Spherical Families of Polynomials: A Graphical Approach to Robust Stability Analysis

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***Abstract*—**This paper is intended to present not so common and frequently used approach to the definition of uncertainty bounding set for systems with parametric uncertainty and related tools for robust stability analysis. More specifically, the work deals with spherical families of polynomials. The set of illustrative examples demonstrates an easy-to-use graphical method of robust stability testing based on the combination of the value set concept and the zero exclusion condition by means of the Polynomial Toolbox for Matlab.

***Keywords*—**Spherical Uncertainty, Weighted Euclidean Norm, Robust Stability Analysis, Value Set Concept, Zero Exclusion Condition.

I. Introduction

Robustness of control systems represents attractive research topic with a countless number of real life applications [1] – [5]. Parametric uncertainty is commonly used tool for the description of real plants as it allows using relatively simple and natural mathematical models for processes which behavior can be much more complicated. The structure (i.e. order) of the models with parametric uncertainty is considered to be fixed, but its parameters can lie within given bounds. Within this contribution, these bounds are going to be assumed in a not so frequently applied way.

The typical, mostly used and naturally comprehensible approach assumes the bounds in the shape of a box. Here, the alternative approach, which uses the bounds in the shape of a sphere (ellipsoid), is going to be studied. The scientific literature contains much more works related to the classical "box" uncertainties than to the spherical ones. However, some basic information, as well as possible extensions and various applications, can be found e.g. in [6] – [10].

This paper is focused on polynomials with parametric uncertainty and spherical uncertainty bounding set. More specifically it deals with the description of a spherical polynomial family and with tools for analysis of its robust stability. Special attention is paid to very universal graphical tool based on the combination of the value set concept and the zero exclusion condition [6]. The described ideas are followed by the set of illustrative examples supported by plots from the Polynomial Toolbox for Matlab [11], [10]. The paper is the extended version of the previously published conference contributions [12], [13].

The paper is organized as follows. In Section 2, basic notation and theoretical background of systems with parametric uncertainty and uncertainty bounding sets are provided. The Section 3 is then focused on the description of spherical polynomial families. Next, tools for robust stability analysis with especial emphasis on the value set concept and the zero exclusion condition are shown in Section 4. The following extensive Section 5 presents three illustrative examples of practical robust stability investigation for selected spherical polynomial families. And finally, Section 6 offers some conclusion remarks.

II. Parametric Uncertainty and Uncertainty Bounding Set

Generally, the systems with parametric uncertainty can be described through a vector of real uncertain parameters (often called just uncertainty) q. The continuous-time uncertain polynomial, which is a typical object of researchers' or engineers' interest, can be written in the form:

$$p(s,q)=\sum_{i=0}^{n}\rho_i(q)s^i \tag{1}$$

where ρ_i are coefficient functions.

Then, so-called family of polynomials combines together the structure of uncertain polynomial given by (1) with the uncertainty bounding set Q. Therefore, the family of polynomials can be denoted as:

$$P=\{p(\cdot,q):q\in Q\} \tag{2}$$

The uncertainty bounding set Q is usually given in advance, typically by user requirements. It is supposed as a ball in an appropriate norm. The most frequently used case utilizes L_∞ norm:

$$\|q\|_\infty=\max_i|q_i| \tag{3}$$

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which means that a ball in this norm is a box. Practically, the box is defined by the components, i.e. by the real intervals which can the uncertain parameters lie within.

Another approach employs L_2 (Euclidean) norm:

$$\|q\|_2 = \sqrt{\sum_{i=1}^{n} q_i^2} \tag{4}$$

or more generally the weighted Euclidean norm:

$$\|q\|_{2,W} = \sqrt{q^T W q} \tag{5}$$

where $q \in \mathbf{R}^k$ and W is a positive definite symmetric matrix (weighting matrix) of size $k \times k$. Such definition of Q means that a ball in the norm can be referred as a sphere, or more generally as an ellipsoid. Under assumption of $r \geq 0$ and $q^0 \in \mathbf{R}^k$, the ellipsoid (in $\mathbf{R}^k$) which is centered at q^0 can be expressed by means of:

$$\left(q - q^0\right)^T W \left(q - q^0\right) \leq r^2 \tag{6}$$

or equivalently by:

$$\left\|q - q^0\right\|_{2,W} \leq r \tag{7}$$

The ellipsoid can be easily visualized in two-dimensional space ($k = 2$) for:

$$W = \begin{bmatrix} w_1^2 & 0 \\ 0 & w_2^2 \end{bmatrix} \tag{8}$$

as it is shown in Fig. 1 [6].

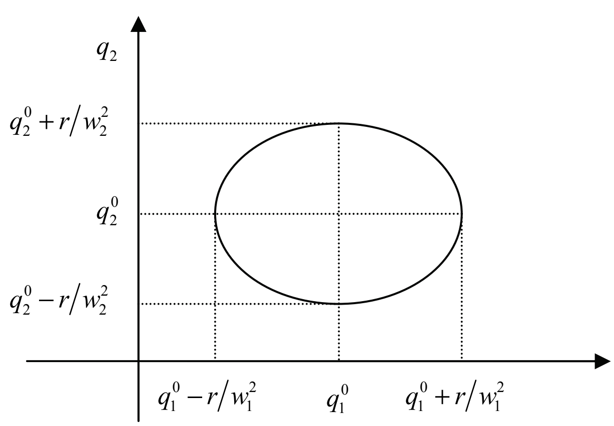

Fig. 1 An ellipsoid defined by weighted Euclidean norm

A decision on what type of norm should be used for uncertainty bounding set Q depends on several factors. In many engineering problems, the real uncertain physical parameters are independent of each other and thus Q should be a box naturally. However, according to [6], the ellipsoids could be useful and justifiable under "imprecise description" of the uncertainty bounds, i.e. if actual Q is located between some minimum and maximum and a suitable ellipsoid can interpolate them. The choice should respect also available tools for solving the specific problem. Besides, the mathematical models obtained on the basis of physical laws usually have Q in the shape of a box, but the identification methods mostly lead to the ellipsoids [14].

III. Spherical Polynomial Family

The family of polynomials given by (2) is called spherical one [6] if $p(s,q)$ has an independent uncertainty structure (all coefficients of the polynomial are independent on each other) and Q is an ellipsoid.

In fact, one can work with two basic representations of spherical polynomial families. The first type assumes that polynomial is centered at zero:

$$\begin{gathered} p(s,q) = \sum_{i=0}^{n} q_i s^i \\ \left\|q - q^0\right\|_{2,W} \leq r \end{gathered} \tag{9}$$

where W is a positive definite symmetric matrix, q_0 is the nominal and $r \geq 0$ means the radius of uncertainty. In the second representation, Q is considered to be centered at zero:

$$\begin{gathered} p(s,q) = p_0(s) + \sum_{i=0}^{n} q_i s^i \\ \|q\|_{2,W} \leq r \end{gathered} \tag{10}$$

where moreover $p_0(s) = p(s,q^0)$.

As an example, suppose a spherical polynomial family:

$$\begin{gathered} p(s,q) = \left(0.5 + q_0\right) + \left(1.5 + q_1\right)s + \left(2.5 + q_2\right)s^2 \\ \|q\|_{2,W} \leq 1 \\ W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \end{gathered} \tag{11}$$

which can be centered on the vector:

$$\tilde{q}^0 = \left(0.5, 1.5, 2.5\right) \tag{12}$$

Then, the resulting spherical polynomial family, equivalent to (11), can be written as:

$$
\begin{aligned}
&p(s,q) = \tilde{q}_0 + \tilde{q}_1 s + \tilde{q}_2 s^2 \\
&\left\| \tilde{q} - \tilde{q}_0 \right\|_{2,W} \leq 1 \\
&W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}
\end{aligned}
\tag{13}
$$

IV. Robust Stability Analysis

Obviously, the most important feature of all control circuits is their stability. Under conditions of parametric uncertainty, this term can be expanded to robust stability, which means that the whole family of closed-loop control systems must remain stable for all possible allowed perturbations of system parameters.

From the practical testing point of view, we are interested in the robust stability of the family of closed-loop characteristic polynomials in the form (2). This family is robustly stable if and only if $p(s,q)$ is stable for all $q \in Q$, i.e. all roots of $p(s,q)$ must be located in the strict left half of the complex plane for all $q \in Q$.

There are many results for robust stability analysis of systems with parametric uncertainty for Q in a shape of a box. Their choice depends mainly on the complexity of the structure of investigated polynomial (or system). Doubtless, the most famous tool is the Kharitonov theorem [15] which is suitable for investigation of the robust stability of interval polynomials (with independent uncertainty structure). Moreover, several modifications and generalizations of classical Kharitonov theorem are also available in the literature [6], [16]. Among other known tools it belong e.g. the edge theorem, the thirty-two edge theorem, the sixteen plant theorem, the mapping theorem, etc. [6]. Furthermore, it exists a graphical method which is applicable for wide range of robust stability analysis problems (from the simplest to the very complicated uncertainty structures, for various stability regions, etc.). This technique combines the value set concept with the zero exclusion condition [6], [17].

Robust stability analysis for systems affected by parametric uncertainty for the case of Q in a shape of an ellipsoid is also relatively well developed and there are several methods available. The Soh-Barger-Dabke theorem [18], [6] represents the analogical tool to Kharitonov theorem for spherical polynomial families. Furthermore, extensions are provided by the theorem of Barmish and Tempo [19], [6] based on the idea of the spectral set and the theorem of Biernacki, Hwang and Bhattacharyya [20], [6] which solves the robust stability for closed-loop system with affine linear uncertainty structure (e.g. a spherical plant family in feedback connection with a fixed controller). Besides, well-known Tsypkin-Polyak function [21] can be used for robust stability testing or actually for computation of robustness margin under spherical uncertainty. In fact, the spherical version of Tsypkin-Polyak criterion is related to the results given by Soh-Berger-Dabke theorem [6].

Nevertheless, the very universal technique based on the value set concept and the zero exclusion condition, which is described in [6], is applicable also to the spherical polynomial families.

The value set at each frequency ω for a spherical polynomial family (2) supposed in the form:

$$
\begin{aligned}
&p(s,q) = p_0(s) + \sum_{i=0}^{n} q_i s^i \\
&p_0(s) = p(s,q^0) = \sum_{i=0}^{n} a_i s^i \\
&\left\| q \right\|_{2,W} \leq r \\
&\deg p(s) \geq 1
\end{aligned}
\tag{14}
$$

is given [6], [22] by an ellipse centered at nominal $p_0(j\omega)$, with the major axis (in the real direction) having the length:

$$
R = 2r\left(\sum_{i\ even} w_i^2 \omega^{2i}\right)^{1/2} \tag{15}
$$

and with minor axis (in the imaginary direction) having the length:

$$
I = 2r\left(\sum_{i\ odd} w_i^2 \omega^{2i}\right)^{1/2} \tag{16}
$$

where W is a weighting matrix:

$$
W = diag\left(w_1^2, w_2^2, \ldots, w_n^2\right) \tag{17}
$$

Moreover, for the special degenerate case of $\omega = 0$, the value set is just the real interval:

$$
p(j0,Q) = \left\langle a_0 - r,\, a_0 + r \right\rangle \tag{18}
$$

The practical visualization of the ellipsoidal value sets can be conveniently performed by means of the Polynomial Toolbox 2.5 [11], [22], [10] by using the "spherplot" command.

Then, the zero exclusion condition can be applied for testing robust stability in the following way: The spherical polynomial family (2) with invariant degree and at least one stable member (e.g. nominal polynomial) is robustly stable if and only if the complex plane origin is excluded from the value set $p(j\omega,Q)$ at all frequencies $\omega \geq 0$, i.e. the spherical polynomial family is robustly stable if and only if:

$$
0 \notin p(j\omega,Q) \quad \forall \omega \geq 0 \tag{19}
$$

Generally, the detailed description, proofs and examples of the zero exclusion principle applications can be found in [6] or for instance in [14], [17].

V. Illustrative Examples

A. Example 1

First, suppose the spherical polynomial family defined by the uncertain polynomial:

$$\begin{aligned} p(s,q) = & \left(1+q_4\right)s^4 + \left(3.5+q_3\right)s^3 + \left(2.5+q_2\right)s^2 + \cdots \\ & + \left(1.5+q_1\right)s + \left(0.5+q_0\right) \end{aligned} \tag{20}$$

and by the uncertainty bounding set:

$$\begin{aligned} & \|q\|_{2,W} \le 0.5 \\ & W = diag\left(5,4,3,2,1\right) \end{aligned} \tag{21}$$

i.e.:

$$5q_0^2 + 4q_1^2 + 3q_2^2 + 2q_3^2 + q_4^2 \le 0.5 \tag{22}$$

The polynomial (20) can be easily expressed in the form (14) as:

$$\begin{aligned} p(s,q) = & s^4 + 3.5s^3 + 2.5s^2 + 1.5s + 0.5 + \cdots \\ & + q_4 s^4 + q_3 s^3 + q_2 s^2 + q_1 s + q_0 \end{aligned} \tag{23}$$

The nominal polynomial is stable and so the family fulfills the condition of at least one stable member. The value sets for the range of frequencies from 0 to 3 with step 0.01 was obtained with the assistance of the Polynomial Toolbox 2.5 for Matlab and its routine "spherplot" [11], [22]. They are plotted in Fig. 2.

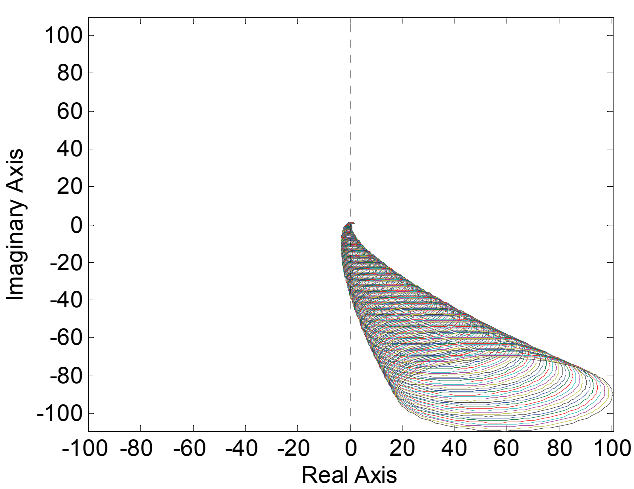

Fig. 2 The value sets for the family (20), (21)

The zoomed version of the same value sets, depicted in Fig. 3, provides better view of the neighborhood of the complex plane origin which is critical area for decision on robust (in)stability.

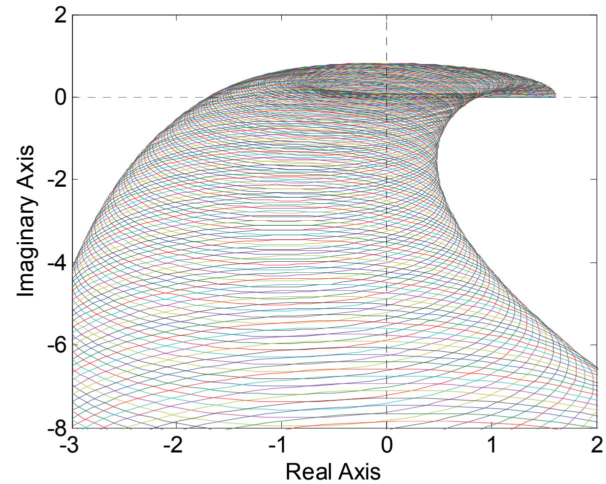

Fig. 3 The value sets for the family (20), (21) – a detailed view near the point [0, 0*j*]

As can be observed, the zero point is included in the value sets which means that the spherical polynomial family (20), (21) is not robustly stable. In other words, not all members of the prescribed family are stable.

The example of robustly stable case can be illustrated e.g. just by using the "narrower" uncertainty bounding set:

$$\begin{aligned} & \|q\|_{2,W} \le 0.1 \\ & W = diag\left(5,4,3,2,1\right) \end{aligned} \tag{24}$$

The full overview of the value sets for the same range of frequencies as in the previous plots can be seen in Fig. 4 and the zoomed version in Fig. 5. Obviously, the family has a stable member and the value sets do not include the origin of the complex plane and consequently the family (20), (24) is robustly stable.

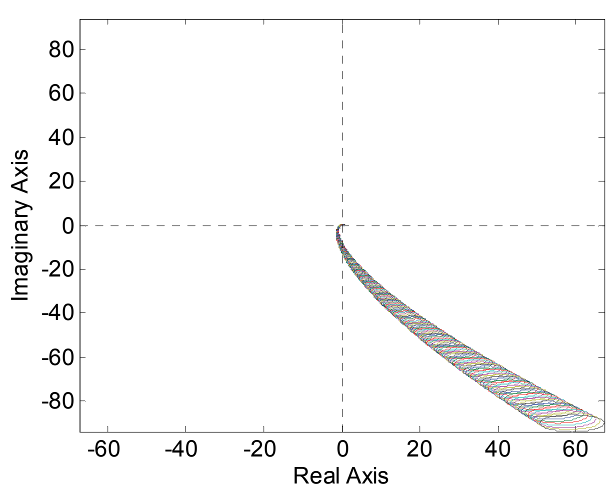

Fig. 4 The value sets for the family (20), (24)

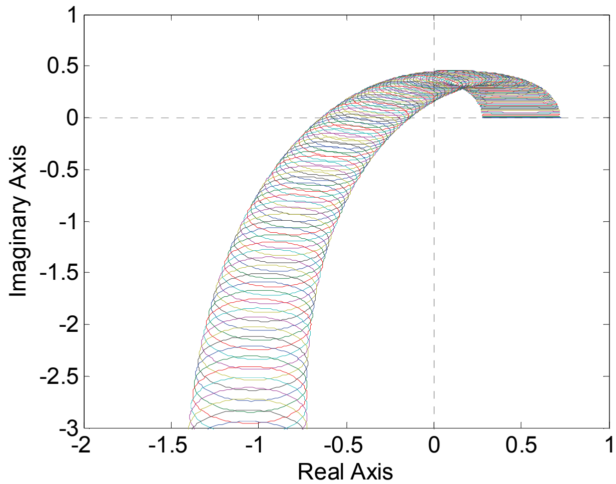

Fig. 5 The value sets for the family (20), (24) – a detailed view near the point [0, 0*j*]

B. Example 2

Now, assume another spherical family of polynomials given by the uncertain polynomial of tenth order:

$$\begin{aligned} p(s,q) &= (1+q_{10})s^{10} + (8+q_9)s^9 + (65+q_8)s^8 + \cdots \\ &+ (250+q_7)s^7 + (680+q_6)s^6 + (1150+q_5)s^5 + \cdots \\ &+ (680+q_4)s^4 + (250+q_3)s^3 + (65+q_2)s^2 + \cdots \\ &+ (8+q_1)s + (1+q_0) \end{aligned} \tag{25}$$

and by the corresponding uncertainty bounding set:

$$\|q\|_{2,W} \le 0.3; \quad W = diag(2,3,1,2,3,1,2,4,3,1,2) \tag{26}$$

The nominal polynomial is stable which means that the condition of at least one stable member is fulfilled. The value sets for the range of frequencies from 0 to 10 with step 0.01 are plotted in Fig. 6.

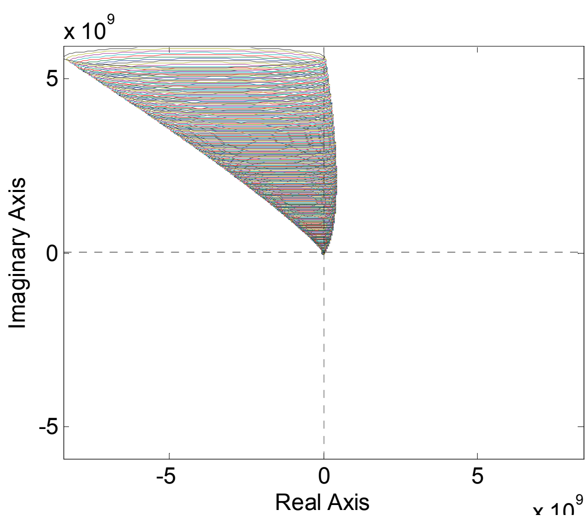

Fig. 6 The value sets for the family (25), (26)

Since the polynomial (25) is of the tenth order, the value sets from Fig. 6 successively go through ten quadrants. Four variously zoomed versions of the full Fig. 6 are shown in Figs. 7-10.

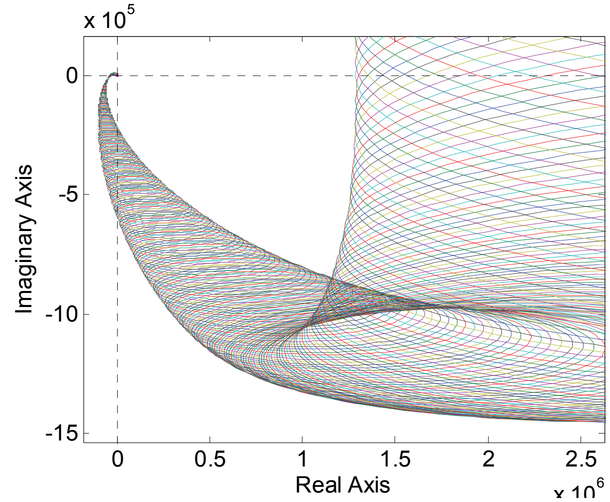

Fig. 7 The value sets for the family (25), (26) – a little zoomed view

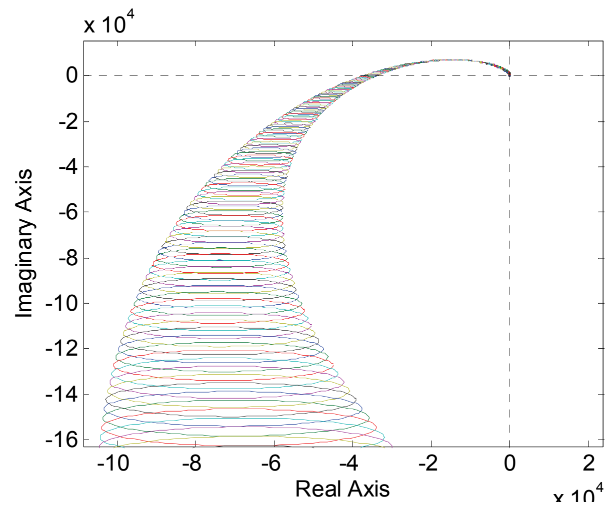

Fig. 8 The value sets for the family (25), (26) – a moderately zoomed view

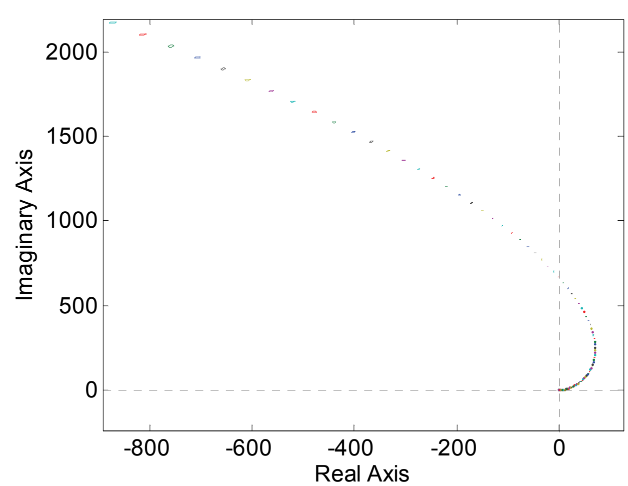

Fig. 9 The value sets for the family (25), (26) – a more zoomed view

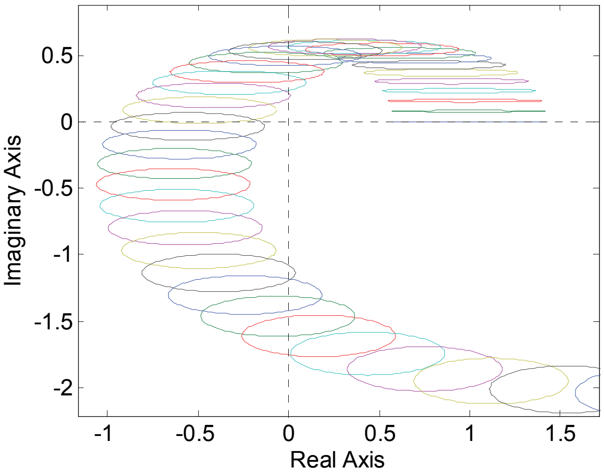

Fig. 10 The value sets for the family (25), (26) – a detailed view near the point [0, 0*j*]

The set of Figs. 6-10 reveals that the zero point is excluded from the value sets. Since the family has a stable member and the value sets do not include the origin, the family (25), (26) is robustly stable.

C. *Example 3*

The last example is intended to demonstrate the importance of at least one stable member precondition fulfillment, because if it is ignored it can lead to the wrong results.

Assume the spherical polynomial family:

$$
\begin{aligned}
&p(s,q) = (1+q_4)s^4 + (1+q_3)s^3 + (1+q_2)s^2 + \cdots \\
&+(1+q_1)s + (1+q_0) \qquad (27)\\
&\|q\|_{2,W} \le 0.1 \\
&W = diag(5,4,3,2,1)
\end{aligned}
$$

The corresponding value sets for the range of frequencies from 0 to 3 with step 0.01 are depicted in Fig. 11 with closer look near the origin in Fig. 12.

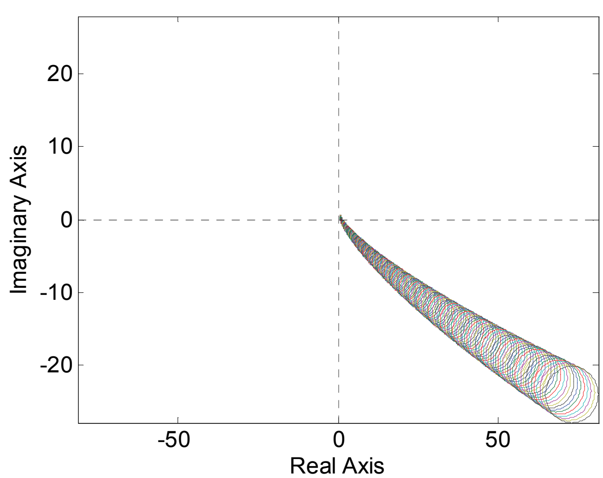

Fig. 11 The value sets for the family (27)

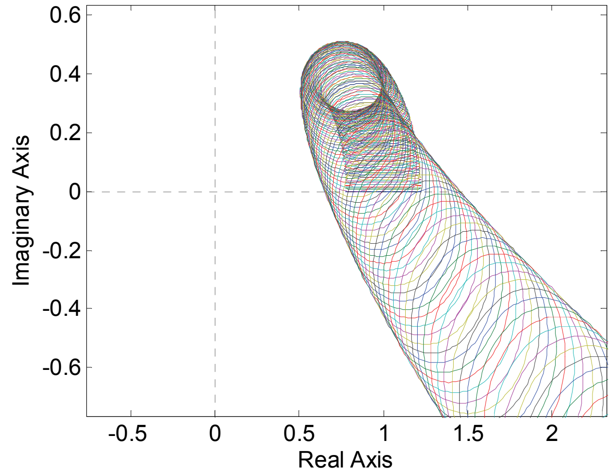

Fig. 12 The value sets for the family (27) – a detailed view near the point [0, 0j]

As the complex plane origin is obviously excluded from the value sets, it could (wrongly) indicate the robust stability of the family. However, the family does not have any stable member and thus the zero exclusion condition is not fulfilled actually. In fact, all members of the family are unstable which is the reason why the stability border is not crossed at all and why the zero point is not included.

VI. Conclusion

The paper has been aimed to an alternative bounding of uncertain parameters in systems with parametric uncertainty, i.e. the main object of interest has been the spherical polynomial family and its robust stability analysis. The basic theoretical descriptions have been accompanied by the set of simple illustrative examples supported by the Polynomial Toolbox for Matlab.

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