$$C(s)=\frac{s^{1.3}+0.5}{s^{1.3}+0.1} \text{ and } G(s)=\frac{280(s+0.5)}{s(s+0.2)(s+5)(s+70)} \quad (54)$$

The open loop transfer function is

$$L(s)=C(s)G(s)= \frac{280s^{2.3}+140s^{1.3}+144s+70}{s^{5.3}+75.2s^{4.3}+0.1s^4+365s^{3.3}+7.52s^3+70s^{2.3}+36.5s^2+7s} \quad (55)$$

Thus, the closed loop transfer function is

$$P(s)=\frac{L(s)}{1+L(s)}= \frac{280s^{2.3}+140s^{1.3}+144s+70}{s^{5.3}+75.2s^{4.3}+0.1s^4+365s^{3.3}+7.52s^3+350s^{2.3}+36.5s^2+140s^{1.3}+151s+70} \quad (56)$$

The step response of $P(s)$ using FSM and the impulse responses of $\frac{1}{s}P(s)$ which is the step response of $P(s)$ using IFTM are plotted in Fig. 8. From zoomed figure given in Fig. 8, one can see that the two plots are the same.

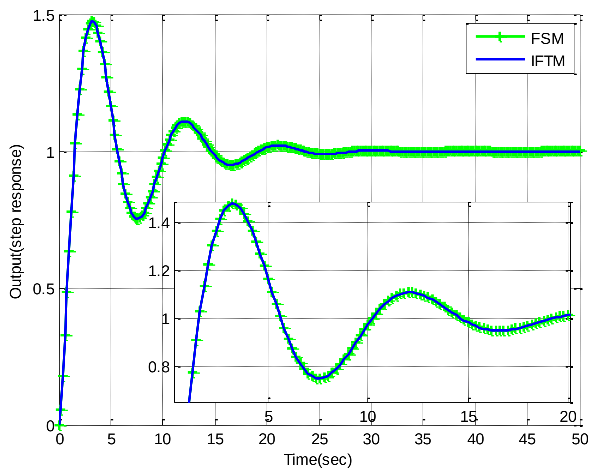


Fig. 8: Step responses obtained from FSM and IFTM.

### D. *Example 4*

Consider the control system of Fig. 2 with

$$G(s)=\frac{6}{s(0.5s+1)(0.1s+1)} \quad (57)$$

The aim is to design a lead controller of the form

$$C(s)=K\frac{s^{\mu}+a}{s^{\mu}+b} \quad (58)$$

which satisfy the following specifications: percentage overshoot must be less than 15%, settling time must be less than 2.8 sec for 2% tolerance band and rise time must be less than 0.8 sec. Using (41)-(43), the parameters of the controller which give these specifications are computed as $K=1.8$, $\mu=0.85$, $a=1.3$ and $b=6$. Thus, the designed controller is

$$C(s)=1.8\frac{s^{0.85}+1.3}{s^{0.85}+6} \quad (59)$$

The step response of the system is shown in Fig. 9 where it can be seen that the percentage overshoot is equal to 12%, the rise time is equal to 0.75 sec and the settling time is 2.5 sec. Thus, with the designed controller the closed loop performance meets the required specifications.

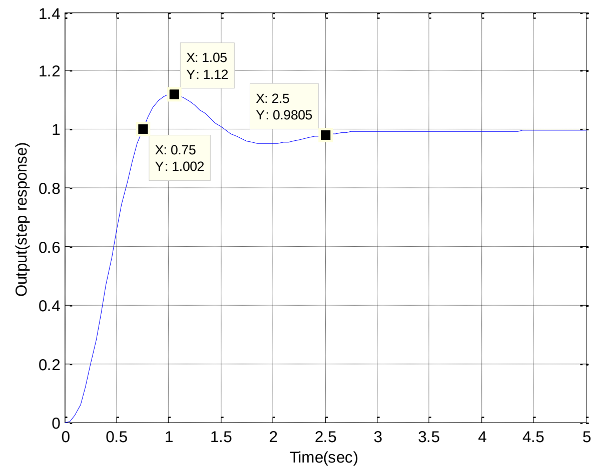


Fig. 9: Step response of the closed loop system with $G(s)$ of (57)and $C(s)$ of Eq. (59).

## VII. CONCLUSIONS

In recent years, there have been many studies in the field of fractional order control systems. Many results have been published related with the frequency and time domain analysis of closed loop fractional order control systems. However, obtaining the exact time response of a fractional order system is a difficult problem since it is not possible to derive the analytical inverse Laplace transform of a fractional order transfer function. In this paper, exact methods have been presented for computation of the time response of a closed loop control system with a fractional order lag or lead compensator using frequency response data of the closed loop system. It has been shown that the unit step and unit impulse responses of a feedback control system including a fractional order lag or lead compensator can be computed exactly using Fourier series of a square wave and inverse Fourier transform of frequency response information namely gain and phase values. Time response equations which are the function of controller parameters have been derived. A design procedure has been presented for estimating the parameters of lag or lead compensator namely $K$, $\mu$, $a$ and $b$. Given examples clearly show that the presented methods provide very useful results in the field of fractional order control systems. Especially the results will be very attractive for the design of fractional order control systems.

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