered as constant values independent of the direction of their propagation, as it was proposed in [21], then the point of the contact force application must coincide with the pole of the curvilinear anisotropy of the plate. If in this case the spatial coordinates coincide with the ray coordinates, then the compatibility condition is written in the following form:

$$G\left[\frac{\partial Z}{\partial r}\right] = -[Z_{,(1)}] + \frac{\delta [Z]}{\delta t}. \qquad (15)$$

It should be noted that the ray coordinates are the system of the orthogonal coordinates connected with the wave surface, namely: two coordinate lines lie on the wave surface, while the third one is directed along the normal to this surface, i.e., along the ray.

The compatibility condition (15) was derived in Rossikhin and Shitikova [30], wherein it was shown that it could be generalized over the physical components of vector and tensor values. This condition has allowed novel investigations to be made in the field of wave dynamics, since with its help it has been made possible to solve a lot of dynamic problems dealing with shock interaction of plates and shells with bodies of finite dimensions [1, 16, 20].

The author of [21] does not realize that each formula and each model has its own validity limits, and one cannot transcend these limits, since outside of the validity limits a useful formula goes over into its antipode and becomes a malignant one.

By the way, he has already used the compatibility condition of incorrect dimension in the problem of impact of a sphere upon a spherical shell of the membrane type [31], resulting in the solution wherein "kilometers" are added with "kilograms", but the surprising thing is that this fact has not been understood by both the authors and the reviewers of this opus (a critique of this paper could be found in [20]).

Since the compatibility condition (15) is of the first order, while the set of Eqs. (1L) is of the second order, then in order to eliminate this drawback we should substitute Z by $\partial Z/\partial r$ in (15). Thus formula (15) takes the form

$$G\left[\frac{\partial^2 Z}{\partial r^2}\right] = -\left[\frac{\partial^2 Z}{\partial r \partial t}\right] + \frac{\delta}{\delta t}\left[\frac{\partial Z}{\partial r}\right],$$

or with due account for (15) we have

$$G^2\left[\frac{\partial^2 Z}{\partial r^2}\right] = [Z_{,(2)}] - 2\frac{\delta [Z_{,(1)}]}{\delta t} + \frac{\delta^2 [Z]}{\delta t^2}. \qquad (16)$$

This formula is the second-order condition of compatibility. How could it be used?

First of all it is needed to rewrite it in the form

$$[Z_{,(2)}] = G^2\left[\frac{\partial^2 Z}{\partial r^2}\right] + 2\frac{\delta [Z_{,(1)}]}{\delta t} - \frac{\delta^2 [Z]}{\delta t^2}. \qquad (17)$$

or in the dimensionless form

$$[\bar{Z}_{,(2)}] = \bar{G}^2\left[\frac{\partial^2 \bar{Z}}{\partial \bar{r}^2}\right] + 2\frac{\delta [\bar{Z}_{,(1)}]}{\delta \tau} - \frac{\delta^2 [\bar{Z}]}{\delta \tau^2}, \qquad (18)$$

where $\bar{Z}$ is the dimensionless form of the value Z, and $\bar{G}^2 = G^2/c_1$ is the dimensionless squared velocity.

The condition of compatibility (17) is valid inside the layer of the width n, within which the desired values change monotonically and continuously from the value Z^+ to the value Z^-. This layer is introduced to interpret the surface of strong discontinuity propagating with the normal velocity G. Since the dimensionless values to be found from Eqs. (1L) are continuous on the wave surface of strong discontinuity, then differentiating all equations in (1L) one time with respect to the time and then substituting all second-order time-derivatives with their values according to (18), after the integration of the obtained set of equations two times with respect r from $-1/2\ n$ to $1/2\ n$ and going to the limit at $n = 0$ we find the following relationships:
from the first and third equations of (1L), with the corrected values of external loads according to (1),

$$(1 + \bar{G}^2)[\dot{\varphi}] - M_r^*[\dot{u}] = 0, \qquad (19)$$

$$M_r^*[\dot{\varphi}] - (1 - \bar{G}^2 - N^*)[\dot{u}] = 0, \qquad (20)$$

from the second equation of (1L)

$$\left(\frac{c_4}{c_1} - N^* - \bar{G}^2\right)[\dot{w}] = 0, \qquad (21)$$

and finally from the forth and the fifth equations

$$\left(\frac{c_3}{c_1} - \bar{G}^2\right)[\dot{v}] = 0, \qquad (22)$$

$$\left(\frac{c_3}{c_1} + \bar{G}^2\right)[\dot{\psi}] = 0. \qquad (23)$$

Supposing the values $[\dot{w}] = [\dot{v}] = [\dot{\psi}] = 0$, while

$$[\dot{\varphi}] \neq 0, \quad [\dot{u}] \neq 0, \qquad (24)$$

then the determinant of the homogeneous set of Eqs. (19) and (20) should be vanished. As a result we have

$$\bar{G}^4 + N^*\bar{G}^2 - \left(1 - N^* - M_r^{*2}\right) = 0. \qquad (25)$$

Solving (25) we have

$$\bar{G}^2_{1,2} = -\frac{1}{2}\,N^* \pm \sqrt{\left(1 - \frac{1}{2}\,N^*\right)^2 - M_r^{*2}}. \qquad (26)$$

Note that in all Loktev's equations it is assumed that negative magnitudes of the longitudinal force correspond to tensile pre-loading, while positive ones refer to compression pre-loading.

If $M_r^* = 0$, then from (26) we find

$$\bar{G}_1^2 = 1 - N^*, \quad \bar{G}_2^2 = -1. \qquad (27)$$

Now assuming that $[\dot{\varphi}] = [\dot{u}] = [\dot{v}] = [\dot{\psi}] = 0$, while

$$[\dot{w}] \neq 0, \qquad (28)$$

then from (21) we have

$$\bar{G}_3^2 = \frac{c_4}{c_1} - N^*. \tag{29}$$

Finally putting in (22) and (23)

$$[\dot{v}] \neq 0, \quad [\dot{\psi}] \neq 0, \tag{30}$$

and considering that $[\dot{w}] = [\dot{\varphi}] = [\dot{u}] = 0$, we find

$$\bar{G}_4^2 = \frac{c_3}{c_1}, \quad \bar{G}_5^2 = -\frac{c_3}{c_1}. \tag{31}$$

Since the velocities could not take on negative magnitudes, then the set of Eqs. (1L) is incorrect one.

4 Existing classification of transient waves propagating in a thin prestressed plate possessing curvilinear orthotropy

In order to correct the situation, it is suffice to change the signs ahead of the values $\partial^2\varphi/\partial\tau^2$ and $\partial^2\psi/\partial\tau^2$ in Eqs. (1L). In other words, Eqs. (5)-(9) should be utilized.

As a result instead of formulas (25) and (26) we obtain

$$\bar{G}^4 - 2\left(1 + \frac{1}{2}\,N^*\right)\bar{G}^2 + 1 + N^* - M_r^{*2} = 0, \tag{32}$$

$$\bar{G}_{1,2}^2 = 1 + \frac{1}{2}\,N^* \pm \sqrt{\frac{1}{4}\,N^{*2} + M_r^{*2}}, \tag{33}$$

where $N^* > 0$ and $N^* < 0$ correspond to the initial tensile and compression longitudinal forces, respectively, while formulas (29) and (31) will take the form

$$\bar{G}_3^2 = \frac{c_4}{c_1} + N^*, \tag{34}$$

$$\bar{G}_4^2 = \bar{G}_5^2 = \frac{c_3}{c_1}. \tag{35}$$

Note that relationship (33) coincides with formula (20b) in [16] if in the latter put the moments of the second order equal to zero and consider N_r to be a compressional force.

Thus, we have obtained four waves of strong discontinuity in the form of diverging circles (while there are five in [21]).

Two of them, the first and the second, are irrotational (longitudinal) waves, and their characteristics are defined by relationships (33) and (24), which show that the velocities of these waves depend on N^* and M_r^* and are independent of M_z^*. As this takes place, the discontinuities in the velocities of particles' displacements are directed along the radius of the circle.

Two other waves, the third and the fourth, are equivoluminal (shear) waves, and their characteristics are defined by relationships (28), (34) and (30), (35), respectively. Moreover, the third wave is the wave of transverse shear, since on its front the discontinuity in the velocity of particles' displacement is directed along the z-axis. On the fourth wave, the discontinuities in the velocities of particles' displacements are directed along the tangent to the circle (to the wave front), i.e., they locate in the plane of the plate.

Alternative graphs of the transient wave velocities as functions of the external force components N^* and M_r^* are presented in Figs. 1 and 2, respectively. The third figure is of no necessity, since all wave velocities are independent of the torsional moment M_z^*.

The velocity $\bar{G}_3^2$ depends only on N^*, and the N^*-dependence of $\bar{G}_3^2$ is linear (see a dotted line in Fig. 1). The velocity $\bar{G}_4^2 = \mathrm{const}$, i.e. it is independent of N^* and M_r^*.

Comparing the corresponding figures with each other, we ascertain that the character of the curves behaviour presented in Figs. 1 and 2 has nothing in common with those shown in Figs. 1L-3L [21]. This result has been proposed in the beginning of our study (maybe more precise to say 'our inquiry'). It seems likely that Figs. 1L-3L are figment of the imagination of the author of [21].

5 Loktev's approach for the analysis of impact response of a prestressed orthotropic plate involving curvilinear anisotropy

Now we proceed to the final part of [21], i.e., to Sect. 3.2 in [21], wherein the solution is constructed for a plate described by the set of five Eqs. (1L) which is impacted by a sphere. This part is the naturality of how the author's incompetence to obtain the lore either shelters behind scientific-like phrases or is replaced by the figment.

Thus, Loktev [21] writes:

> "To determine the contact force and the dynamic buckling of the plate at the point of impact interaction, the system of Eqs. (1L) is written in the Laplace space. For its solution, one can present the unknown displacements and the load $q(\tau, r, \theta)$ caused by the concentrated force of the interaction in the contact area $P(t)$ as a series of expansion in terms of the Legendre polynomial Ref. 16 (paper [32] in our list of references), which satisfy the boundary conditions (2L)

$$\tilde{x} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_{2n,m} P_{2n+1}\left(\cos\frac{\pi r}{2R}\right)\cos(m\theta), \tag{4L}$$

$$\tilde{q}_1 = \frac{P(p)}{\pi R_c^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(4n+3)P_{2n+1}\left(\cos\frac{\pi r_1}{2R}\right) \times P_{2n+1}\left(\cos\frac{\pi r}{2R}\right)\cos(m\theta),$$

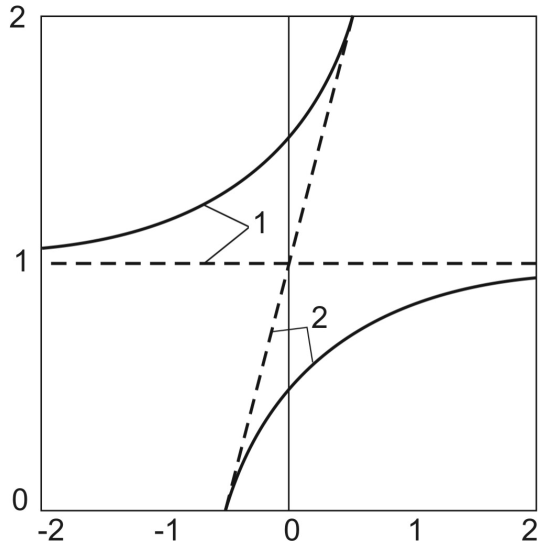

Figure 1: The characteristic curves of the N^*-dependence of squared velocities $\bar{G}^2_{1,2}$. The value M^*_r is chosen as a parameter which is assumed to be smaller than unit; dashed lines correspond to the case when $M^*_r = 0$

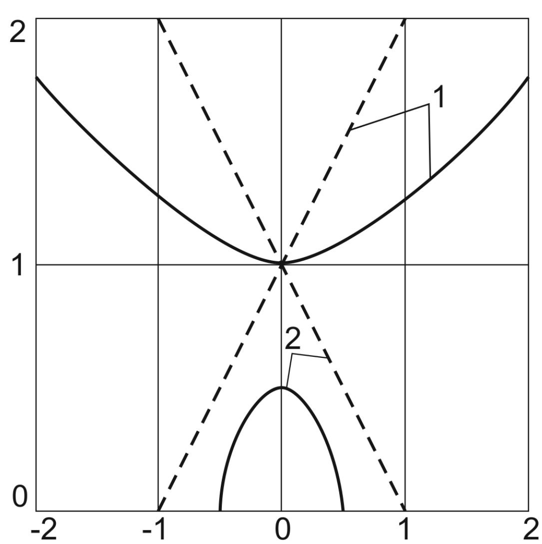

Figure 2: The characteristic curves of the M^*_r-dependence of squared velocities $\bar{G}^2_{1,2}$. The value N^* is chosen as a parameter which is assumed to be smaller than 1/2; dashed lines correspond to the case when $N^* = 0$

where r_1–the coordinate of the point, where the dynamic contact takes place; the tilde above the variable shows that this magnitude is used in the Laplace space."

The authors of this paper argue that the solution constructed by Loktev in Sect. 3.2 of [21] based on Eqs. (1L)-(4L) is no more than fiction. In troth, the following arguments support such a conclusion:

(1) The governing set of Eqs. (1L) is incorrect, and it has been erroneously written in the dimensionless form;

(2) Five governing equations should be subjected to five boundary conditions (3A), where the internal forces and moments are defined by (10)-(13), and seven initial conditions, since two additional initial conditions are needed for the initial value of the local bearing of the plate's material and its velocity at the point of contact interaction with an impactor according to the assumed contact law.

It is amazing that for Loktev [21] it is sufficient only two boundary conditions (2L) (it was shown above that the second condition is invalid for a plate under consideration), when he substituted expansions for five unknown functions $\tilde{x}$ in Eqs. (1L), as well as only two initial conditions (10L)!

(3) Odd spherical functions $P_{2n+1}\left(\cos\frac{\pi r}{2R}\right)$ involve the sums $\cos(2n+1)\frac{\pi r}{2R}$ $(n = 0, 1, 2, ...)$, but the correct boundary conditions (3A) for the plate under consideration contain the internal forces and moments (10)-(13) depending on the first r-derivatives of the five displacement functions. That is why after differentiating one time the functions $P_{2n+1}\left(\cos\frac{\pi r}{2R}\right)$, there would appear $\sin(2n+1)\frac{\pi r}{2R}$, which do not vanish at $r = R$.

It should be emphasized that at first glance the reason to search the solution for a circular plate impacted by a ball in terms of spherical functions but not in terms of cylindrical functions is not understandable at all. In the case of utilizing the latter, the cram would look like more convictive.

In this place, it is necessary to unlock Loktev's small secret. The matter is fact that the problem formulation and the approach for its solution were borrowed by Loktev from the paper by Biryukov and Kadomtsev [33], which for some reason (?!) was not included in the list of references, while only their earlier paper [32] was cited. As this takes place, all deficits of the borrowed paper [33] were automatically copied to the Loktev paper [21]. Thus, for example, formula (5L) for the amplitudes (4L) of the desired values $\tilde{x}$

$$x_{2n,m} = x^0_{2n,m}\varepsilon^0 + x^1_{2n,m}\varepsilon^1 + x^2_{2n,m}\varepsilon^2 + x^3_{2n,m}\varepsilon^3, \quad (5L)$$

where $\varepsilon = p^{-2}$, is written incorrectly (see similar formulas in [33]), because in this case the limiting theorem of the

Laplace transform

$$\lim_{p\to\infty} p\tilde{x} = x(0) = 0$$

does not fulfill.

The correct representation should be the following:

$$x_{2n,m} = \left(x^0_{2n,m}\varepsilon^0 + x^1_{2n,m}\varepsilon^1 + x^2_{2n,m}\varepsilon^2 + x^3_{2n,m}\varepsilon^3\right)\widetilde{P}(p),$$

where $\widetilde{P}(p)$ is the contact force in the Laplace domain.

Moreover, Loktev [21] falls over the scenario of [33] in such a way that he has forgotten about the differences in the objects under investigation: the target in [21] is a circular orthotropic plate possessing curvilinear anisotropy, the motion of which is described by five equations taking the rotary inertia and transverse shear deformations into account; while the target in [33] is a circular sector of a contour-hinged elastic isotropic spherical shell, the motion of which is described by three momentless equations of motion for spherical shells disregarding the rotary inertia and transverse shear deformations. That is why, Biryukov and Kadomtsev [33] used in their problem the spherical functions (4L) subjected to classical boundary conditions and classical Hertz's contact law, but Loktev [21] for his problem had to utilize cylindrical functions subjected to nonclassical boundary conditions (3A). But it was not the case. The spherical functions and classical boundary conditions were transferred from [33] into Loktev's paper [21], as well as the entire description of their algorithm for the solution construction.

(4) As it has been already mentioned above, the contact force should be applied at the pole of curvilinear anisotropy, i.e., in the center of the plate, otherwise it will be needed to introduce a new polar set of coordinates with the pole at the point of impact, what will result in the transformation of the elastic constants into the functions of both new coordinates and the coordinates of the force application point and in the increase of their number till 21. That is why, Loktev's statement about the solution of this problem for the case of the contact force application at any point of the plate is invalid.

(5) The impact on plates involves an interaction between plate deflection and indentation, therefore the choice of the contact law is highly important. As the relationship between the contact force and the indentation, Loktev [21] used

> "the Hertz's *classical* model, which describes the process of the interaction of two bodies at low initial velocities of the impact,
>
> $$\alpha(t) = bP(t)^{2/3}, \qquad (9L)$$
>
> where $b = ((9\pi^2(k_1+k)^2)/16R)^{1/3}$, $k_1 = (1-\sigma_1^2)/E_1$, $k = (1-\sigma_r\sigma_\theta)/E$, σ_1 and E_1– the Poisson's ratio and the module of rigidity of the indenter, respectively."

However, in the strict sense, *classical* formula (9L), which is called by Loktev four lines below as "*the local plastic compression*" (?!), with the contact rigidity coefficient b (this coefficient in Loktev's interpretation involves the plate radius R as the generalized curvature of the interacting bodies instead of $1/R_1$, and an undefined elastic constant E of the plate's material and its two in-plane Poisson's ratios) is inadmissible in this case.

To these authors' knowledge, the analytical function of the contact load in terms of the local bearing of the plate's material at the point of contact interaction has not yet been derived for orthotropic plates in the explicit form, since experimental observations and numerical calculations revealed that the impact force could not be assumed to be concentrated at a point, and the contact domain in the case of an orthotropic plate has an elliptic shape. Sveklo [34] suggested the contact theory for two anisotropic bodies under compression according to which the contact pressure is distributed over an elliptical contact region. The result of Sveklo's theory is the contact law for anisotropic bodies which is formally similar to Hertz's law, i.e.,

$$P(t) = K_H\alpha^{3/2}, \qquad (36)$$

where α is the relative approach between the contacting bodies, and K_H is a parameter to be evaluated through a very complicated procedure [12].

A procedure for determining some of the features of the contact problem for generally anisotropic materials has also been given by Willis [35], that involves numerical contour integration. Recent works by Swanson [36, 37] have shown how stresses and deformations can be determined throughout the contact region for contact of a half-space and for plates of finite thickness using the approach proposed in [35].

However, for engineering applications many approximate approaches have been suggested for evaluation of the contact stiffness coefficient K_H. First, the tentative approximations suitable for orthotropic plates, for example, were suggested in [38, 39], where coefficient K_H depends on the through-the-thickness modulus E_z and the through-thickness Poissons ratios defined by $\sigma_{rz} = \varepsilon_z/\varepsilon_r$ and $\sigma_{zr} = \varepsilon_r/\varepsilon_z$ under uniaxial loading in the r- and z-direction, respectively. It has been noted in [39] that the tentative approximation (which, generally speaking, could be used in (9L))

$$K_H = (1-\sigma_{rz}\sigma_{zr})/E_z \qquad (37)$$

underestimates the contact modulus of typical composite plates by 10–20%.

It is known fact that the features of many orthotropic materials are close to those of transversely isotropic materials. Thus, the problem of impact on transversely isotropic plates often is a suitable homogenized approximation of laminates having many orthotropic plies equally and regularly distributed in at least three directions. Another case, when the approximation by a transversely isotropic plate is appropriate, is that when the shape of the contact region is weakly elliptical [40].

For transversely isotropic plates, the contact region is circular, and thus Hertzian contact law can be used if the isotropic modulus is replaced by a combination of five independent, non-zero components of the stiffness or compliance tensor. It was shown by Turner [41] that the effective modulus for transversely isotropic quasi-static normal contact can be expressed by

$$E^*_{\mathrm{TI}} = \left(\frac{2}{\alpha_1 \alpha_3}\right), \tag{38}$$

where

$$\alpha_1 = \left(\frac{E_r/E_z - \sigma^2_{rz}}{1-\sigma^2_r}\right)^2,$$

$$\alpha_2 = \frac{1 + \left(\frac{E_r}{2G_{rz}} - 1\right) - \sigma_{rz}(1+\sigma_r)}{1-\sigma^2_r},$$

$$\alpha_3 = \left(\frac{\alpha_1 + \alpha_2}{2}\right)^{1/2} \left(\frac{1-\sigma_r}{G_r}\right),$$

what allows to construct the solution for transversely isotropic plates in the closed form.

Some researchers [42, 43, 15] utilized a linear contact law, i.e., considered the linearized Hertzian contact deformations, in order to investigate the influence of plate's orthotropic properties on the dynamic characteristics of the impact process, in so doing using different equations describing the motion of orthotropic plates [1].

(6) At the end of Sect. 3.2 in [21] one could read the following:

> "After the substitution of the expression for the target's buckling (7L) at the given point, i.e., at the fixed values of the coordinates r, θ and *the local plastic compression* (9L), into Eq. (8L) and taking into account the conditions (10L), a nonlinear integro-differential equation with regard to the contact force is obtained, which can be solved by using the iterative scheme [Refs 16 and 17] (papers [32] and [44] in our list of references)."

But neither the governing equation nor the iterative scheme are presented in [21]. Why did not Loktev do this? The answer is very simple: because such an equation could not be derived in principle.

We consider that the above arguments are enough to discredit the solution presented in Sect. 3.2 [21].

Finally, Sect. 4 of [21] terminates the long chain of mystifications, wherein one could read a 'serious discussion' on how the curves in Figs. (1L)-(6L) were constructed (even without indicating the coordinates of the point of impact interaction!). Thus, the fictional solution is finalized by the fictitious figures. Finita la Commedia!

But this is not the end of this story. Mr Loktev inebriated by his success in publishing [21] has submitted and published the same paper in Russian [45] ignoring the Springer's Copyright on [21]. Of course, two published papers are much better than one!

6 Alternative approaches for the analysis of impact response of a pre-stressed orthotropic plate involving curvilinear anisotropy

Since the method described by Loktev [21] for solving the problem of the impact response of a pre-stressed orthotropic plate involving curvilinear anisotropy will take us nowhere, below we propose a simple and fine method allowing us to resolve this problem.

Thus, let us analyze the dynamic behaviour of a circular pre-stressed orthotropic plate possessing curvilinear anisotropy. At the moment $t = 0$ the plate is impacted at its center, i.e., at the pole of curvilinear anisotropy, by a spherical mass moving with the velocity V_0. The radius of the plate is presumed to be sufficiently large, in order that the waves reflected from its edges arrive at the contact region after the rebounce of the impactor from the target. In this case, the boundary conditions (3A) could be neglected, since the plate is assumed to be of infinite extent.

As an axially symmetric problem is considered, the functions w, u, and $\varphi_r = \varphi$ are independent of θ, while $v = \varphi_\theta = 0$, and the set of five Eqs. (5)-(8) is reduced to the following set of three equations with due account for the pre-stress loads N and M_r:

$$\left(1 + \frac{N}{C_r}\right)\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\,\frac{\partial u}{\partial r}\right) + \frac{M_r}{C_r}\left(\frac{\partial^2 \varphi_r}{\partial r^2} + \frac{1}{r}\,\frac{\partial \varphi_r}{\partial r}\right) - \frac{\sigma_\theta}{\sigma_r}\,\frac{u}{r^2} = \frac{\rho}{C_r}\,\ddot{u}, \tag{39}$$

$$\left(1 + \frac{N}{hKG_{rz}}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\,\frac{\partial w}{\partial r}\right) - \frac{\partial \varphi_r}{\partial r} - \frac{\varphi_r}{r} = \frac{\rho}{KG_{rz}}\,\ddot{w}, \tag{40}$$

$$\frac{M_r}{D_r}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\,\frac{\partial u}{\partial r}\right) + \frac{\partial^2 \varphi_r}{\partial r^2} + \frac{1}{r}\,\frac{\partial \varphi_r}{\partial r} - \frac{\sigma_\theta}{\sigma_r}\,\frac{\varphi_r}{r^2} + \frac{hKG_{rz}}{D_r}\left(\frac{\partial w}{\partial r} - \varphi_r\right) = \frac{\rho h^3}{12 D_r}\,\ddot{\varphi}_r. \tag{41}$$

As it has been discussed above, in the general case of contact loading of orthotropic materials the contact zone has an elliptical shape, that is why the fronts of the transient waves, which are generated in the target at the moment of impact, have also elliptical form, resulting in the distortion of rays (normal trajectories to the wave fronts). This, in its turn, results in the fact that the velocities of the transient waves begin to be dependent on the direction of propagation (see, for example, the behaviour of flexural wave during the impact on an orthotropic glass/epoxy composite laminate photographed with pulsed laser holography presented in [46]), what significantly complicates the solution of problems of such a kind and finally does not allow one to obtain the solution in the analytical form.

Thus, the above Eqs. (39)-(41) involving two different in-plane Poisson's rations σ_r and σ_θ subjected to the impact loading will initiate the propagation of transient waves with elliptical-shaped fronts. The analytical solution of this problem via the ray method is impossible due to the distortion of the rays.

But in the case when $\sigma_r \approx \sigma_\theta$ the contact region will be weakly elliptical, and we could use a transversely isotropic plate as a good approximation. The equations of motion of the pre-stressed transversely isotropic circular plate with due account for rotary inertia and transverse shear deformations have the form [47]

$$\left(1+\frac{N}{D}\right)\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\,\frac{\partial u}{\partial r}\right)-\frac{u}{r^2} + \frac{M_r}{D}\left(\frac{\partial^2 \varphi_r}{\partial r^2}+\frac{1}{r}\,\frac{\partial \varphi_r}{\partial r}\right)=\frac{\rho}{E'}\,\ddot{u}, \tag{42}$$

$$\left(1+\frac{N}{hKG_{rz}}\right)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\,\frac{\partial w}{\partial r}\right) - \frac{\partial \varphi_r}{\partial r}-\frac{\varphi_r}{r}=\frac{\rho}{KG_{rz}}\,\ddot{w}, \tag{43}$$

$$\frac{M_r}{D^*}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\,\frac{\partial u}{\partial r}\right)+\frac{\partial^2 \varphi_r}{\partial r^2}+\frac{1}{r}\,\frac{\partial \varphi_r}{\partial r} -\frac{\varphi_r}{r^2}+\frac{hKG_{rz}}{D^*}\left(\frac{\partial w}{\partial r}-\varphi_r\right)=\frac{\rho h^3}{12E'}\,\ddot{\varphi}_r, \tag{44}$$

where $E' = E/(1-\sigma^2)$, $\sigma = \sigma_r = \sigma_\theta$, $D = E'h$, and $D^* = E'h^3/12$.

From the comparison of (39)-(41) and (42)-(44) it is evident that the equations of motion of the pre-stressed orthotropic plate (39)-(41) are reduced to those for the pre-stressed transversely isotropic plate (42)-(44) as $\sigma_r \to \sigma_\theta$. Equations (42)-(44) were used in [16] to study the dynamic response of a pre-stressed transversely isotropic plate impacted by an elastic flat-end rod.

Based on the reasoning presented in Sec.4 it is evident that Eqs. (42)-(44) admit the propagation of three waves of strong discontinuity in the form of diverging circles. Two of them, the first and the second, are irrotational (longitudinal) waves, the velocities $G_{1,2}^2$ of which depend on the values of N^* and M_r^* (33). The third wave is the equivoluminal wave of transverse shear propagating with the velocity $G_3^2 = c_3^2 + N^*$, where $c_3^2 = KG_{rz}\varrho^{-1}$, and $N^* = N(\varrho h)^{-1}$.

The main force acting along the perimeter of the circular contact spot is the transverse force Q_r, which is defined by formula (3) with due account for the condition of compatibility of the first order (15) and has the form

$$Q_r = -\rho h c_3^2 G_3^{-1} W, \tag{45}$$

where $W = \dot{w}$.

The equation of motion of the contact spot with the radius a and the equation of motion of the impactor with the radius $R_{\rm im}$ are written, respectively, as

$$\rho h\pi a^2 \dot{W} = 2\pi a Q_r + P(t), \tag{46}$$

$$m(\ddot{\alpha}+\dot{W}) = -P(t). \tag{47}$$

where m is the mass of the impacting body, $P(t)$ is the force of the contact interaction between the impactor and the target, and the radius a of the contact region is

$$a(t)^2 = R_{\rm im}\alpha(t). \tag{48}$$

The contact force $P(t)$ is defined by Eq. (36) with the contact stiffness K_H evaluated as

$$K_H = \frac{4}{3}\,\sqrt{R_{\rm im}}E^*, \tag{49}$$

where E^* is the effective modulus for normal contact of the elastic impactor and transversely isotropic plate

$$\frac{1}{E^*} = \frac{1-\sigma_{\rm im}^2}{E_{\rm im}} + \frac{1}{E_{\rm TI}}, \tag{50}$$

$E_{\rm im}$ and $\sigma_{\rm im}$ are impactor's elastic modulus and Poisson's ratio, respectively, and $E_{\rm TI}$ could be calculated applying Turner's [41] formula (38).

Eliminating the value W from Eqs. (46) and (47) and considering formula (45), we are led to the functional equation in the value α characterizing the relative approach between the contacting bodies

$$\alpha^{1/2}\left(\ddot{\alpha}+\frac{K_H}{m}\,\alpha^{3/2}\right)+\frac{K_H}{\rho h\pi R_{\rm im}}\,\alpha +dG_3^{-1}\left(\dot{\alpha}+\frac{K_H}{m}\int_0^t \alpha^{3/2}(t_1)dt_1\right) = dG_3^{-1}V_0, \tag{51}$$

where $d = 2R^{-1/2}c_3^2$.

The initial conditions

$$\dot{\alpha}\Big|_{t=0} = V_0, \quad W\Big|_{t=0} = 0 \tag{52}$$

have been considered during the deduction of (51).

We shall seek the solution of Eq. (51) in the form of the series

$$\alpha(t) = V_0 t + \alpha_1 t^{5/2} + b_1 t^3 + \alpha_2 t^{7/2} + b_2 t^4 + ... \quad (53)$$

Substituting (53) in Eq. (51) and equating the coefficients at equal powers of t, we obtain

$$\begin{aligned}
\alpha_1 &= -\frac{4}{15}\,\frac{K_H V_0^{1/2}}{\rho h \pi R_{\text{im}}} < 0, \\
b_1 &= \frac{1}{9}\,\frac{K_H d}{\rho h \pi R_{\text{im}} G_3} > 0, \\
\alpha_2 &= -\frac{4}{35}\,\frac{K_H}{V_0^{1/2}}\left(\frac{V_0^2}{m} + \frac{1}{3}\,\frac{d^2}{\rho h \pi R_{\text{im}} G_3^2}\right) < 0, \\
b_2 &= \frac{1}{90}\,\frac{K_H}{\rho h \pi R_{\text{im}}}\left(\frac{K_H}{\rho h \pi R_{\text{im}}} + \frac{d^3}{V_0 G_3^3}\right) > 0
\end{aligned} \quad (54)$$

Reference to (54) shows that the series (53) is the alternating series, i.e. it describes oscillating motions.

With the increase in the tensile force, in the limiting case when $N \to +\infty$ and $G_3 \to \infty$, the particular solution for α takes the form

$$\alpha(t) = V_0 t - \frac{4}{15}\,\frac{K_H V_0^{1/2}}{\rho h \pi R_{\text{im}}} t^{5/2} - \frac{4}{35}\,\frac{K_H V_0^{3/2}}{m} t^{7/2} + \frac{1}{90}\left(\frac{K_H}{\rho h \pi R_{\text{im}}}\right)^2 t^4. \quad (55)$$

Substituting the found α in Eq. (36), we could calculate the contact force $P(t)$.

Note that the solution for the case of an isotropic plate could be obtained via the substitution of

7 Numerical example

As an example, let us consider the impact of a steel sphere of radius $R_{\text{im}} = 25.4$ mm with the velocity $V_0 = 10$ m/s upon a carbon/epoxy plate of thickness $h = 8$ mm, the elastic constants of which are taken from [37]: $E_r = 51.3$ GPa, $E_z = 12$ GPa, $E^*_{\text{TI}} = 14.61$ GPa, $G_{rz} = 6$ GPa, $\sigma_r = 0.292$, and $\sigma_{rz} = 0.28$.

The dimensionless time $\tilde{t}$-dependence of the dimensionless contact force $\widetilde{P}$ calculated according to Eqs. (36) and (53)-(55) is presented in Fig. 3 for different levels of the press-stress force N^* including the limiting case $N \to +\infty$, where the dimensionless values are the following:

$$\widetilde{P} = \widetilde{K}_H \tilde{\alpha}^{3/2}, \quad \widetilde{K}_H = \frac{K_H}{E_{\text{im}} h^{1/2}}, \quad \tilde{t} = \frac{V_0}{h}\,t, \quad \tilde{\alpha} = \frac{\alpha}{h}.$$

Reference to Fig. 3 shows that the curve corresponding to the plate free from initial stresses, i.e., when $N^* = 0$, is the separatrix dividing the curves constructed for tensile $N^* > 0$ and compression $N^* < 0$ preloading. It is evident

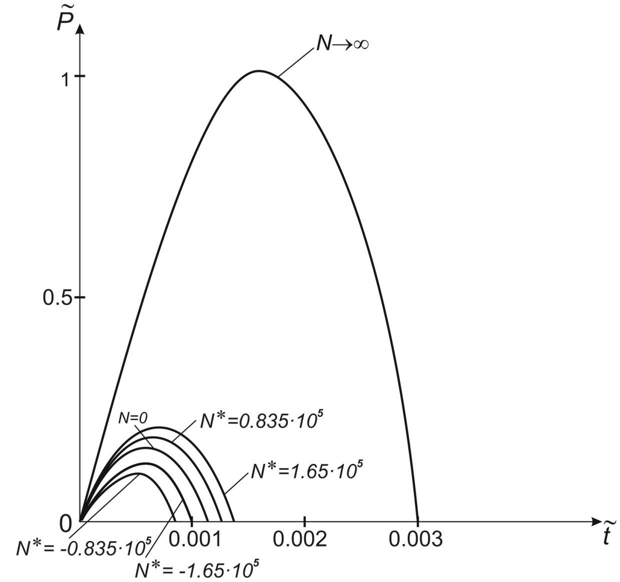

Figure 3: The dimensionless time-dependence of the dimensionless contact force

that the increase in N^* results in the increase of the contact duration and the maximum of the contact force.

It is well known that "the difficulty of compressive preloading is the risk of initial plate buckling as a stability failure of the relatively thin specimens under in-plane compression" [14]. Since the aim of our study is to analyze the impact response of pre-stressed plates, which show no inital deflections, then in order to avoid initial buckling we should limit the amount of pre-stress to a moderate level, what affects the evaluation of the preload effect. That is why the second limiting case, when $N^* \to c_3^2$ and $G_3 \to 0$, is considered to be unachieved in engineering practice.

8 Particular cases

As we have already mentioned above, the review of papers analyzing the impact response of elastic isotropic and anisotropic plates, as well as viscoelastic plates could be found in [1, 2]. However recently our attention was attracted by another paper by Loktev [48] published online by Elsevier on October 2, 2011 to appear soon in the *International Journal of Engineering Science*.

This paper [48] could be divided into two parts. Its first part involves the formulation of the problem of impact response of an elastic plate, the equations of motion of which take the rotary inertia and transverse shear deformation into account, as well as the method of its solution. The Maxwell viscoelastic model has been utilized for describing the process of the contact interaction of an impactor with the plate. Elastoplastic models are discussed in the second part.

8.1 A viscoelastic model of the shock interaction of a rigid body with a plate

We contend that the first part of Loktev's paper [48] is the pure plagiarism, since this problem in more general formulation was solved in the paper by Rossikhin and Shitikova [18], which was submitted to the *Journal of Engineering Mathematics* on September 27, 2005, published online on April 27, 2007, and its hard version appeared at the beginning of 2008.

First of all it should be emphasized that the formulation of the problem of the contact interaction of the impactor with an elastic plate of the Mindlin-Uflyand type, hyperbolic equations of motion of which could be written either in Cartesian or in polar coordinates, was proposed by Rossikhin and Shitikova [49] in 1994 in order to study wave phenomena under transverse impact. The wave approach for solving the problem has been suggested using the ray series expansions, either one-term or multiple-term, proposed by Achenbach and Reddy [24] in 1967 (one again the author of [48] has credited himself the authorship of the ray series, as he had done it before in [21]). The ray method has been extended by Rossikhin since 1968 and further by Rossikhin and Shitikova with respect to dynamic contact and impact problems, resulting in the state-of-the-art article [28] and the D.Sci. thesis by Professor Shitikova [50].

Thus, the problem formulation and one-term ray expansions, i.e., Eqs. (1)-(9) in [48] as well as his Figure 1c, were actually published in 1994 (at that time Mr Loktev was an ordinary schoolboy) by Rossikhin and Shitikova [49] (see Eqs. (2.1)-(2.9) and (6.1)-(6.6) together with Fig. 2), wherein the contact force was determined according to the Hertzian law.

As for the viscoelastic model of the shock interaction of a rigid body with a plate, then for this purpose the generalized fractional-derivative Maxwell model has been suggested in [18], in so doing the ray method is used outside the contact domain, while the Laplace-transform technique is utilized within the contact region. The particular case, when the fractional parameter (the order of the fractional derivative) is equal to the unit, has been considered also, and the solution for the traditional Maxwell model, which is 'studied' in [48], has been obtained and analysed.

Moreover, in Sect. 3.5 "Modeling the Contact Interaction of Thin Bodies via a Linear Elastic Spring in Series with a Damper" of the state-of-the-art article [1], which was published in July of 2007, i.e., earlier than the paper [18], the formulation of this problem and its analytical solution by the Laplace-transform method could be found both for the conventional and fractional-derivative Maxwell models (see Eqs. (156)-(166) together with Figs. 28-31).

Therefore, the problem formulation, the method of solution and governing equations were suggested, analyzed in detail and analytically solved by Rossikhin and Shitikova [1, 18]. The author of [48] has copied completely all these main aspects including notations (e.g. compare the coefficients of the characteristic Eq. (18) from [18] with that in [48] Eq. (15)), and a scheme of the shock interaction (comparison of Figs. 1a,b in [18] and Fig. 1a,b in [48] shows that Mr. Loktev has copied these figures as well with the only difference that the spring-dashpot element is connected with a falling mass, but this fact does not influence the governing equations at all).

Thus, all ideas and derivation of Eqs. (1)-(15) and Fig. 1 from [48] were previously published in [1, 18]. However, the final solution for the contact force in the time domain obtained by Rossikhin and Shitikova [1, 18] is distinguished drastically from that in [48]. This could be explained by the fact the author of [48] is not able to follow out the mathematical treatment thoroughly. Really, the expressions for the displacements of the upper α and lower w ends of the spring-dashpot element in the Laplace domain have been written by Loktev [48] in the form of Eqs. (13) and (14), respectively, as

$$\bar{\alpha} = \frac{V_0[(\zeta+p)(p+B)+A]}{pf(p)}, \qquad \text{(L1)}$$

$$\bar{w} = \bar{\alpha}\,\frac{p(\zeta+p)+C_0}{-p(\zeta+p)+C_0} - \frac{V_0(\zeta+p)}{[-p(\zeta+p)+C_0]p}, \qquad \text{(L2)}$$

where $f(p) = p^3 + (\zeta+B)p^2 + (C+B\zeta)p + BC_0$, p is the transform parameter, an overbar denotes the Laplace transform of the given function, $\zeta = \tau_\varepsilon^{-1}$, τ_ε is the relaxation time, $A = E_1 M^{-1}$, $B = 2r_0^{-1}G_2$, $C = E_1(2M^{-1} + m^{-1})$, $C_0 = E_1 m^{-1}$, $M = \rho\pi r_0^2 h$ and m are the masses of the contact domain and the impactor, respectively, E_1 is the elastic coefficient of the spring, V_0 is the velocity of the impactor at the moment of impact, $G_2 = \sqrt{K\mu/\rho}$ is the velocity of the transient quasi-transverse wave propagating in the target during the process of the contact interaction, ρ and h are the density and the thickness of the plate, respectively, μ is the shear modulus, $K = \pi^2/12$ is the shear coefficient, and r_0 is the radius of the contact region. Loktev's expressions cited from [48] are labeled as L-equations.

If we continue the treatment of the above formula for $\bar{w}$ (L2), then we arrive at the compact formula (see Eq. (14b) in [18] at $\gamma = 1$, where γ is the fractional parameter)

$$\bar{w} = V_0\,\frac{A}{pf(p)}, \qquad (1\mathrm{R}-\mathrm{S})$$

where from hereafter Rossikhin-Shitikova equations are labeled as R-S.

In order to write the relationship for the contact force $F(p)$, first it is necessary to find the difference $\bar{\alpha} - \bar{w}$

$$\bar{\alpha} - \bar{w} = V_0\,\frac{(\zeta+p)(p+B)}{pf(p)}, \qquad (2\mathrm{R}-\mathrm{S})$$

then its substitution into the formula for the contact force in the Laplace domain results in Eq. (17) by Rossikhin and

Shitikova [18]

$$\bar{F}(p) = E_1(\bar{\alpha} - \bar{w})\frac{p}{p+\zeta} = E_1 V_0\, \frac{p+B}{f(p)}. \quad (3\mathrm{R}-\mathrm{S})$$

Inverting from the Laplace domain to the time domain, we could obtain the final solution for the contact force within the region of vibrations (Eq. (33a) in [18])

$$F(t) = g\mathrm{e}^{-\beta t} + Q\omega^{-1}\mathrm{e}^{-\xi t}\sin(\omega t - \varphi), \qquad (4\mathrm{R}-\mathrm{S})$$

where $p_3 = -\beta$ is the real negative root of equation $f(p) = 0$, and $p_{1,2} = -\xi \pm \mathrm{i}\omega$ are its two complex conjugate roots, and in the region of aperiodicity (Eq. (33b) in [18])

$$F(t) = E_1 V_0 \left(a_1\mathrm{e}^{-\alpha_1 t} + b_1\mathrm{e}^{-\beta_1 t} + d_1\mathrm{e}^{-\gamma_1 t}\right), \; (5\mathrm{R}-\mathrm{S})$$

where $p_1 = -\alpha_1$, $p_2 = -\beta_1$, and $p_3 = -\gamma_1$ are three real negative roots of equation $f(p) = 0$, and the coefficients involving in these two formulas are the following:

$$g = E_1 V_0\, \frac{B-\beta}{\beta^2 - 2\beta\xi + \xi^2 + \omega^2},$$

$$d_1 = \frac{B-\gamma_1}{(\alpha_1-\gamma_1)(\beta_1-\gamma_1)},$$

$$Q = E_1 V_0\sqrt{a^2\omega^2 + (\xi a - b)^2}, \quad \tan\varphi = \frac{a\omega}{\xi a - b},$$

$$a = -g, \quad b = 1 - a\beta - 2\xi g, \qquad (6\mathrm{R}-\mathrm{S})$$

$$a_1 = \frac{B-\alpha_1}{(\beta_1-\alpha_1)(\gamma_1-\alpha_1)}, \quad b_1 = \frac{B-\beta_1}{(\alpha_1-\beta_1)(\gamma_1-\beta_1)}.$$

For the comparison purpose, we shall now reproduce formulas for the contact force presented by Loktev [48]: his Eq. (25) for the vibratory regime

$$\begin{aligned} F(t) = E_1 \Bigg\{ &\mathrm{e}^{-at/2}\left\{\xi^{-1/2}\sin\left(\frac{1}{2}\xi^{1/2}t\right)[2(B_3 - B_5)\right. \\ &\left.+a(A_5 - A_3)] - (A_5 - A_3)\cos\left(\frac{1}{2}\xi^{1/2}t\right)\right\} + (C_3 - C_5)\mathrm{e}^{a_3 t} \\ &+(D_3 - D_5 - G_5) - (E_5 + H_5)\mathrm{e}^{a_9 t} - (F_5 + K_5)\mathrm{e}^{a_{10} t}\Bigg\} \\ &-\frac{E_1}{\tau_1}\int_0^t \Bigg\{ \mathrm{e}^{-at/2}\left\{\xi^{-1/2}\sin\left(\frac{1}{2}\xi^{1/2}t\right)[2(B_3 - B_5)\right. \\ &\left.+a(A_5 - A_3)] - (A_5 - A_3)\sin\left(\frac{1}{2}\xi^{1/2}t\right)\right\} \\ &+(C_3 - C_5)\mathrm{e}^{a_3 t} + (D_3 - D_5 - G_5) - (E_5 + H_5)\mathrm{e}^{a_9 t} \\ &-(F_5 + K_5)\mathrm{e}^{a_{10} t}\Bigg\}\mathrm{e}^{-(t-t')/\tau}dt', \end{aligned} \qquad (\mathrm{L3})$$

and his Eq. (24) for the aperiodic regime

$$\begin{aligned} F(t) = E_1 \big[&(A_2 - A_4)\mathrm{e}^{a_1 t} + (B_2 - B_4)\mathrm{e}^{a_2 t} \\ &+(C_2 - C_4)\mathrm{e}^{a_3 t} + (D_2 - D_4 - G_4) \\ &-(E_4 + H_4)\mathrm{e}^{a_7 t} - (F_4 + K_4)\mathrm{e}^{a_8 t}\big] \\ -\frac{E_1}{\tau_1}\int_0^t \big\{ &(A_2 - A_4)\mathrm{e}^{a_1 t} + (B_2 - B_4)\mathrm{e}^{a_2 t} \\ &+(C_2 - C_4)\mathrm{e}^{a_3 t} + (D_2 - D_4 - G_4) \\ &-(E_4 + H_4)\mathrm{e}^{a_7 t} - (F_4 + K_4)\mathrm{e}^{a_8 t}\big\}\,\mathrm{e}^{-(t-t')/\tau}dt', \end{aligned} \quad (\mathrm{L4})$$

where all cumbersome coefficients could be found in [48].

For the first glance, these two cumbersome relationships (L3) and (L4), which are inconvenient for engineering applications, differ significantly from two compact expressions (4R-S) and (5R-S) obtained in [1, 18]. But it is not the case. If one integrates the second parts in (L3) and (L4), what could be easily carried out, and further gathers the similar terms, then Rossikhin-Shitikova formulas (4R-S) and (5R-S) could be obtained, the physical meaning of which is understandable for any engineer. This brings up two questions: (1) Why Loktev did not do this integration (since all terms are table integrals) and leave off at Eqs. (L3) and (L4)? and (2) How could he manage to calculate the contact force directly using (L3) and (L4) (no mention has been made of the procedure used for this purpose)?

It seems likely that the author of [48] wrote his cumbersome expressions (L3) and (L4) in order to cover the tracks indicating the presence of plagiarism and carried out all calculations presented graphically in his Fig. 3 on the basis of Rossikhin-Shitikova formulas (4R-S) and (5R-S).

8.2 Impact response of isotropic plates of the Uflyand-Mindlin type

As for the second part, then its content is constructed on the hanky-panky tricks. In order to show this, let us analyze the deduction of its governing equation, i.e., Eq. (26) in [48],

$$\alpha\left(-\frac{F(t)}{m} - \ddot{\alpha}\right) = -\frac{2G_2}{R_{\mathrm{im}}^{1/2}}\alpha^{1/2}\left[-\frac{1}{m}\int_0^t F(t)dt - \dot{\alpha}\right] + \frac{F(t)}{\rho\pi R_{\mathrm{im}} h}. \qquad (\mathrm{L5})$$

To derive the functional equation, one should start from the equations of motion of the impactor (Eq. (1) in [18])

$$m(\ddot{\alpha} + \ddot{w}) = -F(t), \qquad (7\mathrm{R}-\mathrm{S})$$

and of the contact region (Eq. (2) in [18])

$$\rho\pi r_1^2 h\ddot{w} = 2\pi r_1 Q_r(t)|_{r=r_1} + F(t), \qquad (8\mathrm{R}-\mathrm{S})$$

subjected to the initial conditions

$$\alpha|_{t=0} = w|_{t=0} = \dot{w}|_{t=0} = 0, \quad \dot{\alpha}|_{t=0} = V_0, \quad (9\mathrm{R}-\mathrm{S})$$

where $r_1(t)$ is the radius of the contact region, and the transverse force acting at the boundary of the contact domain is defined by the dynamic condition of compatibility [49]

$$Q_r = -\rho G_2 h\dot{w}. \qquad (10\mathrm{R}-\mathrm{S})$$

It should be emphasized here that Eq. (2) in [18] has a typing error, since a minus sign was misprinted after the sign of equality ahead of the term involving the transverse force Q_r, while all further mathematical treatment was carried out correctly with a positive sign before this term, and in our Fig. 1 the direction of the transverse forces coincides with that of the contact force. Even this our misprint was carefully copied by Loktev [48] in his Eq. (2)!

Integrating Eq. (8R-S) twice with due account for the initial conditions (9R-S) yields

$$w = V_0 t - \alpha - \frac{1}{m}\int_0^t dt_1 \int_0^{t_1} F(t_1) dt_2, \quad (11\mathrm{R-S})$$

Substituting Q_r (10R-S) and the above found expression (11R-S) for w in Eq. (7R-S), we could obtain

$$r_1^2\left(-\frac{F(t)}{m} - \ddot{\alpha}\right) = -2G_2 r_1 \left[V_0 - \frac{1}{m}\int_0^t F(t)dt - \dot{\alpha}\right] + \frac{F(t)}{\rho\pi h}. \quad (12\mathrm{R-S})$$

If we utilize the formula for the radius of the contact domain r_1, which was used in [48], i.e.,

$$r_1^2 = R_{\mathrm{im}}\alpha, \quad (13\mathrm{R-S})$$

then the contact force $F(t)$ should be substituted by

$$F(t) = k\alpha^{3/2}, \quad (14\mathrm{R-S})$$

because these two relationships are interconnected since they are obtained from the Hertz problem about elastic compression of two bodies [51]. Thus, Eq. (12R-S) takes the form

$$\alpha\left(\frac{k\alpha^{3/2}}{m} + \ddot{\alpha}\right) + \frac{2G_2}{R_{\mathrm{im}}^{1/2}}\alpha^{1/2}\left[\frac{k}{m}\int_0^t \alpha^{3/2}dt + \dot{\alpha} - V_0\right] + \frac{k\alpha^{3/2}}{\rho\pi h R_{\mathrm{im}}} = 0, \quad (15\mathrm{R-S})$$

where k is the contact stiffness coefficient depending on the properties of the interacting bodies, which is evaluated as

$$k = \frac{4}{3}\sqrt{R}E^*, \quad (16\mathrm{R-S})$$

and E^* is the effective modulus for normal contact of the elastic impactor and elastic isotropic plate

$$\frac{1}{E^*} = \frac{1-\sigma_{\mathrm{im}}^2}{E_{\mathrm{im}}} + \frac{1-\sigma^2}{E}, \quad (17\mathrm{R-S})$$

and E and σ are plate's elastic modulus and Poisson's ratio, respectively.

The solution of Eq. (15R-S) for the isotropic plate could be easily obtained from that for the transversely isotropic plate, i.e., in the form of the series (53) with the coefficients (54), wherein R_H and d/G_3 should be replaced, respectively, by k defined in (16R-S) and $2G_2R_{\mathrm{im}}^{-1/2}$.

And now attention, please! What has Mr. Loktev performed? Yenning to generalize this approach with the purpose of considering elastoplastic deformations within the contact area and trying to suppress the fact that Eq. (15R-S) has been derived only for the case of elastic interactions, the author of [48] has performed a neat conjuring trick substituting $k\alpha^{3/2}$ by $F(t)$. As this takes place, the initial conditions (9R-S) have been forgotten. As a result of such manipulations, the functional equation takes a rather strange form of Eq. (5L), which withal lacks the initial velocity V_0 (compare it with the functional Eq. (15R-S)).

Therefore, Eq. (5L), which is utilized in [48], is valid *only* for the elastic contact interaction (of course if the boundary conditions (9R-S) are taken correctly into account), and it is unsuitable for considering plastic deformations, as this has been done in [48] via the substitution of α by the elastoplastic models, which Loktev [48] had borrowed once again from Biryukov and Kadomtsev [32].

All absurdities discussed above are enhanced by an illiterate paper review presented in Introduction of [48]. It seems likely that the author of [48] did not even see the majority of the cited papers. Thus, he wrote in the second sentence of Introduction that an elastoplastic model of contact interaction was considered in [26], but neither contact nor impact problems were discussed by Uflyand [26].

Further in the fourth sentence, to our great surprise we find the statement that the impact response of “the classical Kirchhoff plate was investigated by Filippov [52], Loktev [25], Mittal [53], Rossikhin and Shitikova [49]”, while “dynamic characteristics of the nonclassical Uflyand-Mindlin plate were determined in [44]”. As for these authors, we are dealing only with the wave equations of thin bodies, since we develop the wave approach for solving impact problems proposed in our papers many years ago (see our review article [1], and the Kirchhoff plate equations do not belong to this class. Filippov [52] considered the transverse elastic impact upon a circular elastic isotropic plate with due account for the rotary inertia and transverse shear deformations, i.e., he studied the nonclassical plate. Mittal [53] investigated the impact response of a transversely isotropic plate considering the transverse shear deformations but ignoring the rotary inertia, i.e., he also studied the nonclassical plate. As for Zheng and Binienda [44], then their analysis was based on Kirchhoff's plate theory for specially orthotropic composite plates, and the authors clear formulated this during the derivation of the governing equations.

Thus, the majority of references are incorrect, and its full enumeration will be boring for a reader.

But the main mystification concerns Loktev's paper [25], wherein he considered the *elastic impact* of a falling mass against an elastic specially orthotropic plate, equations of motion of which take the rotary inertia and transverse shear deformations into account, i.e., a nonclassical plate model was utilized. Moreover, he cited his paper [25] several times attributing it non-existing facts. In Introduction of [48], he mentioned that

> "as the experimental and theoretical results show (Loktev, 2005) (paper [25] in our list of references) if the initial velocity of impact exceeds a certain value, the elastoplastic and viscoelastic properties of the interacting bodies substantially influence the dynamic characteristics",

but neither experimental research or discussion about the influence of the impact velocity on the choice of the elastoplastic or viscoelastic models of contact interaction could be found in [25]. Last but not least, in the caption to Fig. 4, wherein time dependences of the contact force are presented for the *viscoelastic isotropic model* of contact interaction, one could read that "curve 5 is obtained from Loktev (2005) (paper [25] in our list of references) taking into account five terms of the ray series for $\tau = 0.001$, where τ is the relaxation time". Isn't it surprising that in 2011 the author of [48] 'has forgotten' what he did in 2005?!

It should be added that the paper under consideration [48] abounds both the principal errors [see e.g. Eqs. (10), (26)] and misprints [e.g. in Eqs. (2), (3), (28), in the coefficients after Eq. (29)] in equations throughout the paper. We wonder, for example, how Fig. 5 could be constructed, since wire-drawn governing Eq. (5L) lacks the initial velocity V_0! It seems likely that Fig. 5 is simply the figment of Loktev's imagination [48].

In the conclusion it should be emphasized once again that (1) the first part of Loktev's paper [48] is the cribbage of the papers by Rossikhin and Shitikova [1, 18], and (2) its second part is the trash based on the incorrect governing equation.

One more intriguing fact, which characterizes the author of [48] in a full measure: the second part of Loktev's paper was published by him in 2007 [54] in Russian journal *Pis'ma v Zhurnal Technicheskoi Fiziki* which is translated into English as *Technical Physics Letters*, in so doing all absurdities and errors made in Loktev [54] were irreflectively copied in Loktev [48]. For an obvious reason Mr. Loktev did not include his previous publication [54] in the list of references in Loktev [48].

The surprising thing is that such a paper could receive positive response from a reviewer, or reviewers. It has been formed an impression that persons reviewed Loktev's paper are not the experts in the field of the impact theory and they are not acquainted with the current literature in this field. But nowadays an editor of any research journal could find qualified experts in each, even narrow, field without difficulties due to the presence of modern data bases such as SCOPUS and others.

Papers of such a kind as Loktev's [48] and [54] are harmful for the mechanical research community and first of all for young and untutored researchers.

9 Conclusion

The problem on normal low-velocity impact of an elastic falling body upon a circular pre-stressed orthotropic plate possessing curvilinear anisotropy has been studied using the wave theory of impact with due account for the changes in the geometrical dimensions of the contact domain. At the moment of impact, shock waves (surfaces of strong discontinuity) are generated in the target, which then propagate along its median surface as 'diverging circles' during the process of impact. The classification of transient waves propagating in a thin pre-stressed plate possessing curvilinear orthotropy has been presented.

Behind the wave fronts upto the boundary of the contact domain, the analytical solution has been constructed with the help of the theory of discontinuities and one-term ray expansions. Nonlinear Hertz's theory has been employed within the contact region. For the analysis of the processes of shock interaction of the elastic sphere with the pre-stressed orthotropic plate, nonlinear integro-differential equation has been obtained with respect to the value characterizing the local indentation of the impactor into the target, which has been solved analytically in terms of time series with integer and fractional powers. The particular case of a pre-stressed transversely isotropic plate has been analyzed in detail.

As the analysis carried out in the present paper shows, the velocity of the transient wave of transverse shear begins to decrease with the increase in the compression force, resulting in reducing magnitudes of the stress discontinuities on this wave and 'locking' of its energy within the contact region. If reaching the critical magnitude of the compression force, the transverse shear wave could be 'locked' at all in the contact domain.

The occurrence of such an effect may be attributable to the fact that under intense compression of the plate material in the critical state, atoms are brought closer together, and atomic lattice is compressed. There is no way of shearing the atoms in such a lattice, therefore the transient shear wave attenuates quickly losing all its energy at the moment of its generation. Similar effect is observed in a highly compressed gas during the propagation of short compression waves through the gas, which is known as Landau attenuation. Thus, since all shear energy of the impactor is concentrated in the contact domain, then this may result in plate damage within this region, and hence in the increase of the peak deflection at this place [15]. But luckily to engineers, this limiting case is hardly to be achieved in engineering practice, because it is well known that 'the difficulty of compressive preloading is the risk of initial plate buckling as a stability failure of the relatively thin specimens under in-plane compression' [14]. Since the aim of our study was to analyze the impact response of pre-stressed plates, which show no initial deflections, then in order to avoid initial buckling we have limited the amount of pre-stress to a moderate level, what affects the evaluation of the preload effect.

As for the impact response of the pretensioned plate, then an opposite situation takes place, namely: a higher initial tensile force elevates the velocity of the transient wave of transverse shear, resulting in an increase in magnitudes of the stress discontinuities on this wave. Therefore, the bulk energy of shock interaction is imparted to the transient wave of transverse shear, and the lesser part of energy is passed on the contact zone. Because of this, damage within the contact region and the magnitude of the maximal deflection in the case of the pretensioned plate are less than those of the precompressed plate. This conclusion is supported by experiments [5].

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