

# New Design Methodology for Adaptive 2-D (Two-Dimensional) IIR Notch filters

Nikos E. Mastorakis, Dimitris Tseles, Lazaros Vryzides

**Abstract** — A new method for the design of 2-D (Two-Dimensional) notch filters is proposed. We discuss the advantages of our methods in comparison with previously published methods in the 2-D (Two-Dimensional) Systems literature. An appropriate transformation is considered. Numerical examples illustrate the validity and the efficiency of the method.

**Keywords** — Adaptive Filters, Notch Filters, 2-D Notch Filters, Multidimensional Systems, Multidimensional Filters, Filter Design, Modeling, Simulation.

## I. INTRODUCTION

Adaptive Notch filters are adaptive band-stop filters with extremely narrow stop band. Adaptive techniques are characterized by flexibility and accuracy. Among the processing operations, linear filtering is probably the most common and important. It is adaptive if its parameters, i.e. the coefficients are varied according to a specified criterion as new information becomes available, [11]. Adaptive Notch filters are used in live sound reproduction, in instrument amplifiers design, in electrocardiogram (ECG) signal processing etc. So, for every case in signal processing and communications that an elimination of an undesirable frequency is necessary, an appropriate Notch Filter is necessary in order to cut-off this unwanted frequency. For example in ECG signal processing there is a need to eliminate the power line noise as it is added in the bandwidth of the ECG signal. In the one-dimensional (1-D) case, several methods for the design and performance analysis of IIR and FIR notch filters have been developed [1]–[3]. In this paper, we use the results of [3] and an attempt to extend them in 2-D case via appropriate transformations is presented. On the other hand, the adaptation here is achieved by using a 2-D adaptation law. This paper is organized as follows: Section II presents

First-Order 2-D IIR Notch Filters design together with a numerical example. In Section III, the design of a family of Second-Order 2-D IIR Notch Filters is presented. Some remarks can be found in IV and finally there is a

Conclusion. The adaptation law is an LMS based algorithm and is given in Section IV.

## II. THE PROPOSED METHOD FOR FIRST-ORDER IIR 2-D NOTCH FILTERS WITHOUT ADAPTATION

Consider the 1-D transfer function

$$H(z^{-1}) = K \frac{1 - z^{-1}}{1 - rz^{-1}} \quad (1)$$

with  $z^{-1} = e^{j\omega T}$   $-\pi \leq \omega \leq \pi$ ,  $T$  is the Sampling Period, and  $0 << r < 1$ . For  $0 << r < 1$  this 1-D transfer function is stable [3].

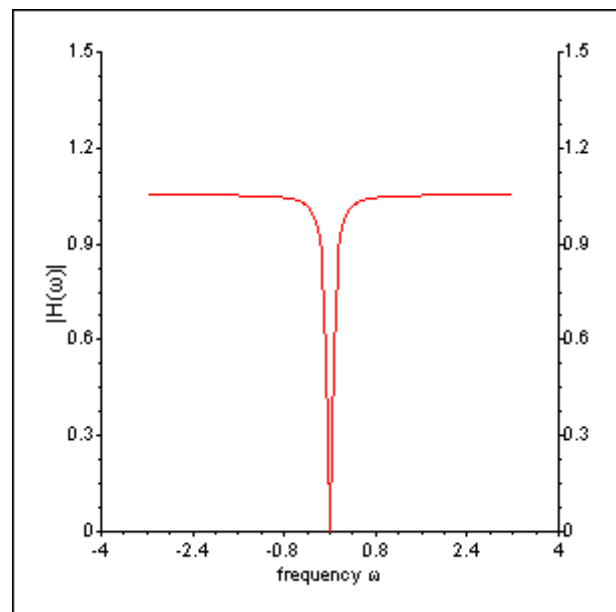


Fig. 1.a

$K$  is a scaling factor such that the maximum gain of the filter to be equal to 1. With the pole radius almost equal to 1, the pole almost cancels the effect of the zero except in the case  $z=1$ . So, this filter is an all-pass filter that rejects the frequency of  $\omega=0$  (e.g. DC frequency). The magnitude response is illustrated in Fig.1.a in the case of  $r=0.9$ , ( $T=1$  without loss of generality).

Nikos E. Mastorakis is with the Hellenic Naval Academy, Terma Hatzikyriakou, 18539, Piraeus, Greece [mastor@hna.gr](mailto:mastor@hna.gr)  
Dimitris Tseles and Lazaros Vryzides are with the University of West Attica, Egaleo, Athens, Greece [dtseles@puas.gr](mailto:dtseles@puas.gr), [proedros@puas.gr](mailto:proedros@puas.gr)

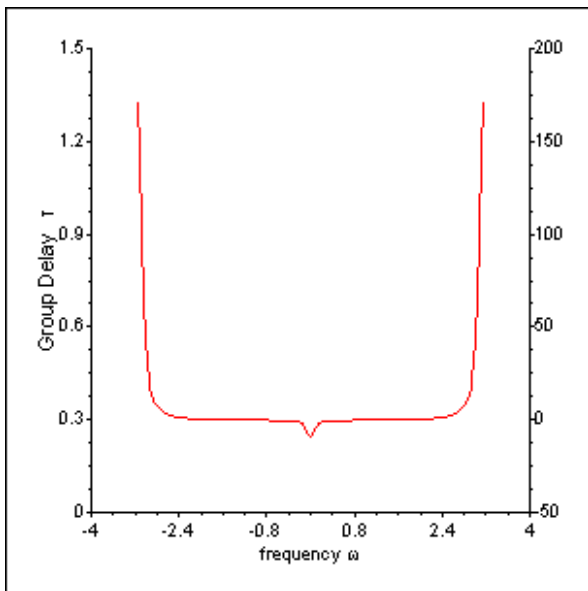


Fig. 1.b

The Group Delay  $\tau = -\frac{\partial \text{Arg}H(j\omega)}{\partial \omega}$  is depicted in Fig.

1.b and shows almost linear behavior in a big part of the frequency domain. The Notch filter of (1) is presented in [3]. In this section, we extend it to 2-D case as follows:

In this paper, we propose 2-D filters, based on the results of [3] by applying appropriate transformations. So, for the first-order notch filter of (1) considering the transformation

$$z^{-1} = \frac{z_1^{-1} + z_2^{-1}}{2} \tag{2}$$

we take

$$H(z^{-1}) = K \frac{2 - (z_1^{-1} + z_2^{-1})}{2 - r(z_1^{-1} + z_2^{-1})}$$

with  $z_1^{-1} = e^{j\omega_1 T_1}$ ,  $-\pi \leq \omega_1 \leq \pi$ ,  $z_2^{-1} = e^{j\omega_2 T_2}$ ,  $-\pi \leq \omega_2 \leq \pi$   $T_1, T_2$  are the sampling periods to horizontal and vertical direction whereas:  $0 << r < 1$

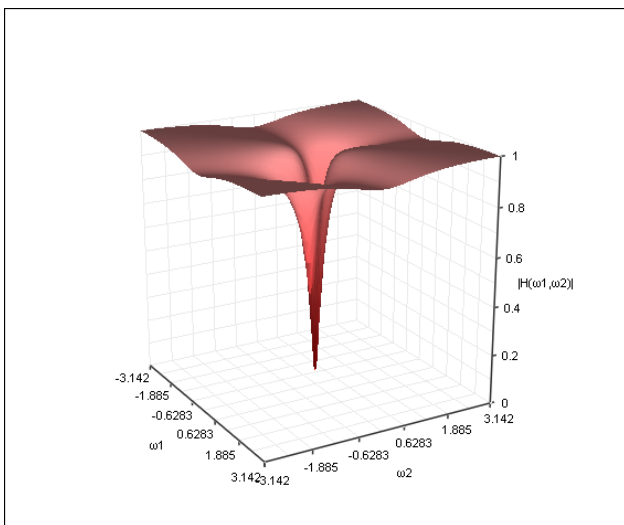


Fig. 2.a

Transformation (2) is remarkable because the Equation  $2 - (z_1^{-1} + z_2^{-1}) = 0$  has the unique solution  $z_1^{-1} = 1, z_2^{-1} = 1$  since  $z_1^{-1} = e^{j\omega_1 T_1}$ ,  $z_2^{-1} = e^{j\omega_2 T_2}$  and can be easily extended to a family of transformations as follows. Introducing

$$z^{-1} = \frac{\lambda_1 z_1^{-1} + \lambda_2 z_2^{-1}}{\lambda_1 + \lambda_2} \text{ with } \lambda_1, \lambda_2 \text{ real numbers or simply}$$

$$z^{-1} = \lambda z_1^{-1} + (1 - \lambda) z_2^{-1} \text{ with } 0 < \lambda < 1$$

one obtains

$$H(z^{-1}) = K \frac{1 - (\lambda z_1^{-1} + (1 - \lambda) z_2^{-1})}{1 - r(\lambda z_1^{-1} + (1 - \lambda) z_2^{-1})} \tag{3}$$

with  $z_1^{-1} = e^{j\omega_1 T_1}$ ,  $-\pi \leq \omega_1 \leq \pi$ ,  $z_2^{-1} = e^{j\omega_2 T_2}$ ,  $-\pi \leq \omega_2 \leq \pi$  ( $0 << r < 1$ )

The  $r < 1$  condition guarantees the 1-D and 2-D filter stability in all the above cases.

**Numerical Example 1:**

Consider without loss of generality  $T_1, T_2$  equal to 1. Then,

for  $r = 0.9$  and  $\lambda = \frac{1}{2}$  in (3), one finds  $K = 1.05260$ ,

the magnitude response is depicted in Fig.2.a, while the Group Delays

$$\tau_1 = -\frac{\partial \text{Arg}H(j\omega_1, j\omega_2)}{\partial \omega_1}, \tau_2 = -\frac{\partial \text{Arg}H(j\omega_1, j\omega_2)}{\partial \omega_2}$$

are depicted in Fig.2.b and Fig.2.c.

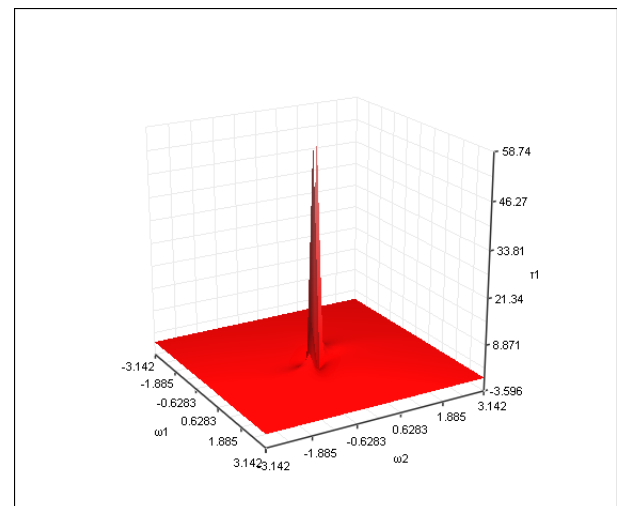


Fig.2.b

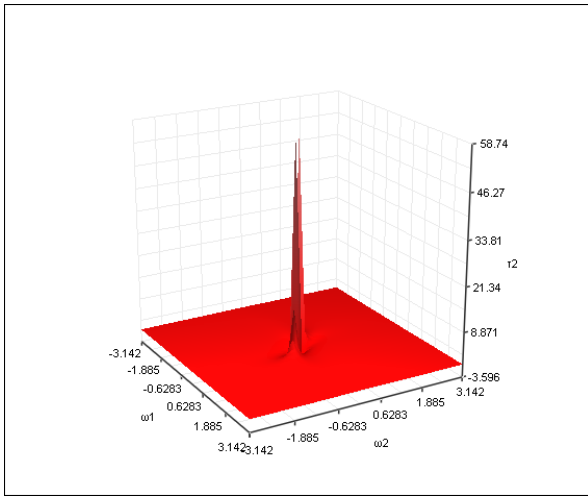


Fig.2.c

It is apparent that the family of the filters of (3) eliminates the 2-D frequency  $(\omega_1, \omega_2) = (0, 0)$ . Using this First-Order 2-D Notch filter, the only frequency that can be eliminated is  $(\omega_1, \omega_2) = (0, 0)$ . If elimination of another 2-D frequency  $(\omega_1, \omega_2) = (\omega_{10}, \omega_{20}) \neq (0, 0)$  is necessary, a second-order 2-D IIR notch filter must be used. As we prove in Section IV the 2-D first-order Notch filter is also Stable for  $0 << r < 1$ .

### III. THE PROPOSED METHOD FOR SECOND -ORDER IIR 2-D NOTCH FILTERS WITHOUT ADAPTATION

In this session, we extend (3) as follows in order to create a filter for rejection  $(\omega_1, \omega_2) = (\omega_{10}, \omega_{20})$

$$H(z_1^{-1}, z_2^{-1}) = K \frac{\lambda e^{j\omega_{10}T_1} + (1-\lambda)e^{j\omega_{20}T_2} - (\lambda z_1^{-1} + (1-\lambda)z_2^{-1})}{\lambda e^{j\omega_{10}T_1} + (1-\lambda)e^{j\omega_{20}T_2} - r(\lambda z_1^{-1} + (1-\lambda)z_2^{-1})} \cdot \frac{\lambda e^{-j\omega_{10}T_1} + (1-\lambda)e^{-j\omega_{20}T_2} - (\lambda z_1^{-1} + (1-\lambda)z_2^{-1})}{\lambda e^{-j\omega_{10}T_1} + (1-\lambda)e^{-j\omega_{20}T_2} - r(\lambda z_1^{-1} + (1-\lambda)z_2^{-1})} \quad (4)$$

with  $0 < \lambda < 1$ ,  $0 << r < 1$  and  $K$  is a scaling factor such that the maximum gain of the filter to be equal to 1. For  $0 << r < 1$  this 1-D transfer function is stable [3]. Eq. (4) can be written also as

$$H(z_1^{-1}, z_2^{-1}) = K \frac{A(z_1^{-1}, z_2^{-1})}{B(z_1^{-1}, z_2^{-1})}$$

where

$$A(z_1^{-1}, z_2^{-1}) = \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)\cos(\omega_{10}T_1 - \omega_{20}T_2) - 2(\lambda z_1^{-1} + (1-\lambda)z_2^{-1})(\lambda \cos \omega_{10}T_1 + (1-\lambda)\cos \omega_{20}T_2) + (\lambda z_1^{-1} + (1-\lambda)z_2^{-1})^2$$

and

$$B(z_1^{-1}, z_2^{-1}) = \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)\cos(\omega_{10}T_1 - \omega_{20}T_2) - 2r(\lambda z_1^{-1} + (1-\lambda)z_2^{-1})(\lambda \cos \omega_{10}T_1 + (1-\lambda)\cos \omega_{20}T_2) + r^2(\lambda z_1^{-1} + (1-\lambda)z_2^{-1})^2$$

Let's examine for what  $(z_1^{-1}, z_2^{-1})$  we have  $H(z_1^{-1}, z_2^{-1}) = 0$ . Because for the frequency response  $|z_1^{-1}| = 1, |z_2^{-1}| = 1$ , we have to examine the frequencies  $(\omega_1, \omega_2)$  for which

$$\lambda e^{j\omega_1 T_1} + (1-\lambda)e^{j\omega_2 T_2} - \lambda e^{j\omega_{10} T_1} - (1-\lambda)e^{j\omega_{20} T_2} = 0$$

$$\lambda e^{j\omega_1 T_1} + (1-\lambda)e^{j\omega_2 T_2} - \lambda e^{-j\omega_{10} T_1} - (1-\lambda)e^{-j\omega_{20} T_2} = 0$$

Using now:  $c = \frac{1-\lambda}{\lambda}$

$$(e^{j\omega_1 T_1} + c e^{j\omega_2 T_2}) - e^{j\omega_{10} T_1} - c e^{j\omega_{20} T_2} = 0 \quad (5.1)$$

$$(e^{j\omega_1 T_1} + c e^{j\omega_2 T_2}) - e^{-j\omega_{10} T_1} - c e^{-j\omega_{20} T_2} = 0 \quad (5.2)$$

We examine two cases

- $c = 1$ , that means  $\lambda = \frac{1}{2}$  and
- $c \neq 1$ , that means  $\lambda \neq \frac{1}{2}$

a) The first case yields the two equations:

$$e^{j\omega_1 T_1} + e^{j\omega_2 T_2} - e^{j\omega_{10} T_1} - e^{j\omega_{20} T_2} = 0 \quad (6.1)$$

$$e^{j\omega_1 T_1} + e^{j\omega_2 T_2} - e^{-j\omega_{10} T_1} - e^{-j\omega_{20} T_2} = 0 \quad (6.2)$$

From (6.1) one obtains the notch frequencies

$$\omega_1 = \omega_{10}, \omega_2 = \omega_{20}$$

and the symmetric solution

$$\omega_1 = \frac{T_2}{T_1} \omega_{20}, \omega_2 = \frac{T_1}{T_2} \omega_{10}$$

while from (6.2) two other couple of notch frequencies, i.e.  $\omega_1 = -\omega_{10}, \omega_2 = -\omega_{20}$

$$\omega_1 = -\frac{T_2}{T_1} \omega_{20}, \omega_2 = -\frac{T_1}{T_2} \omega_{10}$$

are obtained.

b) The second case yields also two equations:

$$e^{j\omega_1 T_1} + c e^{j\omega_2 T_2} - e^{j\omega_{10} T_1} - c e^{j\omega_{20} T_2} = 0 \quad (7.1)$$

$$e^{j\omega_1 T_1} + c e^{j\omega_2 T_2} - e^{j\omega_{10} T_1} - c e^{-j\omega_{20} T_2} = 0 \quad (7.2)$$

with  $c \neq 1$ .

So, from (7.1) one obtains the notch frequencies

$\omega_1 = \omega_{10}, \omega_2 = \omega_{20}$ , and from (7.2) the notch frequencies

$$\omega_1 = -\omega_{10}, \omega_2 = -\omega_{20}$$

Evidently, as 2-D IIR filter we can use only the case b) since the elimination of the “symmetric frequencies”

$(\omega_1 = \frac{T_2}{T_1} \omega_{20}, \omega_2 = \frac{T_1}{T_2} \omega_{10})$  is not required.

Therefore our 2-D IIR Notch Filter is given by (4) that can be also written as

$$H(z_1^{-1}, z_2^{-1}) = K \frac{A(z_1^{-1}, z_2^{-1})}{B(z_1^{-1}, z_2^{-1})}$$

where

$$A(z_1^{-1}, z_2^{-1}) = \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)\cos(\omega_{10}T_1 - \omega_{20}T_2) - 2(\lambda z_1^{-1} + (1-\lambda)z_2^{-1})(\lambda \cos(\omega_{10}T_1) + (1-\lambda)\cos(\omega_{20}T_2)) + (\lambda z_1^{-1} + (1-\lambda)z_2^{-1})^2$$

and

$$B(z_1^{-1}, z_2^{-1}) = \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)\cos(\omega_{10}T_1 - \omega_{20}T_2) - 2r(\lambda z_1^{-1} + (1-\lambda)z_2^{-1})(\lambda \cos(\omega_{10}T_1) + (1-\lambda)\cos(\omega_{20}T_2)) + r^2(\lambda z_1^{-1} + (1-\lambda)z_2^{-1})^2$$

with  $0 \ll r < 1 < \lambda < 1$ ,  $\lambda \neq 0.5$  and  $K$  is a scaling factor such that the maximum gain of the filter to be equal to 1.

Using  $c = \frac{1-\lambda}{\lambda}$  now, a further simplification of the second-order 2-D IIR Notch Filter transfer function is

$$H(z_1^{-1}, z_2^{-1}) = K \frac{1^2 + c^2 + 2c \cos(\omega_{10}T_1 - \omega_{20}T_2) - 2(z_1^{-1} + cz_2^{-1})(\cos(\omega_{10}T_1) + c \cos(\omega_{20}T_2)) + (z_1^{-1} + cz_2^{-1})^2}{1^2 + c^2 + 2c \cos(\omega_{10}T_1 - \omega_{20}T_2) - 2r(z_1^{-1} + cz_2^{-1})(\cos(\omega_{10}T_1) + c \cos(\omega_{20}T_2)) + r^2(z_1^{-1} + cz_2^{-1})^2} \tag{8}$$

where  $c \neq 1$

**Numerical Example 2:**

Consider the 2-D IIR Notch Filter of (8). Suppose that we want the cancellation of  $\omega_{10} = \frac{\pi}{2}, \omega_{20} = \frac{\pi}{4}$  (and of course the symmetric  $\omega_{10} = -\frac{\pi}{2}, \omega_{20} = -\frac{\pi}{4}$ ). One can choose for example  $c = 2, r = 0.9$ . Consider also without loss of generality  $T_1, T_2 = 1$ . Then

$$H(z_1^{-1}, z_2^{-1}) = K \frac{(z_1^{-1} + 2z_2^{-1} + 3j)(z_1^{-1} + 2z_2^{-1} - 3j)}{(rz_1^{-1} + r2z_2^{-1} + 3j)(rz_1^{-1} + r2z_2^{-1} - 3j)}$$

with  $z_1^{-1} = e^{j\omega_1} - \pi \leq \omega_1 \leq \pi, z_2^{-1} = e^{j\omega_2}, -\pi \leq \omega_2 \leq \pi$

Hence, the magnitude response is depicted in Fig.3.a, while the Group Delays

$$\tau_1 = -\frac{\partial \text{Arg}H(j\omega_1, j\omega_2)}{\partial \omega_1}, \tau_2 = -\frac{\partial \text{Arg}H(j\omega_1, j\omega_2)}{\partial \omega_2}$$

are depicted in Fig.3.b and Fig.3.c

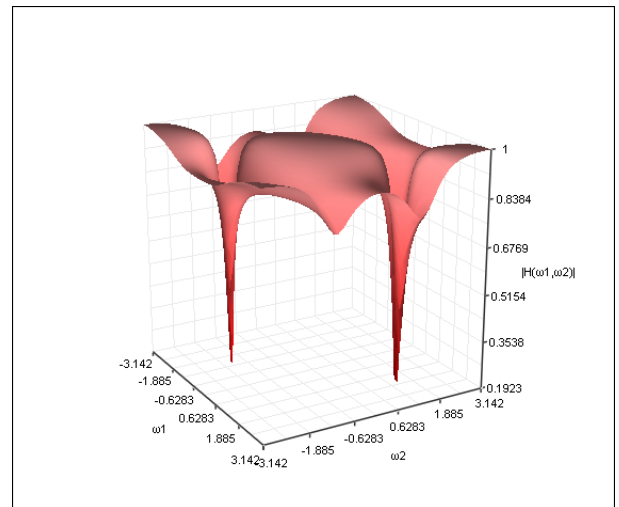


Fig.3.a

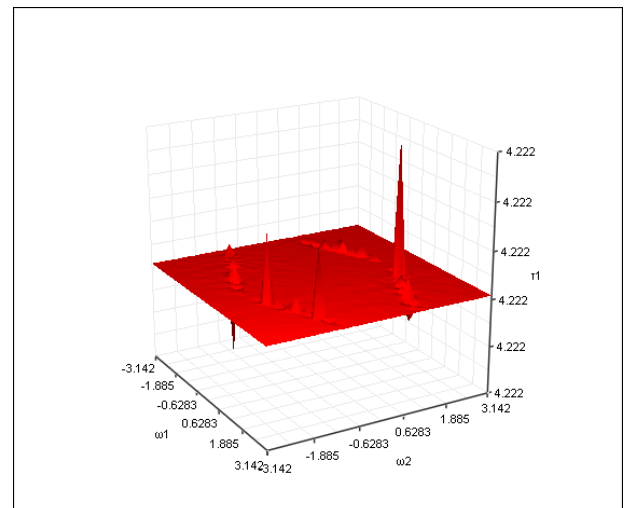


Fig.3.b

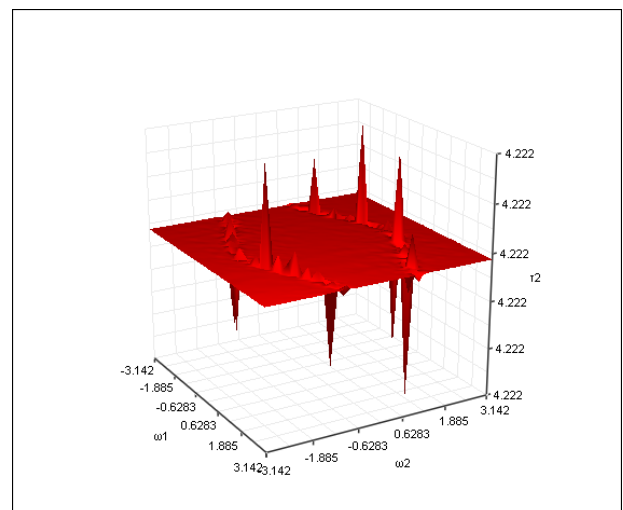


Fig.3.c

As we prove in the next Section, our 2-D second-order Notch filter is also Stable for  $0 << r < 1$ .

#### REMARKS

A first remark is that 2-D filters with several notch frequencies can be easily implemented by cascade design, while by using the new transformations

$z_1^{-1} = z_1^{-P_1}$  and  $z_2^{-1} = z_2^{-P_2}$  where  $P_1, P_2$  are positive integers, except the notch frequencies

$$\omega_1 = \pm\omega_{10}, \omega_2 = \pm\omega_{20}$$

the following notch frequencies are obtained

$$\omega_1 = \pm \frac{k_1}{P_1} \omega_{10}, \omega_2 = \pm \frac{k_2}{P_2} \omega_{20}$$

$$k_1 = 1, 2, \dots, P_1 \text{ and } k_2 = 1, 2, \dots, P_2$$

Therefore periodic 2-D notch filters can easily implemented.

#### IV. ADAPTIVE 2-D NOTCH FILTERS

The parameters in the aforementioned filters can be estimated from the input signal by an LMS adaptation Law. Suppose that our 2-D filter has the parameters  $a_i$  with  $i=1,2,\dots$  for example  $a_1 = \omega_{10}$ ,  $a_2 = \omega_{20}$  for the second order 2-D filter.

Then, a gradient LMS adaptation algorithm is

$$a_i(m+1) = a_i(m) - \delta \frac{\partial e(m)}{\partial a_i(m)} e(m)$$

where the parameter  $m$  declares time.

Taking into account that

$$e(m) = \frac{1}{2} \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} (y(n_1, n_2) - y_d(n_1, n_2))^2$$

where  $y_d(n_1, n_2)$  is the desired (reference signal) the

derivative  $\frac{\partial e(m)}{\partial a_i(m)}$  can be easily calculated as follows

$$\frac{\partial e(m)}{\partial a_i(m)} = \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} (y(n_1, n_2) - y_d(n_1, n_2)) \frac{\partial y(n_1, n_2)}{\partial a_i(m)}$$

Therefore the adaptation LMS law can be easily formulated in each case.

#### V. CONCLUSION

A new efficient and elegant technique for adaptive 2-D Notch Filter Design is investigated in this paper. Some other studies of the author for the stability of  $m$ -D systems can be found in [4] ÷ [10]. Work is in progress by the author towards of statement new  $m$ -D design techniques better and more effective than the McClellan Transformations.

#### REFERENCES

- [1] R. Carney, "Design of a digital notch filter with tracking requirements," IEEE Trans. Space Electron. Telem., vol. SET-9, pp. 109-114, Dec. 1963.
- [2] K. Hirano, S. Nishimura, and S. K. Mitra, "Design of digital notch filters," IEEE Trans. Circuits Syst., vol. CAS-21, pp. 540-546, July 1974.
- [3] T.I.Laakso, J. Ranta, S.J. Ovaska, "Design and implementation of efficient IIR notch filters with quantization error feedback", IEEE Transactions on Instrumentation and Measurement, Vol.43, No.3, pp. 449 - 456, Jun 1994
- [4] N.E.Mastorakis, "A method for computing the 2-D stability margin based on a new stability test for 2-D systems", Multidimensional Systems and Signal Processing, Vol.10, pp.93-99, 1998.
- [5] N.E.Mastorakis, "New Necessary Stability Conditions for 2-D Systems", IEEE Transactions on Circuits and Systems, Part I, Vol.47, No.7, pp.1103-1105, July 2000.
- [6] N.E.Mastorakis, "A method for computing the 2-D stability margin", IEEE Transactions on Circuits and Systems, Part II, Vol.45, No.3, pp.376-379, March 1998.
- [7] N.E.Mastorakis, "Recursive Algorithms for Two-Dimensional Filters' Spectral Transformations", IEEE Transactions on Signal Processing. Vol.44, No.10, pp.2647-2651, Oct.1996.
- [8] V. Mladenov, and N. Mastorakis, "Design of 2-Dimensional Recursive Filters by using Neural Networks", IEEE Trans. on Neural Networks.
- [9] Mastorakis, N.E., Gonos, I.F., and Swamy, M.N.S., "Stability of multidimensional systems using genetic algorithms", IEEE Trans. on Circuits and Systems, Part I: Fundamental Theory and Applications, Volume 50, Issue 7, July 2003, pp.962 - 965.
- [10] Mastorakis, N.E. Swamy, M.N.S., "Spectral transformations for two-dimensional filters via FFT" IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications, Vol.49, Issue 6, pp.827 - 831, Jun 2002
- [11] Maurice Bellanger, "Adaptive Digital Filters and Signal Analysis", Marcel Dekker, Basel, Switzerland, 2001.