Exact time response computation of control systems with fractional order lag and lead compensators

Nusret Tan, Derek P. Atherton, Ali Yuce, and Furkan Nur Deniz

***Abstract*—**In this paper, two exact methods are developed for the computation of unit step and unit impulse responses of closed loop control systems with fractional order lag and lead compensators. The methods are based on using the frequency response data of the closed loop fractional order control system. It is shown that the unit step and unit impulse responses of a feedback control system including a fractional order lag or lead controller can be computed exactly using Fourier series of a square wave and inverse Fourier transform of frequency response information namely gain and phase values. Time response equations which are the function of controller parameters are derived. A design procedure is given for estimating the parameters of a fractional order lag or lead compensator which give specified performance values of the closed loop system. Numerical examples are provided to show the success of the presented method.

***Keywords*—**Fractional order control system; time response; frequency response; step response; impulse response; time response from frequency response, fractional order lag and lead compensators

I. Introduction

THE fractional calculus has been known since 1695 [1-2]. However, it has become an important research area due to the increase of computational facilities and tools [3-4]. Nowadays, it can be seen that the application of the fractional calculus provides great opportunity in many field of science and many studies have been made in the different areas of science such as engineering, chemistry, physical, mechanical and other sciences [1-3]. One of the main reasons for this is that a fractional order differential equation can represent a real physical system more adequately [3]. This feature of fractional order reality can be certainly very advantageous in control theory and open an area of developing new analysis and design methods. Therefore, there has been considerable interest in recent years in studying the performance of fractional order feedback systems and many results have been developed [1-12].

A fractional order system can be simply defined by a differential equation whose derivative orders do not have to be an integer number instead it can be any real number. Thus, if the Laplace transform of such a differential equation is obtained then a transfer function with fractional order Laplace complex variable s such as s^{μ}, $\mu \in R$ can be obtained. This kind of transfer function is called a Fractional Order Transfer Function(FOTF) [1-3]. The frequency response of such a transfer function can be obtained exactly since the gain and phase values of the fractional order transfer function can be exactly obtained by replacing the Laplace complex variable s in the transfer function with $(j\omega)^{\mu} = \omega^{\mu}[\cos(\mu\pi/2) + j\sin(\mu\pi/2)]$. However, it is well known that it is not possible to calculate the exact time response of a fractional order transfer function since analytical inverse Laplace transforms do not normally exist. Generally, for computing the time responses namely impulse and step responses of a FOTF, integer order approximation methods such as Continued Fractional Expansion (CFE) method, Oustaloup's method, Carlson's method, Matsuda's method, Chareff's method, least square methods [13-19] and numerical approximation methods such as Grünwald-Letnikov approximations [2] are used. There are also some methods based on Mittag-Leffler and Gamma functions [20-21]. However, in recent papers by the authors [22-23] using frequency response data of a FOTF, the computation problem of the time response of a fractional order system was solved. Two methods were given in [22-23] one is based on Fourier series of a square wave and therefore it is called Fourier Series Method(FSM) and the other is based on Inverse Fourier Transform Method (IFTM). The results obtained from these methods are exact since both methods use frequency response information such as gain and phase data.

In feedback control theory an integer order model of the plant transfer function can be obtained using identification methods. However, the main task is to design a suitable controller which gives required performance specifications such as rise time, settling time, percentage overshoot, gain margin and phase margin. For this purpose the well known classical controller structures namely P, PI, PID, Lag or Lead controllers are used. It has been shown that the fractional order

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PI, PID, Lag or Lead controllers can be more advantageous than classical integer order controllers [21, 24-28].

The classical lag or lead compensator can be represented in a general form with the transfer function $C(s)=K(s+a)/(s+b)$ where the controller is a lead controller if $a<b$ and it is a lag controller if $a>b$. In some text books [29] the lag or lead controller are given by different transfer functions such as $C(s)=(\tau s+1)/(\alpha\tau s+1)$ which is a lead controller when $\alpha<1$ and a lag controller when $\alpha>1$ or $C(s)=k(\alpha\tau s+1)/(\tau s+1)$ which is a lead controller when $\alpha>1$ and a lag controller when $\alpha<1$. The representation given by $C(s)=(\tau s+1)/(\alpha\tau s+1)$ is similar to the general form $C(s)=K(s+a)/(s+b)$ if one takes $K=a/b$, $\tau=1/a$ and $\alpha\tau=1/b$. On the other hand the fractional order lag and lead compensators are given by two forms in the literature [3, 30] such as $C(s)=k[(\alpha\tau s+1)/(\tau s+1)]^{\mu}$ where $\mu\in(0,\infty)$ or $C(s)=k(\alpha\tau s^{\mu}+1)/(\tau s^{\mu}+1)$ where $\mu\in(0,2)$. In this paper, we take the fractional order lag or lead controller as $C(s)=K(s^{\mu}+a)/(s^{\mu}+b)$. The preliminary version of the results given in this paper has been given in [31] where we used the more general case $C(s)=K(s^{\lambda}+a)/(s^{\mu}+b)$. However, in this structure if $\lambda\neq\mu$ the phase of the controller can be positive for low frequency values and negative for high frequency values or vice versa. It can be considered as a lag-lead or lead-lag controller. However, this structure does not correspond to the classical lag-lead or lead–lag controller features for $\lambda\neq\mu$ since the phase will not approach zero when the frequency approaches zero and infinity. Thus, it is more appropriate to represent a fractional order lag or lead controller with $\lambda=\mu$.

In this paper, we propose exact methods for computation of step and impulse responses of a feedback control structure with an integer order model of a plant and a fractional order lag or lead controller of the form $C(s)=K(s^{\mu}+a)/(s^{\mu}+b)$. The derived equations depend on controller parameters and frequency response data of the closed loop control system. The proposed methods are based on the results obtained in [22-23]. Since the results obtained in the paper use the frequency response such as gain and phase values at each frequency, the time response computations of the closed loop system with a fractional lag or lead controller are accurate. The obtained results can be helpful for fractional order control design.

The paper is organized as follows: In Section II, a short review of fractional order system is given. In Section III, a closed loop control system with fractional order lag or lead controller is introduced. The exact methods using FSM and IFTM for computation of step and impulse responses of closed loop control systems with fractional order lag or lead controller are introduced in Section IV. In Section V, a procedure to design lag or lead compensator using FSM is given. In Section VI, examples are provided to show the importance of the method presented. Concluding remarks are given in Section VII.

II. Fractional Order Systems

Fractional order derivative and integrator are widely taken into account as an extension of integer order derivative and integrator operators to the case of non-integer orders and it is defined in general form as [15],

$$ {}_aD_t^{\mu}=\begin{cases}\dfrac{d^{\mu}}{dt^{\mu}} & \mu>0\\ 1 & \mu=0\\ \displaystyle\int_a^t (d\tau)^{(-\mu)} & \mu<0\end{cases} \tag{1}$$

where ${}_aD_t^{\mu}$ represents fundamental non-integer order derivative operator of fractional calculus. Parameters a and t are the lower and upper bounds of integration, and $\mu\in R$ denotes the fractional-order. A fractional order control system with input $r(t)$ and output $y(t)$ can be described by a fractional differential equation of the form [13],

$$\begin{aligned}&p_nD^{\alpha_n}y(t)+p_{n-1}D^{\alpha_{n-1}}y(t)+\cdots+p_0D^{\alpha_0}y(t)=\\&q_mD^{\beta_n}r(t)+q_{n-1}D^{\beta_{n-1}}r(t)+\cdots+q_0D^{\beta_0}r(t)\end{aligned} \tag{2}$$

or applying $L\{D^{\mu}f(t)\}=s^{\mu}F(s)$ by a fractional order transfer function of the form,

$$G_f(s)=\frac{Y(s)}{R(s)}=\frac{q_ms^{\beta_m}+q_{m-1}s^{\beta_{m-1}}+\cdots+q_0s^{\beta_0}}{p_ns^{\alpha_n}+p_{n-1}s^{\alpha_{n-1}}+\cdots+p_0s^{\alpha_0}} \tag{3}$$

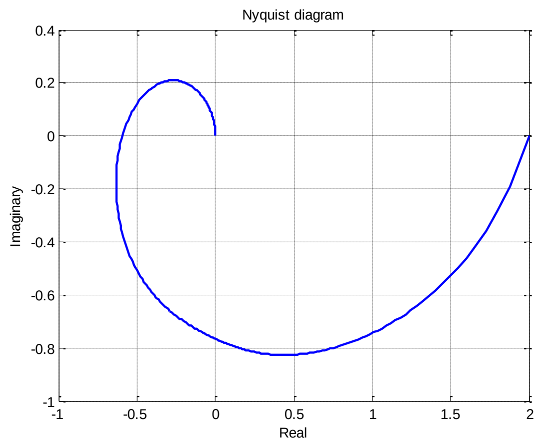
where p_i, q_j ($i=0,1,2,...,n$ and $j=0,1,2,...,m$) are real parameters and α_i, β_j are real positive numbers with $\alpha_0<\alpha_1<\cdots<\alpha_n$ and $\beta_0<\beta_1<\cdots<\beta_m$. Thus, a transfer function including fractional powered s terms can be called a fractional order transfer function, FOTF. For example, with the FOTF

$$G_f(s)=\frac{2}{s^{3.2}+3s^{1.8}+3s^{0.7}+1} \tag{4}$$

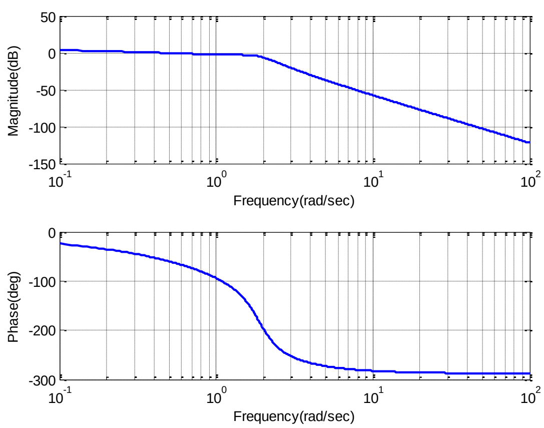
replacing s by $j\omega$ and using $(j\omega)^{\mu}=\omega^{\mu}[\cos(\mu\pi/2)+j\sin(\mu\pi/2)]$, one obtains

$$G(j\omega)=\frac{2}{\begin{aligned}&(0.309\omega^{3.2}-2.8533\omega^{1.8}+1.362\omega^{0.7}+1)+\\&j(-0.9511\omega^{3.2}+0.927\omega^{1.8}+2.673\omega^{0.7})\end{aligned}} \tag{5}$$

Bode and Nyquist diagrams of this equation can then be obtained as shown in Fig. 1 (a) and (b). However, obtaining the exact time response of (5) is a difficult problem since it is not possible to derive the analytical inverse Laplace transform of a fractional order transfer function.



a)



b)

Fig. 1: a) Nyquist plot of $G(s)$ of (6) b) Bode plots of $G(s)$ of (6)

III. CONTROL SYSTEMS WITH FRACTIONAL ORDER LAG OR LEAD COMPENSATOR

A fractional order closed loop control system is shown in Fig. 2 where

$$C(s) = K\frac{s^{\lambda}+a}{s^{\mu}+b} \tag{6}$$

is a fractional controller which is a lead or lag controller when $\lambda = \mu$ and

$$G(s) = \frac{q_m s^m + q_{m-1}s^{m-1} + \cdots + q_0}{p_n s^n + p_{n-1}s^{n-1} + \cdots + p_0} \tag{7}$$

is the plant transfer function which is an integer order transfer function. The Bode plots of the following controller transfer functions are shown in Fig. 3.

$$C_1(s) = \frac{s^{0.8}+1}{s^{0.8}+4} \tag{8}$$

$$C_2(s) = \frac{s^{1.2}+5}{s^{1.2}+2} \tag{9}$$

$$C_3(s) = \frac{(s^{0.8}+1)}{(s^{0.8}+4)}\frac{(s^{1.2}+5)}{(s^{1.2}+2)} = \frac{s^2+s^{1.2}+5s^{0.8}+5}{s^2+4s^{1.2}+2s^{0.8}+8} \tag{10}$$

$$C_4(s) = \frac{s^{1.2}+10}{s^{0.8}+2} \tag{11}$$

From Fig. 3 it can be seen that $C_1(s)$ is a lead controller where $\lambda = \mu = 0.8$, $C_2(s)$ is a lag controller where $\lambda = \mu = 1.2$, $C_3(s)$ is a lead-lag controller and $C_4(s)$ can be considered as lag-lead controller where $\lambda = 1.2$ and $\mu = 0.8$.

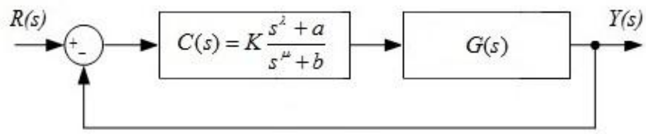
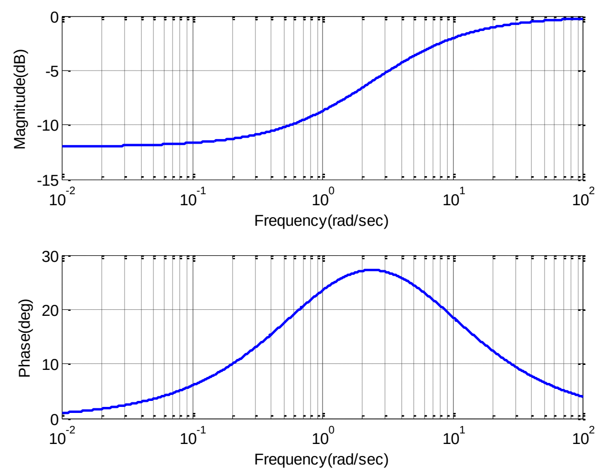
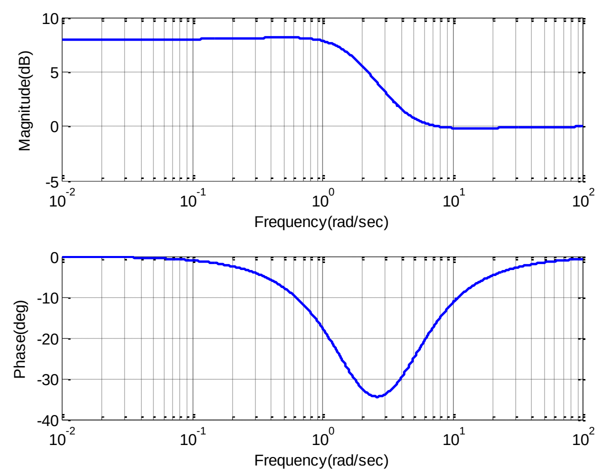


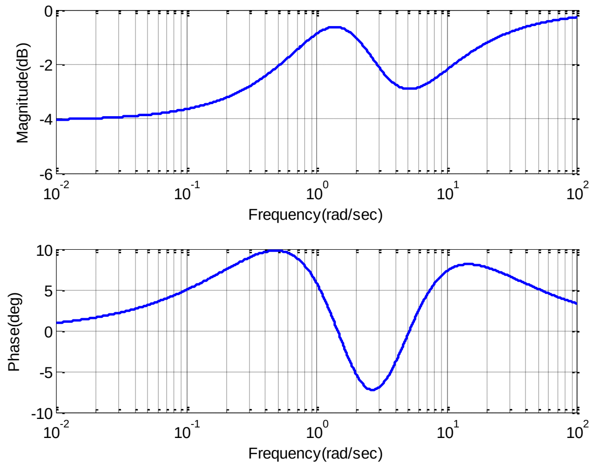
Fig. 2: A closed loop control system with fractional order controller



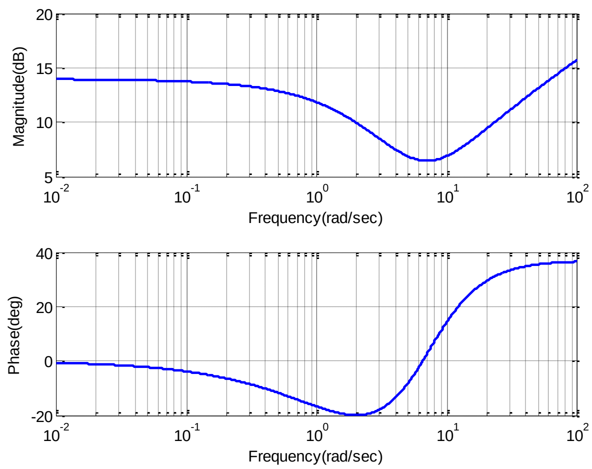
a)



b)



c)



d)

Fig. 3: Bode plots of a) $C_1(s)$ b) $C_2(s)$ c) $C_3(s)$ d) $C_4(s)$

For $\lambda = \mu$, the open loop transfer function of the system is

$$L(s) = C(s)G(s) = K\frac{(s^{\mu}+a)}{(s^{\mu}+b)}G(s) \tag{12}$$

and the closed loop transfer function can be written as

$$P(s) = \frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1+C(s)G(s)} = \frac{K(s^{\mu}+a)G(s)}{(s^{\mu}+b)+K(s^{\mu}+a)G(s)} \tag{13}$$

replacing s by $j\omega$, the following equation can be obtained

$$P(j\omega) = \frac{K((j\omega)^{\mu}+a)G(j\omega)}{((j\omega)^{\mu}+b)+K((j\omega)^{\mu}+a)G(j\omega)} = \frac{U(\omega)+jV(\omega)}{Z(\omega)+jQ(\omega)} \tag{14}$$

Using $G(j\omega) = \mathrm{Re}[G(j\omega)] + j\,\mathrm{Im}[G(j\omega)]$ and $(j\omega)^{\mu} = \omega^{\mu}[\cos(\mu\pi/2) + j\sin(\mu\pi/2)]$ in (14), one obtains

$$U(\omega) = \mathrm{Re}[G(j\omega)](K\omega^{\mu}\cos\mu\frac{\pi}{2} + Ka) - \mathrm{Im}[G(j\omega)](K\omega^{\mu}\sin\mu\frac{\pi}{2}) \tag{15}$$

$$V(\omega) = \mathrm{Re}[G(j\omega)](K\omega^{\mu}\sin\mu\frac{\pi}{2}) + \mathrm{Im}[G(j\omega)](K\omega^{\mu}\cos\mu\frac{\pi}{2} + Ka) \tag{16}$$

$$Z(\omega) = \omega^{\mu}\cos\mu\frac{\pi}{2} + b + \mathrm{Re}[G(j\omega)](K\omega^{\mu}\cos\mu\frac{\pi}{2} + Ka) - \mathrm{Im}[G(j\omega)](K\omega^{\mu}\sin\mu\frac{\pi}{2}) \tag{17}$$

$$Q(\omega) = \omega^{\mu}\sin\mu\frac{\pi}{2} + \mathrm{Re}[G(j\omega)](K\omega^{\mu}\sin\mu\frac{\pi}{2}) + \mathrm{Im}[G(j\omega)](K\omega^{\mu}\cos\mu\frac{\pi}{2} + Ka) \tag{18}$$

Similarly, for $\lambda \neq \mu$, the open loop transfer function is

$$L(s) = C(s)G(s) = K\frac{(s^{\lambda}+a)}{(s^{\mu}+b)}G(s) \tag{19}$$

and the closed loop transfer function can be written as

$$P(s) = \frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1+C(s)G(s)} = \frac{K(s^{\lambda}+a)G(s)}{(s^{\mu}+b)+K(s^{\lambda}+a)G(s)} \tag{20}$$

replacing s by $j\omega$, the following equation can be obtained

$$P(j\omega) = \frac{K((j\omega)^{\lambda}+a)G(j\omega)}{((j\omega)^{\mu}+b)+K((j\omega)^{\lambda}+a)G(j\omega)} = \frac{U(\omega)+jV(\omega)}{Z(\omega)+jQ(\omega)} \tag{21}$$

Using $G(j\omega) = \mathrm{Re}[G(j\omega)] + j\,\mathrm{Im}[G(j\omega)]$, $(j\omega)^{\lambda} = \omega^{\lambda}[\cos(\lambda\pi/2) + j\sin(\lambda\pi/2)]$ and $(j\omega)^{\mu} = \omega^{\mu}[\cos(\mu\pi/2) + j\sin(\mu\pi/2)]$ in (21), one obtains

$$U(\omega) = \mathrm{Re}[G(j\omega)](K\omega^{\lambda}\cos\lambda\frac{\pi}{2} + Ka) - \mathrm{Im}[G(j\omega)](K\omega^{\lambda}\sin\lambda\frac{\pi}{2}) \tag{22}$$

$$V(\omega) = \mathrm{Re}[G(j\omega)](K\omega^{\lambda}\sin\lambda\frac{\pi}{2}) + \mathrm{Im}[G(j\omega)](K\omega^{\lambda}\cos\lambda\frac{\pi}{2} + Ka) \tag{23}$$

$$Z(\omega) = \omega^{\mu}\cos\mu\frac{\pi}{2} + b + \mathrm{Re}[G(j\omega)](K\omega^{\lambda}\cos\lambda\frac{\pi}{2} + Ka) - \mathrm{Im}[G(j\omega)](K\omega^{\lambda}\sin\lambda\frac{\pi}{2}) \tag{24}$$

$$Q(\omega) = \omega^{\mu}\sin\mu\frac{\pi}{2} + \mathrm{Re}[G(j\omega)](K\omega^{\lambda}\sin\lambda\frac{\pi}{2}) + \mathrm{Im}[G(j\omega)](K\omega^{\lambda}\cos\lambda\frac{\pi}{2} + Ka) \tag{25}$$

Thus, $P(j\omega)$ can be written as

$$P(j\omega) = \frac{[U(\omega)Z(\omega)+V(\omega)Q(\omega)] + j[V(\omega)Z(\omega)-U(\omega)Q(\omega)]}{Z(\omega)^2+Q(\omega)^2} \tag{26}$$

From this equation, the real part and imaginary part of $P(j\omega)$ are obtained as

$$\mathrm{Re}[P(j\omega)] = \frac{[U(\omega)Z(\omega)+V(\omega)Q(\omega)]}{Z(\omega)^2+Q(\omega)^2} \tag{27}$$

$$\mathrm{Im}[P(j\omega)] = \frac{[V(\omega)Z(\omega) - U(\omega)Q(\omega)]}{Z(\omega)^2 + Q(\omega)^2} \tag{28}$$

The controller of (6) with $\lambda = \mu$ has a phase of

$$\angle C(j\omega) = \tan^{-1}\left(\frac{\omega^{\mu}\sin(\mu\pi/2)}{\omega^{\mu}\cos(\mu\pi/2)+a}\right) - \tan^{-1}\left(\frac{\omega^{\mu}\sin(\mu\pi/2)}{\omega^{\mu}\cos(\mu\pi/2)+b}\right) \tag{29}$$

which is a lead controller if it is greater than zero and a lag controller if it is less than zero.

IV. TIME RESPONSE OF CLOSED LOOP CONTROL SYSTEMS WITH FRACTIONAL ORDER LAG OR LEAD COMPENSATOR

A. Fourier Series Method(FSM)

The Fourier series for the square wave of -1 to 1 with frequency $\omega_s = 2\pi/T$ can be written as

$$x(t) = \frac{4}{\pi}\sum_{k=1(2)}^{\infty}\frac{1}{k}\sin(k\omega_s t) \tag{30}$$

where T is the period of the square wave. If $x(t)$ passes through the transfer function $P(s)$ then the output, which is the unit step response if T is sufficiently large, can be written as

$$y_s(t) \cong \frac{4}{\pi}\sum_{k=1(2)}^{\infty}\frac{1}{k}\mathrm{Re}\left[P(jk\omega_s)\right]\sin(k\omega_s t) \tag{31}$$

The proof of this can be done using convolution. The details of the proof can be found in [22] where it has been shown that the step response can be written as

$$y_s(t) = \frac{4}{\pi}\sum_{k=1(2)}^{\infty}\begin{pmatrix}\frac{1}{k}\mathrm{Re}\left[P(jk\omega_s)\right]\sin(k\omega_s t)\\ +\frac{1}{k}\mathrm{Im}[P(jk\omega_s)]\cos(k\omega_s t)\end{pmatrix} \tag{32}$$

As $T \to \infty$ and $\omega_s \to 0$ the numerator of the imaginary part of $P(jk\omega_s)$ is multiplied by ω_s so that $\lim_{\omega_s \to 0}\mathrm{Im}P(jk\omega_s) = 0$ and (32) becomes

$$y_s(t) \cong \frac{4}{\pi}\sum_{k=1(2)}^{\infty}\frac{1}{k}\mathrm{Re}\left[P(jk\omega_s)\right]\sin(k\omega_s t) \tag{33}$$

For the closed loop control system of Fig. 2 with fractional order lag or lead controller, substituting (27) into (33) gives

$$y_s(t) \cong \frac{4}{\pi}\sum_{k=1(2)}^{\infty}\frac{1}{k}\frac{[U(k\omega_s)Z(k\omega_s)+V(k\omega_s)Q(k\omega_s)]}{[Z(k\omega_s)^2+Q(k\omega_s)^2]}\sin(k\omega_s t) \tag{34}$$

which is the unit step response of the closed loop system of Fig. 2. Similarly, the impulse response, which is the derivative of the step response is given by

$$y_i(t) = \frac{dy_s(t)}{dt} = \frac{4}{\pi}\sum_{k=1(2)}^{\infty}\begin{pmatrix}\omega_s\mathrm{Re}\left[P(jk\omega_s)\right]\cos(k\omega_s t)\\ -\omega_s\mathrm{Im}[P(jk\omega_s)]\sin(k\omega_s t)\end{pmatrix} \cong \frac{4}{\pi}\sum_{k=1(2)}^{\infty}\omega_s\mathrm{Re}\left[P(jk\omega_s)\right]\cos(k\omega_s t) \tag{35}$$

For the system of Fig. 2 again substituting Eq. (27) into (35), one can obtain the equation of the impulse response as

$$y_i(t) \cong \frac{4}{\pi}\sum_{k=1(2)}^{\infty}\omega_s\frac{[U(k\omega_s)Z(k\omega_s)+V(k\omega_s)Q(k\omega_s)]}{[Z(k\omega_s)^2+Q(k\omega_s)^2]}\cos(k\omega_s t) \tag{36}$$

Thus the step and impulse responses of the system can be obtained using (34) and (36).

B. Inverse Fourier Transform Method(IFTM)

The impulse response, $p(t)$, corresponding to the transfer function $P(s)$ of (13) or (20) is given by $p(t) = L^{-1}\left(P(s)\right)$ where L^{-1} denotes the inverse Laplace transform. Assuming the impulse response is that of a stable system so that $\lim_{t\to\infty} p(t) = 0$ then the Fourier transform can be evaluated. It has been shown in [22] that the impulse response, $p(t)$, can be written as

$$p(t) = \frac{2}{\pi}\int_0^{\infty}\mathrm{Re}[P(j\omega)]\cos(\omega t)d(\omega) \tag{37}$$

or

$$p(t) = -\frac{2}{\pi}\int_0^{\infty}\mathrm{Im}[P(j\omega)]\sin(\omega t)d(\omega) \tag{38}$$

Using (27) in (37), the following equations can be written

$$p(t) = \frac{2}{\pi}\int_0^{\infty}\frac{[U(\omega)Z(\omega)+V(\omega)Q(\omega)]}{[Z(\omega)^2+Q(\omega)^2]}\cos(\omega t)d(\omega) \tag{39}$$

Similarly, substituting (28) into (38), one obtains

$$p(t) = -\frac{2}{\pi}\int_0^{\infty}\frac{[V(\omega)Z(\omega)-U(\omega)Q(\omega)]}{[Z(\omega)^2+Q(\omega)^2]}\sin(\omega t)d(\omega) \tag{40}$$

Thus, $p(t)$ can be computed by numerical integration using Eq. (39) or Eq. (40).

Remark: The results given in this section are derived for the integer order plant transfer function given in (7). However, the results can be directly used when the plant transfer function is a fractional order transfer function in the form of (3) since exact frequency response data of a fractional order transfer function can be calculated.

V. DESIGN OF LAG OR LEAD COMPENSATOR USING FSM

Classical controller design involves the choice of a suitable transfer function for controller so that the closed loop performance meets the required specifications. This can often be achieved with well known transfer functions with three common ones being the phase lead controller, the phase lag controller and the PID controllers. For a closed loop control system it is required that there should be no steady state error to a step input therefore the phase lead and phase lag controllers are generally used for plant transfer functions with an integral term since the phase lead and lag controllers do not include an integral term [29].

The closed loop performance of a control system in the time domain can be determined with the specifications such as percentage overshoot (PO) value, rise time (t_r) and settling time (t_s). In the previous section, it was shown that the unit step response of the closed loop control system given in Fig. 2 can be determined with the (34). This equation can be used for designing the parameters of the lag or lead controller for which the closed loop performance meets the required specifications. Let the required specifications for the feedback system in Fig. 2 be PO, t_r and t_s. Then from the following equations the required parameters of the phase lag or phase lead compensator can be computed.

$$\frac{\max y_s(t) - y_s(\infty)}{y_s(\infty)} \times 100 \le PO \tag{41}$$

where

$$\max y_s(t) = \max\left(\frac{4}{\pi}\sum_{k=1(2)}^{\infty}\frac{1}{k}\frac{[U(k\omega_s)Z(k\omega_s)+V(k\omega_s)Q(k\omega_s)]}{[Z(k\omega_s)^2+Q(k\omega_s)^2]}\sin(k\omega_s t)\right)$$

and

$$y_s(\infty) = \frac{4}{\pi}\sum_{k=1(2)}^{\infty}\frac{1}{k}\frac{[U(k\omega_s)Z(k\omega_s)+V(k\omega_s)Q(k\omega_s)]}{[Z(k\omega_s)^2+Q(k\omega_s)^2]}\sin(k\omega_s \infty)$$

For rise time t_r,

$$y(t_r) = \frac{4}{\pi}\sum_{k=1(2)}^{\infty}\frac{1}{k}\frac{[U(k\omega_s)Z(k\omega_s)+V(k\omega_s)Q(k\omega_s)]}{[Z(k\omega_s)^2+Q(k\omega_s)^2]}\sin(k\omega_s t_r) = 1 \tag{42}$$

and settling time for the tolerance band 2% and $t \in [t_s, \infty)$,

$$0.98 \le \frac{4}{\pi}\sum_{k=1(2)}^{\infty}\frac{1}{k}\frac{[U(k\omega_s)Z(k\omega_s)+V(k\omega_s)Q(k\omega_s)]}{[Z(k\omega_s)^2+Q(k\omega_s)^2]}\sin(k\omega_s t) \le 1.02 \tag{43}$$

VI. Examples

A. Example 1

The aim of this example is to show the validity of the method by considering the control system of Fig. 2 with integer order transfer function

$$G(s) = \frac{1}{s(s+2)(s+3)} \tag{44}$$

and the controller of the form

$$C(s) = \frac{s^{\mu}+0.1}{s^{\mu}+0.01} \tag{45}$$

The open loop transfer function is

$$L(s) = C(s)G(s) = \frac{s^{\mu}+0.2}{s^{3+\mu}+5s^{2+\mu}+6.s^{1+\mu}+0.02s^3+0.1s^2+0.12s} \tag{46}$$

and the closed loop transfer function is

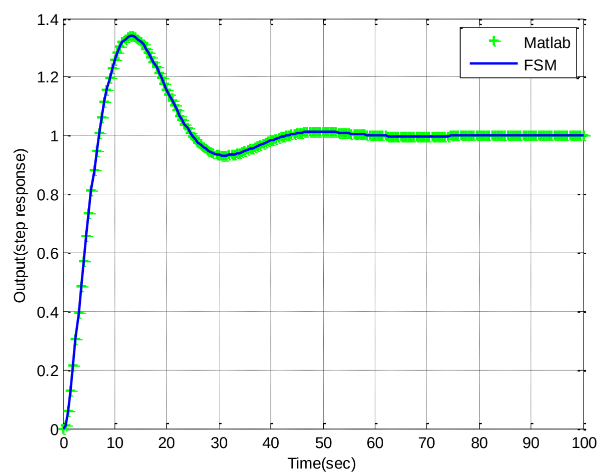
$$P(s) = \frac{L(s)}{1+L(s)} = \frac{s^{\mu}+0.2}{s^{3+\mu}+5s^{2+\mu}+6s^{1+\mu}+0.02s^3+0.1s^2+0.12s+s^{\mu}+0.2} \tag{47}$$

For $\mu = 1$, the controller and the plant transfer functions are integer order transfer functions and the closed loop transfer function of the system is

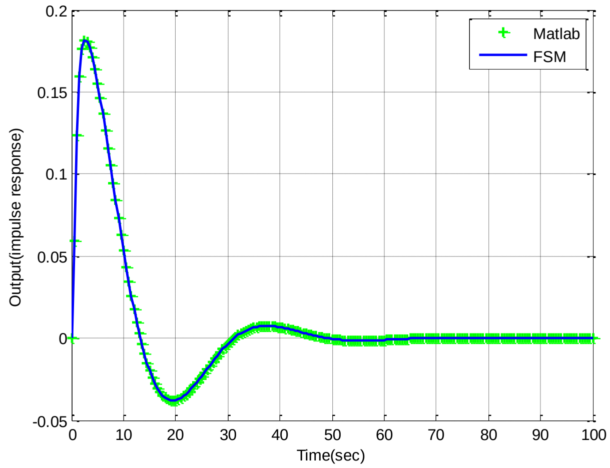
$$P(s) = \frac{s+0.2}{s^4+5.02s^3+6.1s^2+0.12s+0.12} \tag{48}$$

The step responses of the system obtained by Matlab step function and the FSM program are shown in Fig. 4 (a) where it can be seen that both results are same since the maximum error between two plots was computed as 5.4×10^{-9}. Similarly, the impulse responses of the closed loop system using Matlab impulse function and FSM are plotted in Fig. 4 (b) where it was estimated that the maximum error between two plots are less than 1.02×10^{-6}. These errors occur at initial time and then the error between two plots becomes zero. Also, the initial time errors (5.4×10^{-9} and 1.02×10^{-6}) are very small. Therefore, the presented method gives exact step and impulse responses of the closed loop system since the approach is based on using the frequency response data of the system which can be computed exactly.

The step responses for different values of μ are shown in Fig. 5 where it can be seen that the step responses become oscillatory when μ is increased. The step response can be changed by altering μ to hopefully a suitable one.



a)



b)

Fig. 4 : a) Step responses b) Impulse responses

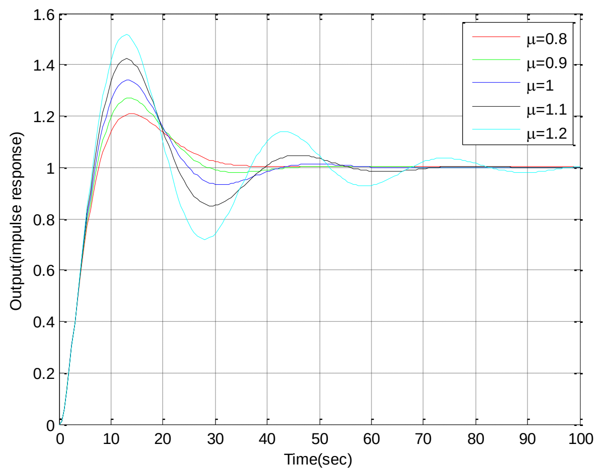


Fig. 5 : Step responses for different values of μ

B. Example 2

Consider Fig. 2 with

$$C(s) = 10\frac{(s^{0.7}+5)}{(s^{0.7}+8)} \text{ and } G(s) = \frac{2}{s(s+1)} \tag{49}$$

The open loop transfer function is

$$L(s) = C(s)G(s) = \frac{20s^{0.7}+100}{s^{2.7}+8s^2+s^{1.7}+8s} \tag{50}$$

and the closed loop transfer function is

$$\begin{aligned} P(s) &= \frac{L(s)}{1+L(s)} = \frac{20s^{0.7}+100}{s^{2.7}+8s^2+s^{1.7}+8s+20s^{0.7}+100} \\ &= \frac{20(s^{0.7})+100}{s^2(s^{0.7})+8s^2+s(s^{0.7})+8s+20(s^{0.7})+100} \end{aligned} \tag{51}$$

Here, comparisons results with Oustaloup method and Grünwald-Letnikov (GL) are given. Oustaloup's fifth order integer approximations for $s^{0.7}$ is

$$s^{0.7} \cong \frac{25.12s^5+623.6s^4+2117s^3+1111s^2+90.14s+1}{s^5+90.14s^4+1111s^3+2117s^2+623.6s+25.12} \tag{52}$$

Using this approximation in (51), one can obtain the following 7th order closed loop transfer function

$$P_{ous5}(s) = \frac{\begin{array}{l}600s^5+21490s^4+153440s^3+233920s^2 \\ +64160s+2530\end{array}}{\begin{array}{l}33.12s^7+1377.7s^6+12950s^5+50542s^4 \\ +176570s^3+239200s^2+64362s+2530\end{array}} \tag{53}$$

The exact step response of the system obtained from FSM and the step responses of $P_{ous5}(s)$ are shown in Fig 6. It can be seen that the difference between the two plots is very small.

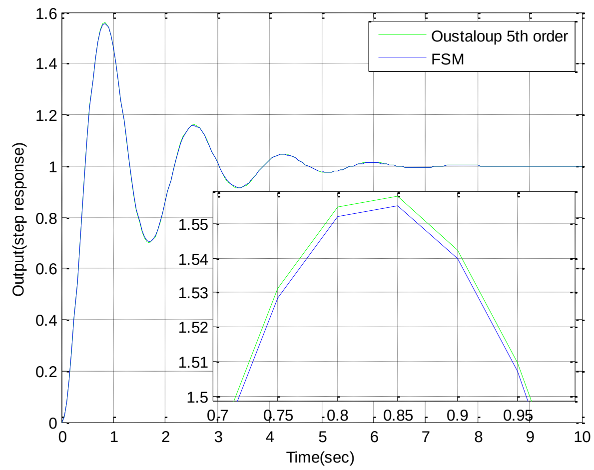


Fig. 6: Step response of the system obtained from FSM and the step responses of $P_{ous5}(s)$

One can compute the step response of (51) using Grünwald-Letnikov(GL) approximation method. We used the Matlab GL program given in [2] and obtained Fig. 7. The difference between the plots can be seen more clearly on the small figure. The result from FSM is the exact result.

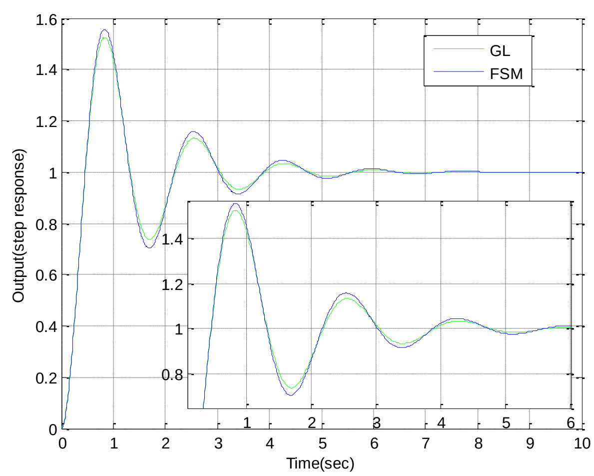


Fig. 7: Step responses obtained from GL method and FSM.

C. Example 3

Consider the control system of Fig. 2 with

$$C(s)=\frac{s^{1.3}+0.5}{s^{1.3}+0.1} \text{ and } G(s)=\frac{280(s+0.5)}{s(s+0.2)(s+5)(s+70)} \tag{54}$$

The open loop transfer function is

$$L(s)=C(s)G(s)= \frac{280s^{2.3}+140s^{1.3}+144s+70}{s^{5.3}+75.2s^{4.3}+0.1s^{4}+365s^{3.3}+7.52s^{3}+70s^{2.3}+36.5s^{2}+7s} \tag{55}$$

Thus, the closed loop transfer function is

$$P(s)=\frac{L(s)}{1+L(s)}= \frac{280s^{2.3}+140s^{1.3}+144s+70}{s^{5.3}+75.2s^{4.3}+0.1s^{4}+365s^{3.3}+7.52s^{3}+350s^{2.3}+36.5s^{2}+140s^{1.3}+151s+70} \tag{56}$$

The step response of $P(s)$ using FSM and the impulse responses of $\frac{1}{s}P(s)$ which is the step response of $P(s)$ using IFTM are plotted in Fig. 8. From zoomed figure given in Fig. 8, one can see that the two plots are the same.

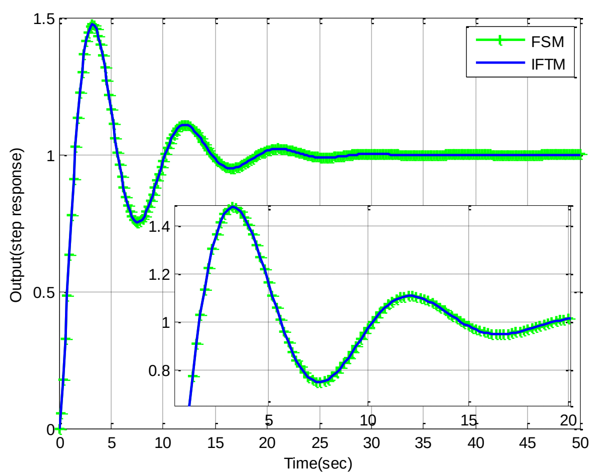


Fig. 8: Step responses obtained from FSM and IFTM.

D. Example 4

Consider the control system of Fig. 2 with

$$G(s)=\frac{6}{s(0.5s+1)(0.1s+1)} \tag{57}$$

The aim is to design a lead controller of the form

$$C(s)=K\frac{s^{\mu}+a}{s^{\mu}+b} \tag{58}$$

which satisfy the following specifications: percentage overshoot must be less than 15%, settling time must be less than 2.8 sec for 2% tolerance band and rise time must be less than 0.8 sec. Using (41)-(43), the parameters of the controller which give these specifications are computed as $K=1.8$, $\mu=0.85$, $a=1.3$ and $b=6$. Thus, the designed controller is

$$C(s)=1.8\frac{s^{0.85}+1.3}{s^{0.85}+6} \tag{59}$$

The step response of the system is shown in Fig. 9 where it can be seen that the percentage overshoot is equal to 12%, the rise time is equal to 0.75 sec and the settling time is 2.5 sec. Thus, with the designed controller the closed loop performance meets the required specifications.

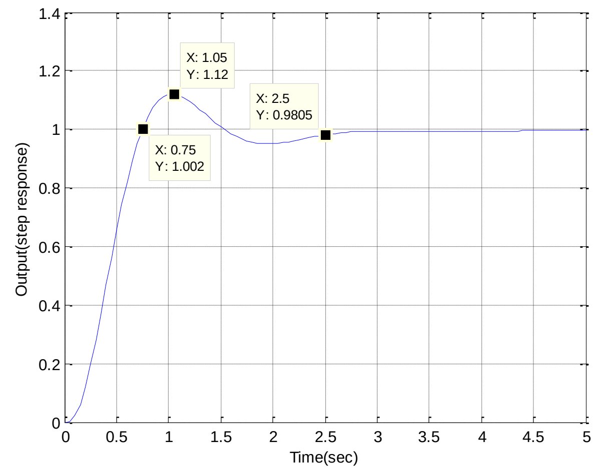


Fig. 9: Step response of the closed loop system with $G(s)$ of (57)and $C(s)$ of Eq. (59).

VII. CONCLUSIONS

In recent years, there have been many studies in the field of fractional order control systems. Many results have been published related with the frequency and time domain analysis of closed loop fractional order control systems. However, obtaining the exact time response of a fractional order system is a difficult problem since it is not possible to derive the analytical inverse Laplace transform of a fractional order transfer function. In this paper, exact methods have been presented for computation of the time response of a closed loop control system with a fractional order lag or lead compensator using frequency response data of the closed loop system. It has been shown that the unit step and unit impulse responses of a feedback control system including a fractional order lag or lead compensator can be computed exactly using Fourier series of a square wave and inverse Fourier transform of frequency response information namely gain and phase values. Time response equations which are the function of controller parameters have been derived. A design procedure has been presented for estimating the parameters of lag or lead compensator namely K, μ, a and b. Given examples clearly show that the presented methods provide very useful results in the field of fractional order control systems. Especially the results will be very attractive for the design of fractional order control systems.

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