The Solution of Boundary Value Problems with Mixed Boundary Conditions via Boundary Value Methods

Grace O. Akinlabi, Raphael B. Adeniyi, and Enahoro A. Owoloko

***Abstract*—** Boundary Value Methods (BVMs) are methods based on Linear Multistep Methods (LMMs), which are used for the numerical approximation of Differential Equations (DEs). These methods were introduced to overcome the weaknesses of the LMMs.

In this paper, we introduce a new class of BVMs – Hybrid Boundary Value Methods (HBVMs) and used them to solve first order systems BVPs with mixed boundary conditions by using the specific cases: 2, 4 and 6. These methods are also based on LMMs where data are used at both step and off-step points.

The maximum errors and rate of convergence (ROC) of the solutions are reported for these cases to illustrate the effectiveness of these new class of methods.

***Keywords*—** Boundary value methods, boundary value problems, hybrid formula, linear multistep method.

I. Introduction

NUMERICAL analysis continues to be an active field of study in science and engineering as Numerical Analysts have introduced and continues to develop new and better numerical methods for solving Differential problems resulting from the modelisation of real world phenomena [1] – [4].

The Boundary Value Methods (BVMs) were introduced to overcome the limitations suffers by the Linear Multistep Methods (LMMs). Some of these problems are highlighted in [5] – [7].

Several BVMs have been introduced and their stability analysis fully investigated. See [7] – [18] for comprehensive work on BVMs.

In this work, we present the hybrids of the BVMs namely Hybrid Boundary Value Methods (HBVMs). As hybrid methods share the characteristic property of Runge-Kutta methods, which are more flexible than the LMMs in the way they are used [19] – [22]. Our intention is to develop BVMs that share this characteristic.

This is done by using the Adam Moulton Methods at both step and off-step points. These methods are then applied as BVMs and used to solve the two-point BVP of the form:

$$\begin{aligned} y_1'(x) &= f_1(x, y_1(x), y_2(x)) \\ y_2'(x) &= f_2(x, y_1(x), y_2(x)) \\ a_0 y(0) - b_0 y(0) &= \alpha_0, \quad a_1 y(1) - b_1 y(1) = \alpha_1 \end{aligned} \tag{1}$$

where all $f : \mathbb{R}^2 \to \mathbb{R}^2$ are continuous functions that satisfy the existence and uniqueness conditions, guaranteed by Henrici in [23]

The numerical integration of BVPs by BVMs were first considered by Brugnano and Trigiante in [24] where they used the two symmetric schemes: Extended Trapezoidal Rule (ETR) of order 4 and Top Order Method (TOM) of order 6.

II. Overview of the Boundary Value Methods [25]- [26]

In this section, we give a brief description of the BVMs. Consider the IVP:

$$y' = f(x, y), \quad y(x_0) = y_0, \quad x \in [x_0, x_N] \tag{2}$$

To approximate this problem, we consider the *k*-step LMF:

$$\sum_{r=0}^{k} \alpha_r y_{n+r} = h \sum_{r=0}^{k} \beta_r f_{n+r} \tag{3}$$

This discrete problem needs k independent conditions to be imposed so as to get the discrete solution $\{y_n\}$. Now, the first $k-1$ values need to be generated, since the IVP (2) has provided the first value y_0. Hence, we are to obtain the $k-1$ values: $y_0, \ldots, y_{k-1}$ of the discrete solution.

By this process, we say that the given continuous IVP has been approximated by means of a discrete IVP and this is what is known as IVM.

On the other hand, if we decide to fix the first k_1 values of the discrete solution, $y_0, \ldots, y_{k_1-1}$ and the last k_2 values of the discrete solution, $y_N, \ldots, y_{N+k_2-1}$ such that k_1, k_2 are integers and $k_1 + k_2 = k$. The discrete problem becomes.

$$\sum_{r=-k_1}^{k_2} \alpha_{r+k_1} y_{n+r} = h \sum_{r=-k_1}^{k_2} \beta_{r+k_1} f_{n+r} \tag{4}$$

By this, we have succeeded in fixing the first k_1 and final k_2 values of the discrete solution.

By this process, we say that the continuous IVP has been approximated by means of a discrete BVP and this approach is

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what is called BVM.

III. DERIVATION OF METHODS (HBVMS)

We shall construct, via interpolation and collocation, methods of the form:

$$y_{n+v} - y_{n+v-1} = h \sum_{r=0\left(\frac{1}{2}\right)}^{k} \beta_r f_{n+r}$$

$$\text{where} \quad v = \begin{cases} \dfrac{k+1}{2}, & \text{for odd } k \\ \dfrac{k}{2}, & \text{for even } k \end{cases}$$

For example for $k=1$, $v=1$ we have the formula

$$y_{n+1} - y_n = h\left[\beta_0 f_n + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_0 f_{n+1}\right]$$

After the derivation we implement these LMMs as BVMs while considering two specific cases: $k = 4$ and 6 .

A. *For case k=2*

The main method is as follows:

$$y_{n+2} - y_n = \frac{h}{45}\left[7 f_n + 12 f_{n+1} + 7 f_{n+2} + 32\left(f_{n+\frac{1}{2}} + 7 f_{n+\frac{3}{2}}\right)\right]$$

which is used together with the following initial methods:

$$y_{\frac{1}{2}} - y_0 = \frac{h}{1400}\left[251 f_0 - 264 f_1 - 19 f_2 + 646 f_{\frac{1}{2}} + 106 f_{\frac{3}{2}}\right]$$

and the final methods

$$y_{N-1} - y_N = -\frac{h}{180}\left[29 f_N + 24 f_{N-1} - f_{N-2} + 124 f_{N-\frac{1}{2}} + 4 f_{N-\frac{3}{2}}\right]$$

$$y_{N-\frac{3}{2}} - y_N = h\left[\frac{27}{160} f_N + \frac{9}{20} f_{N-1} - \frac{3}{160} f_{N-2} + \frac{51}{80} f_{N-\frac{1}{2}} + \frac{21}{80} f_{N-\frac{3}{2}}\right]$$

B. *For case k=4*

The main method is as follows:

$$y_{n+3} - y_{n+1} = \frac{h}{28350}\left[\begin{array}{l} 13 f_n + 5494 f_{n+1} + 10870 f_{n+2} \\ +5494 f_{n+3} + 13 f_{n+4} \\ -32\left(7 f_{n+\frac{1}{2}} - 551\left(f_{n+\frac{3}{2}} + f_{n+\frac{5}{2}}\right) + 7 f_{n+\frac{7}{2}}\right) \end{array}\right]$$

which is used together with the following initial methods:

$$y_{\frac{1}{2}} - y_1 = \frac{h}{7257600}\left[\begin{array}{l} 33953 f_0 - 3244786 f_1 - 1317280 f_2 \\ -294286 f_3 - 7297 f_4 - 1375594 f_{\frac{1}{2}} \\ +1752542 f_{\frac{3}{2}} + 755042 f_{\frac{5}{2}} + 68906 f_{\frac{7}{2}} \end{array}\right]$$

$$y_{\frac{3}{2}} - y_1 = \frac{h}{7257600}\left[\begin{array}{l} 7297 f_0 + 1638286 f_1 - 833120 f_2 \\ -142094 f_3 - 3233 f_4 - 99626 f_{\frac{1}{2}} \\ +2631838 f_{\frac{3}{2}} + 397858 f_{\frac{5}{2}} + 31594 f_{\frac{7}{2}} \end{array}\right]$$

$$y_{\frac{5}{2}} - y_1 = \frac{h}{89600}\left[\begin{array}{l} 81 f_0 + 19118 f_1 + 44640 f_2 - 2862 f_3 - 49 f_4 \\ -1098 f_{\frac{1}{2}} + 50814 f_{\frac{3}{2}} + 23234 f_{\frac{5}{2}} + 552 f_{\frac{7}{2}} \end{array}\right]$$

and the final methods

$$y_{N-4} - y_{N-1} = -\frac{h}{2800}\left[\begin{array}{l} -9 f_N + 158 f_{N-1} - 360 f_{N-2} + 18 f_{N-3} \\ +401 f_{N-4} + 8\left(\begin{array}{l} 9 f_{N-\frac{1}{2}} + 333 f_{N-\frac{3}{2}} \\ +403 f_{N-\frac{5}{2}} + 279 f_{N-\frac{7}{2}} \end{array}\right) \end{array}\right]$$

$$y_{N-\frac{7}{2}} - y_{N-1} = -\frac{5h}{290304}\left[\begin{array}{l} 85 f_N + 13606 f_{N-1} + 10546 f_{N-\frac{7}{2}} \\ +5\left(\begin{array}{l} 6560 f_{N-2} + 7442 f_{N-3} - 49 f_{N-4} \\ -202 f_{N-\frac{1}{2}} + 6014 f_{N-\frac{3}{2}} + 4418 f_{N-\frac{5}{2}} \end{array}\right) \end{array}\right]$$

$$y_{N-2} - y_{N-1} = -\frac{h}{226800}\left[\begin{array}{l} 127 f_N + 44446 f_{N-1} + 43480 f_{N-2} \\ -494 f_{N-3} - 23 f_{N-4} - 1976 f_{N-\frac{1}{2}} \\ +141928 f_{N-\frac{3}{2}} - 872 f_{N-\frac{5}{2}} + 184 f_{N-\frac{7}{2}} \end{array}\right]$$

$$y_N - y_{N-1} = -\frac{h}{226800}\left\{\begin{array}{l} -32377 f_N + 42494 f_{N-1} + 116120 f_{N-2} \\ +31154 f_{N-3} + 833 f_{N-4} \\ -8\left(\begin{array}{l} 22823 f_{N-\frac{1}{2}} + 15011 f_{N-\frac{3}{2}} \\ +9341 f_{N-\frac{5}{2}} + 953 f_{N-\frac{7}{2}} \end{array}\right) \end{array}\right\}$$

C. *For case k=6*

The main method is as follows:

$$y_{n+4} - y_{n+2} = \frac{h}{2554051500}\left[\begin{array}{l} 28151 f_n + 4721736 f_{n+1} + 529200405 f_{n+2} \\ +1047943344 f_{n+3} + 529200405 f_{n+4} \\ +4721736 f_{n+5} + 28151 f_{n+6} \\ -64\left(\begin{array}{l} 7923 f_{n+\frac{1}{2}} + 531095 f_{n+\frac{3}{2}} \\ -23916042 f_{n+\frac{5}{2}} - 23916042 f_{n+\frac{7}{2}} \\ +531095 f_{n+\frac{9}{2}} + 7923 f_{n+\frac{11}{2}} \end{array}\right) \end{array}\right]$$

which is used together with the following initial methods:

$$y_{\frac{1}{2}} - y_2 = \frac{h}{21525504000}\left[\begin{array}{l} 50840663 f_0 - 15631690812 f_1 \\ -17564506125 f_2 - 13516516608 f_3 \\ -4999623795 f_4 - 510865092 f_5 \\ -6279127 f_6 - 3507456066 f_{\frac{1}{2}} \\ -3243018230 f_{\frac{3}{2}} + 15178447404 f_{\frac{5}{2}} \\ +9451486164 f_{\frac{7}{2}} + 1927727350 f_{\frac{9}{2}} \\ +83198274 f_{\frac{11}{2}} \end{array}\right]$$

$$y_{\frac{3}{2}} - y_2 = \frac{h}{5230697472000}\left[\begin{array}{l} 456196373 f_0 + 72649122828 f_1 \\ -2008959454935 f_2 - 576826591488 f_3 \\ -171945526185 f_4 - 15894332172 f_5 \\ -184329877 f_6 - 7728247206 f_{\frac{1}{2}} \\ -1152341705090 f_{\frac{3}{2}} + 826951939524 f_{\frac{5}{2}} \\ +353393854524 f_{\frac{7}{2}} + 62574497410 f_{\frac{9}{2}} \\ +2505840294 f_{\frac{11}{2}} \end{array}\right]$$

$$y_{\frac{5}{2}} - y_2 = \frac{h}{5230697472000}\left[\begin{array}{l} 184329877 f_0 + 22105977612 f_1 \\ +1284137567145 f_2 - 510641870592 f_3 \\ -116161302825 f_4 - 9856152588 f_5 \\ -109551893 f_6 - 2852484774 f_{\frac{1}{2}} \\ -125367467650 f_{\frac{3}{2}} + 1771726903236 f_{\frac{5}{2}} \\ +260516522556 f_{\frac{7}{2}} + 40149664130 f_{\frac{9}{2}} \\ +1516601766 f_{\frac{11}{2}} \end{array}\right]$$

$$y_{\frac{7}{2}}-y_2=\frac{h}{21525504000}\left[\begin{array}{l}688087f_0+80355012f_1\\+4938122355f_2+10933456128f_3\\-824424435f_4-51176388f_5\\-521303f_6-10514754f_{\frac{1}{2}}\\-451692790f_{\frac{3}{2}}+11828012076f_{\frac{5}{2}}\\+5609039316f_{\frac{7}{2}}\\+229447670f_{\frac{9}{2}}+7465026f_{\frac{11}{2}}\end{array}\right]$$

$$y_{\frac{9}{2}}-y_2=\frac{5h}{41845579776}\left[\begin{array}{l}387173f_0+40903116f_1\\+2009196729f_2+4356823296f_3\\+4948419015f_4-100766412f_5\\-637669f_6-5670918f_{\frac{1}{2}}\\-211497890f_{\frac{3}{2}}+4450127364f_{\frac{5}{2}}\\+3692434428f_{\frac{7}{2}}+1732368034f_{\frac{9}{2}}\\+10703622f_{\frac{11}{2}}\end{array}\right]$$

$$y_{\frac{11}{2}}-y_2=-\frac{7h}{106748928000}\left\{\begin{array}{l}4616563f_0+390117588f_1\\+6701781375f_2+2481846474f_{\frac{11}{2}}\\-7\left(\begin{array}{l}2261849856f_3+2229061815f_4\\+1586077044f_5-5121461f_6\\-8852838f_{\frac{1}{2}}-224093890f_{\frac{3}{2}}\\+349026372f_{\frac{5}{2}}-230586948f_{\frac{7}{2}}\\+299226050f_{\frac{9}{2}}\end{array}\right)\end{array}\right\}$$

and the final methods

$$y_{N-3}-y_{N-2}=-\frac{h}{40864824000}\left[\begin{array}{l}584203f_N+83659728f_{N-1}\\+8440941375f_{N-2}+8383546752f_{N-3}\\+26265105f_{N-4}-8111952f_{N-5}\\-133787f_{N-6}-9718596f_{N-\frac{1}{2}}\\-561950380f_{N-\frac{3}{2}}+24975451224f_{N-\frac{5}{2}}\\-485424216f_{N-\frac{7}{2}}+18109100f_{N-\frac{9}{2}}\\+1605444f_{N-\frac{11}{2}}\end{array}\right]$$

$$y_{N-5}-y_{N-2}=-\frac{h}{168168000}\left[\begin{array}{l}-6887f_N-402672f_{N-1}+27874005f_{N-2}\\+51015552f_{N-3}+53970075f_{N-4}\\+30828528f_{N-5}+44983f_{N-6}+82644f_{N-\frac{1}{2}}\\+455900f_{N-\frac{3}{2}}+113824584f_{N-\frac{5}{2}}+117908664f_{N-\frac{7}{2}}\\+109885220f_{N-\frac{9}{2}}-976596f_{N-\frac{11}{2}}\end{array}\right]$$

$$y_{N-6}-y_{N-2}=\frac{2h}{638512875}\left\{\begin{array}{l}739276f_N+58489176f_{N-1}\\+491088915f_{N-2}+1267922544f_{N-3}\\+1006809120f_{N-4}+171401976f_{N-5}\\-42194069f_{N-6}\\-32\left(\begin{array}{l}302481f_{N-\frac{1}{2}}+6723935f_{N-\frac{3}{2}}\\+37948986f_{N-\frac{5}{2}}+51102126f_{N-\frac{7}{2}}\\+27054245f_{N-\frac{9}{2}}+9095811f_{N-\frac{11}{2}}\end{array}\right)\end{array}\right\}$$

$$y_{N-1}-y_{N-2}=\frac{h}{40864824000}\left\{\begin{array}{l}10480453f_N+7415784528f_{N-1}\\+4647521745f_{N-2}-4370314368f_{N-3}\\-1693823265f_{N-4}-173397072f_{N-5}\\-2123957f_{N-6}\\+4\left(\begin{array}{l}-57299919f_{N-\frac{1}{2}}+6811487435f_{N-\frac{3}{2}}\\+1040444586f_{N-\frac{5}{2}}+792336726f_{N-\frac{7}{2}}\\+163656245f_{N-\frac{9}{2}}+7048911f_{N-\frac{11}{2}}\end{array}\right)\end{array}\right\}$$

$$y_N-y_{N-2}=-\frac{h}{2554051500}\left\{\begin{array}{l}-337524401f_N+1375937544f_{N-1}\\+8583673365f_{N-2}+11191323696f_{N-3}\\+4457911725f_{N-4}+472635144f_{N-5}\\+5942359f_{N-6}\\-64\left(\begin{array}{l}36391167f_{N-\frac{1}{2}}+108748075f_{N-\frac{3}{2}}\\+180492462f_{N-\frac{5}{2}}+127879902f_{N-\frac{7}{2}}\\+27426835f_{N-\frac{9}{2}}+1217847f_{N-\frac{11}{2}}\end{array}\right)\end{array}\right\}$$

IV. NUMERICAL EXAMPLES

In this section, we apply the proposed methods to linear and nonlinear first order systems BVPs, which were converted from the second order BVPs in [27]. In Table I and Table II, the maximum errors and the convergence rates of the solutions for case 1 and case 2 are presented respectively. Also, Fig. 1 and Fig. 2 show the graphs of their exact solutions.

Case 1: Consider the linear BVP [27]:

$$y_1'=y_2$$

$$y_2'=\frac{y_1+xy_2}{1+x}$$

for $x\in(0,1)$

with boundary conditions:

$$y_1(0)-2y_2(0)=-1\,,\quad y_1(1)+2y_2(1)=3e$$

with exact solutions:

$$y_1(x)=e^x,\quad y_2(x)=e^x$$

Case 2: Consider the nonlinear BVP [27]:

$$y_1'=y_2$$

$$y_2'=\frac{e^{2y_1}+(y_2)^2}{2}$$

for $x\in(0,1)$

with boundary conditions:

$$y_1(0)-y_2(0)=1\,,\quad y_1(1)+y_2(1)=-ln2-\frac{1}{2}$$

with exact solutions:

$$y_1(x)=\log\frac{1}{1+x},\quad y_2(x)=-\frac{1}{1+x}$$

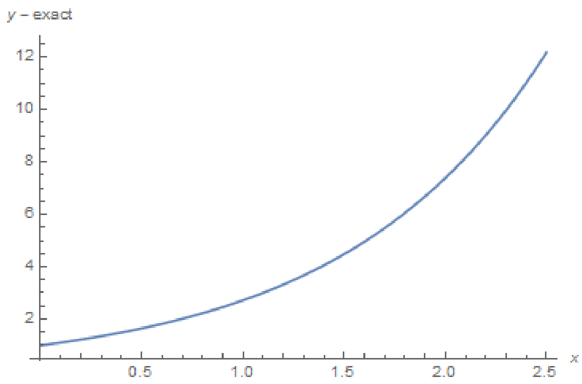

Fig. 1: Exact Solution of Case 1

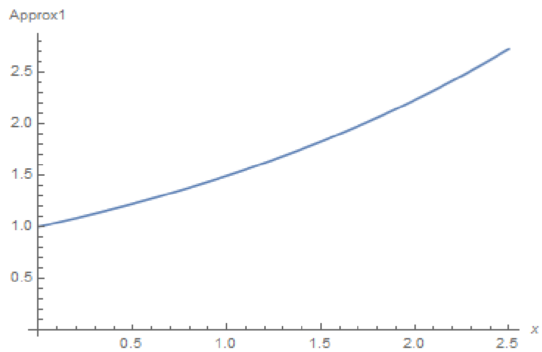

Fig. 2: Approximate Solution of Case 1 computed with HBVM (k = 2, h = 0.025)

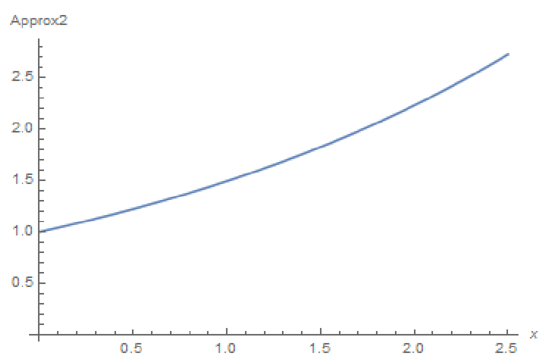

Fig. 3: Approximate Solution of Case 1 computed with HBVM (k = 4, h = 0.025)

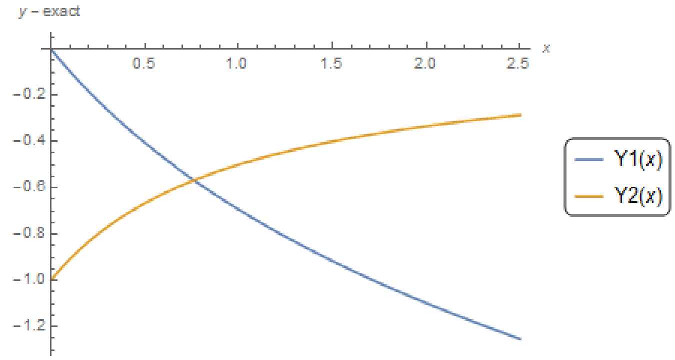

Fig. 4: Exact Solution of Case 2

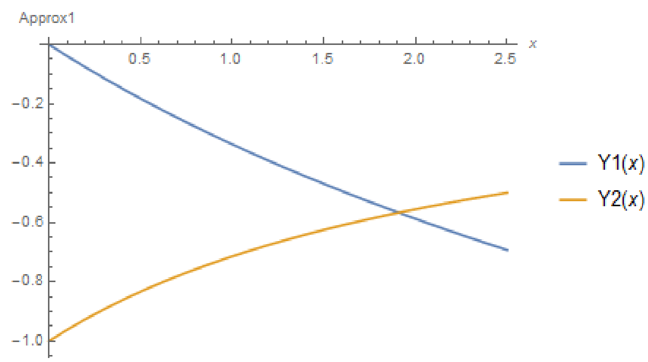

Fig. 5: Approximate Solution of Case 2 computed with HBVM (k = 4, h = 0.025)

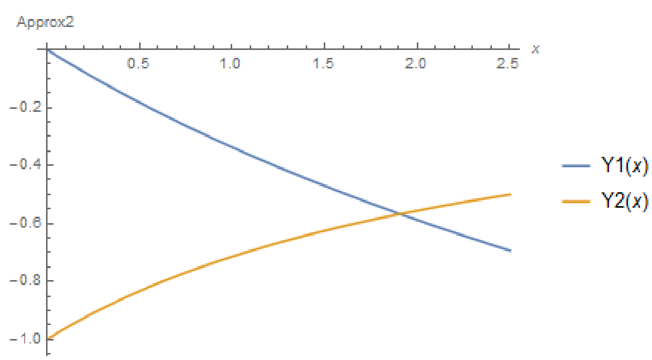

Fig. 6: Approximate Solution of Case 2 computed with HBVM (k = 6, h = 0.025)

V. Conclusion

A new class of BVMs: HBVMs were introduced with cases 2, 4 and 6 and applied to two-point BVPs with mixed boundary conditions. The maximum error and rate of convergence of the solutions were presented to illustrate the efficiency of these new methods.

Acknowledgment

The authors are grateful to Covenant University for the full support and also for the provision of good working environment. They also express their gratitude to the anonymous reviewer(s) for their helpful and constructive contributions.

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Table I: Maximum errors for HBVMs of order 2, 4 and 6 (Case 1)

h	HBVM of order 2		HBVM of order 4		HBVM of order 6	
	$\|e\|_\infty$	Rate	$\|e\|_\infty$	Rate	$\|e\|_\infty$	Rate
1e-1	1.24993e-09	-	1.50576e-11	-	9.07355e-09	-
5e-2	1.96973e-11	5.99	5.29879e-12	1.51	8.89963e-09	0.03
2.5e-2	3.09246-13	5.99	6.25188e-12	0.24	4.24738e-09	1.07
1.25e-2	5.24044e-15	5.88	5.40719e-12	0.21	6.98025e-09	0.72
6.25e-3	1.60119e-15	1.71	1.30816e-11	1.27	1.09309e-08	0.65

Table II: Maximum errors for HBVMs of order 2, 4 and 6 (Case 2)

h	HBVM of order 2		HBVM of order 4		HBVM of order 6	
	$\|e\|_\infty$	Rate	$\|e\|_\infty$	Rate	$\|e\|_\infty$	Rate
1e-1	1.35839e-07	-	3.70304e-07	-	4.51940e-08	-
5e-2	2.54603e-09	5.74	1.94881e-09	7.57	1.79255e-09	4.66
2.5e-2	4.39060-11	5.86	5.98631e-12	8.35	6.75463e-10	1.41
1.25e-2	7.21997e-13	5.93	4.08598e-12	0.55	1.73838e-09	1.36
6.25e-3	1.15875e-14	5.96	7.28745e-12	0.83	9.31409e-10	0.90