Reichenbach and f-generated implications in fuzzy database relations

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***Abstract*—** Applying a definition of attribute conformance based on a similarity relation, we introduce an interpretation as a function associated to some fuzzy relation instance and defined on the universal set of attributes. As a consequence, the attributes become fuzzy formulas. Conjunctions, disjunctions and implications between the attributes become fuzzy formulas as well in view of the requirement that the interpretation has to agree with the minimum t-norm, the maximum t-conorm and appropriately chosen fuzzy implication. The purpose of this paper is to derive a number of results related to these fuzzy formulas if the fuzzy implication is selected so to be either Reichenbach or some f-generated fuzzy implication.

***Keywords*—** Conformance, Fuzzy implications, Interpretations, Similarity relations.

I. Introduction

In this paper we relate fuzzy dependencies and fuzzy logic theories by joining fuzzy formulas to fuzzy functional and fuzzy multivalued dependencies.

We research the concept of fuzzy relation instance that actively satisfies some fuzzy multivalued dependency. We determine the necessary and sufficient conditions needed to given two-element fuzzy relation instance actively satisfies some fuzzy multivalued dependency. In particular, for Reichenbach and some f-generated fuzzy implication operators, we prove that a two-element fuzzy relation instance actively satisfies given fuzzy multivalued dependency if and only if:

1) tuples of the instance are conformant on certain, well known set of attributes with degree of conformance greater than or equal to some explicitly known constant,

2) related fuzzy formula is satisfiable in appropriate interpretations.

Finally, for Reichenbach and some f-generated fuzzy implication operators, we prove that any two-element fuzzy relation instance which satisfies all dependencies from the set F satisfies the dependency f if and only if satisfiability of all formulas from the set F' implies satisfiability of the formula f'. Here, $f \notin \mathrm{F}$ is a fuzzy functional or a fuzzy multivalued dependency, F is a set of fuzzy functional and fuzzy multivalued dependencies, F' resp. f' denote the set of fuzzy formulas resp. the fuzzy formula related to F resp. f.

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II. Preliminaries

We introduce the minimum t-norm (see. e.g., [11], [9], [12]), the maximum t-conorm (see. e.g., [11], [7]) as follows

$$\mathrm{T}(p \& q) = min(\mathrm{T}(p), \mathrm{T}(q)),$$

$$\mathrm{T}(p \| q) = max(\mathrm{T}(p), \mathrm{T}(q)),$$

where $0 \leq \mathrm{T}(p)$, $\mathrm{T}(q) \leq 1$. Here, $\mathrm{T}(m)$ is the truth value od m.

An interpretation $\mathcal{I}$ is said to satisfy resp. falsify formula f if $\mathrm{T}(f) \geq \frac{1}{2}$ resp. $\mathrm{T}(f) \leq \frac{1}{2}$ under $\mathcal{I}$ (see. e.g., [13]).

We introduce the notation following similarity-based fuzzy relational database approach [16] (see also, [3]-[5]).

A similarity relation on D is a mapping $\mathrm{s}: \mathrm{D} \times \mathrm{D} \to [0,1]$ such that (see, [21])

$$\mathrm{s}(x,x) = 1,$$

$$\mathrm{s}(x,y) = \mathrm{s}(y,x),$$

$$\mathrm{s}(x,z) \geq \max_{y \in \mathrm{D}}\left(min(\mathrm{s}(x,y), \mathrm{s}(y,z))\right),$$

where D is a set and $x,y,z \in \mathrm{D}$.

Let $\mathrm{R}(\mathrm{U}) = \mathrm{R}(\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n)$ be a scheme on domains $\mathrm{D}_1, \mathrm{D}_2, ..., \mathrm{D}_n$ where U is the set od all attributes $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n$ on $\mathrm{D}_1, \mathrm{D}_2, ..., \mathrm{D}_n$ (we say that U is the universal set of attributes). Here, we assume that the domain of $\mathcal{B}_i$ is the finite set D_i, $i = 1,2,...,n$.

A fuzzy relation instance r on $\mathrm{R}(\mathrm{U})$ is defined as a subset of the cross product of the power sets $2^{\mathrm{D}_1}, 2^{\mathrm{D}_2}, ..., 2^{\mathrm{D}_n}$ of the domains of the attributes. A member of a fuzzy relation instance corresponding to a horizontal row of the table is called a tuple. More precisely, a tuple is an element t of r of the form $(\mathrm{d}_1,\mathrm{d}_2,...,\mathrm{d}_n)$, where $\mathrm{d}_i \subseteq \mathrm{D}_i$, $\mathrm{d}_i \neq \varnothing$ (see also, [8]). Here, we consider d_i as the value of $\mathcal{B}_i$ on t.

Recall that the similarity based database approach allows each domain to be equipped with a similarity relation.

The conformance of attribute $\mathcal{B}$ defined on domain D for any two tuples t_1 and t_2 present in relation instance r and denoted by $\mathcal{B}^{t_1,t_2}$ is defined by

$$\mathcal{B}^{t_1,t_2} = min\left\{ \min_{x\in \mathrm{d}_1}\left\{ \max_{y\in \mathrm{d}_2}\left\{ \mathrm{s}(x,y)\right\}\right\}, \min_{x\in \mathrm{d}_2}\left\{ \max_{y\in \mathrm{d}_1}\left\{ \mathrm{s}(x,y)\right\}\right\}\right\},$$

where d_I denote the value od attribute $\mathcal{B}$ for tuple t_i, $i=1,2$ and $\mathrm{s}:\mathrm{D}\times\mathrm{D}\rightarrow[0,1]$ is a similarity relation on D.

If $\mathcal{B}^{t_1,t_2} \geq q$, where $0 \leq q \leq 1$, than the tuples t_1 and t_2 are said to be conformant on attribute $\mathcal{B}$ with q.

The conformance of attribute set $\mathcal{X}$ for any two tuples t_1 and t_2 present in fuzzy relation instance r and denoted by $\mathcal{X}^{t_1,t_2}$ is defined by

$$\mathcal{X}^{t_1,t_2} = \min_{\mathcal{B}\in\mathcal{X}}\left\{\mathcal{B}^{t_1,t_2}\right\}.$$

Obviously: 1) $\mathcal{X}^{t,t}=1$ for any t in r,

2) If $\mathcal{X}\supseteq\mathcal{Y}$, then $\mathcal{Y}^{t_1,t_2} \geq \mathcal{X}^{t_1,t_2}$ for any t_1 and t_2 in r,

3) If $\mathcal{X}=(\mathcal{B}_1,\mathcal{B}_2,..,\mathcal{B}_m)$ and $\mathcal{B}_k^{t_1,t_2} \geq q$ for all $k\in\{1,2,...,m\}$, then $\mathcal{X}^{t_1,t_2} \geq q$ for any t_1 and t_2 in r.

Let r be any fuzzy relation instance on scheme $\mathrm{R}(\mathcal{B}_1,\mathcal{B}_2,...,\mathcal{B}_n)$, U be the universal set of attributes $\mathcal{B}_1,\mathcal{B}_2,...,\mathcal{B}_n$ and $\mathcal{X},\mathcal{Y}$ be subsets of U.

Fuzzy relation instance r is said to satisfy the fuzzy functional dependency $\mathcal{X} \xrightarrow{\theta}_F \mathcal{Y}$ if for every pair of tuples t_1 and t_2 in r, $\mathcal{Y}^{t_1,t_2} \geq min(\theta,\mathcal{X}^{t_1,t_2})$.

Fuzzy relation instance r is said to satisfy the fuzzy multivalued dependency $\mathcal{X} \rightarrow\xrightarrow{\theta}_F \mathcal{Y}$ if for every pair of tuples t_1 and t_2 in r, there exists a tuple t_3 in r such that:

$$\begin{aligned} &\mathcal{X}^{t_3,t_1} \geq min(\theta,\mathcal{X}^{t_1,t_2}),\\ &\mathcal{Y}^{t_3,t_1} \geq min(\theta,\mathcal{X}^{t_1,t_2}), \qquad (1)\\ &\mathcal{Z}^{t_3,t_2} \geq min(\theta,\mathcal{X}^{t_1,t_2}),\end{aligned}$$

where $\mathcal{Z}=\mathrm{U}-\mathcal{XY}$. Here $\mathrm{U}-\mathcal{XY}$ means $\mathrm{U}\setminus(\mathcal{X}\cup\mathcal{Y})$. Moreover, $0\leq\theta\leq 1$ describes the linguistic strength of the dependency. Namely, some dependencies are precise, some of them are not, some dependencies are more precise than the other ones. Therefore, the linguistic strength of the dependency gives us a method for describing imprecise dependencies as well as precise ones.

Fuzzy relation instance r is said to satisfy the fuzzy multivalued dependency $\mathcal{X} \rightarrow\xrightarrow{\theta}_F \mathcal{Y}$, θ-actively if r satisfies that dependency and if $\mathcal{B}^{t_1,t_2} \geq \theta$ for all $\mathcal{B}\in\mathcal{X}$ and all $t_1,t_2 \in \mathrm{r}$.

It follows immediately that the instance r satisfies the dependency $\mathcal{X} \rightarrow\xrightarrow{\theta}_F \mathcal{Y}$, θ-actively if and only if r satisfies $\mathcal{X} \rightarrow\xrightarrow{\theta}_F \mathcal{Y}$ and $\mathcal{X}^{t_1,t_2} \geq \theta$ for all $t_1,t_2 \in \mathrm{r}$.

Let $\mathrm{r}=\{t_1,t_2\}$ be any two-element fuzzy relation instance on scheme $\mathrm{R}(\mathcal{B}_1,\mathcal{B}_2,...,\mathcal{B}_n)$ and $0\leq\varepsilon\leq 1$.

A mapping $\mathrm{v}_\varepsilon^{\mathrm{r}} : \{\mathcal{B}_1,\mathcal{B}_2,...,\mathcal{B}_n\} \rightarrow [0,1]$ such that

$$\mathrm{v}_\varepsilon^{\mathrm{r}}(\mathcal{B}_k) > \frac{1}{2} \text{ if } \mathcal{B}^{t_1,t_2} \geq \varepsilon,$$

$$\mathrm{v}_\varepsilon^{\mathrm{r}}(\mathcal{B}_k) \leq \frac{1}{2} \text{ if } \mathcal{B}^{t_1,t_2} < \varepsilon,$$

$k=1,2,...,n$, is called a valuation (or an interpretation) joined to r and ε.

III. Results

Let $\mathcal{X} \xrightarrow{\theta}_F \mathcal{Y}$ $\left(\mathcal{X} \rightarrow\xrightarrow{\theta}_F \mathcal{Y}\right)$ be some fuzzy functional dependency (fuzzy multivalued dependency) on U, where U is the universal set of attributes $\mathcal{B}_1,\mathcal{B}_2,..,\mathcal{B}_n$ and $\mathrm{R}(\mathcal{B}_1,\mathcal{B}_2,...,\mathcal{B}_n)$ is a scheme.

In this paper we associate the fuzzy formula

$$\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right) \rightarrow \left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)$$

to $\mathcal{X} \xrightarrow{\theta}_F \mathcal{Y}$ and the fuzzy formula

$$\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right) \rightarrow \left(\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right) \| \left(\underset{\mathcal{C}\in\mathcal{Z}}{\&}\mathcal{C}\right)\right)$$

to $\mathcal{X} \rightarrow\xrightarrow{\theta}_F \mathcal{Y}$, where $\mathcal{Z}=\mathrm{U}-\mathcal{XY}$.

Through the rest of the section, we assume that the fuzzy implication operator is given either by

$$\mathrm{T}(p\rightarrow q) = \mathrm{T}(q)^{\mathrm{T}(p)}$$

if $\mathrm{T}(p)\neq 0$ or $\mathrm{T}(q)\neq 0$, $\mathrm{T}(p\rightarrow q)=1$ if $\mathrm{T}(p)=0$ and $\mathrm{T}(q)=0$, or by

$$\mathrm{T}(p\rightarrow q) = 1-\mathrm{T}(p)+\mathrm{T}(p)\mathrm{T}(q).$$

Note that the first fuzzy implication operator is known as Yager's (Y) operator (see, [19]). It is a typical example of f-generated fuzzy implication operator (see, [15], [20]). The second fuzzy implication operator is widely-known as Kleene-

Dienes-Lukasiewicz operator or Reichenbach (R) operator (see, [14]). It represents a classical example of strong (S) and quantum logic (QL) implication (see, [15], [17], [18]). We refer to [6], [10] and [1] as well. In general, classes of fuzzy implication operators are very nicely described in [2] and [15].

Theorem 1. Let $r=\{t_1,t_2\}$ be any two-element fuzzy relation instance on scheme $R(\mathcal{B}_1,\mathcal{B}_2,...,\mathcal{B}_n)$, U be the universal set of attributes $\mathcal{B}_1,\mathcal{B}_2,...,\mathcal{B}_n$ and $\mathcal{X},\mathcal{Y}$ be subsets of U. Let $\mathcal{Z}=U-\mathcal{X}\mathcal{Y}$. Then, r satisfies the fuzzy multivalued dependency $\mathcal{X}\to\xrightarrow{\theta}_F\mathcal{Y}$, θ-actively if and only if $\mathcal{X}^{t_1,t_2}\geq\theta$ and $v_\theta^r(\mathcal{K})>\frac{1}{2}$, where $\mathcal{K}$ denotes the fuzzy formula $(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})\to((\&_{\mathcal{B}\in Y}\mathcal{B})\|(\&_{\mathcal{C}\in\mathcal{Z}}\mathcal{C}))$ associated to $\mathcal{X}\to\xrightarrow{\theta}_F\mathcal{Y}$.

Proof: **(for Y)** First, we prove that r satisfies $\mathcal{X}\to\xrightarrow{\theta}_F\mathcal{Y}$, θ-actively if and only if $\mathcal{X}^{t_1,t_2}\geq\theta$, $\mathcal{Y}^{t_1,t_2}\geq\theta$ or $\mathcal{X}^{t_1,t_2}\geq\theta$, $\mathcal{Z}^{t_1,t_2}\geq\theta$.

Suppose that the instance r satisfies the dependency $\mathcal{X}\to\xrightarrow{\theta}_F\mathcal{Y}$, θ-actively. Now, $\mathcal{X}^{t_1,t_2}\geq\theta$ and there is a tuple $t_3\in r$ such that the conditions given by (1) hold true, i.e., that $\mathcal{X}^{t_3,t_1}\geq\theta$, $\mathcal{Y}^{t_3,t_1}\geq\theta$, $\mathcal{Z}^{t_3,t_2}\geq\theta$. Hence, if $t_3=t_1$, then $\mathcal{Z}^{t_1,t_2}\geq\theta$. Else, if $t_3=t_2$, then $\mathcal{Y}^{t_1,t_2}\geq\theta$.

Let $\mathcal{X}^{t_1,t_2}\geq\theta$, $\mathcal{Y}^{t_1,t_2}\geq\theta$. Hence, $min(\theta,\mathcal{X}^{t_1,t_2})=\theta$. Now, there is $t_3\in r$, $t_3=t_2$ such that $\mathcal{X}^{t_3,t_1}\geq\theta$, $\mathcal{Y}^{t_3,t_1}\geq\theta$, $\mathcal{Z}^{t_3,t_2}=1\geq\theta$, i.e., (1) holds true. Analogously, if $\mathcal{X}^{t_1,t_2}\geq\theta$, $\mathcal{Z}^{t_1,t_2}\geq\theta$, then $min(\theta,\mathcal{X}^{t_1,t_2})=\theta$. Moreover, there is $t_3\in r$, $t_3=t_1$ such that $\mathcal{X}^{t_3,t_1}=1\geq\theta$, $\mathcal{Y}^{t_3,t_1}=1\geq\theta$, $\mathcal{Z}^{t_3,t_2}\geq\theta$. Therefore, (1) holds true. Now, since r satisfies the dependency $\mathcal{X}\to\xrightarrow{\theta}_F\mathcal{Y}$ and $\mathcal{X}^{t_1,t_2}\geq\theta$, it follows that the instance r satisfies the dependency $\mathcal{X}\to\xrightarrow{\theta}_F\mathcal{Y}$, θ-actively.

Now, we prove the main assertion.

$(\Rightarrow)$ Suppose that r satisties $\mathcal{X}\to\xrightarrow{\theta}_F\mathcal{Y}$, θ-actively. We have, $\mathcal{X}^{t_1,t_2}\geq\theta$, $\mathcal{Y}^{t_1,t_2}\geq\theta$ or $\mathcal{X}^{t_1,t_2}\geq\theta$, $\mathcal{Z}^{t_1,t_2}\geq\theta$.

Suppose that $\mathcal{X}^{t_1,t_2}\geq\theta$, $\mathcal{Y}^{t_1,t_2}\geq\theta$. Now,

$$\min_{\mathcal{A}\in\mathcal{X}}\left\{\mathcal{A}^{t_1,t_2}\right\}=\mathcal{X}^{t_1,t_2}\geq\theta,$$

$$\min_{\mathcal{B}\in\mathcal{Y}}\left\{\mathcal{B}^{t_1,t_2}\right\}=\mathcal{Y}^{t_1,t_2}\geq\theta.$$

Hence, $\mathcal{A}^{t_1,t_2}\geq\theta$ for all $\mathcal{A}\in\mathcal{X}$ and $\mathcal{B}^{t_1,t_2}\geq\theta$ for all $\mathcal{B}\in\mathcal{Y}$. Therefore, $v_\theta^r(\mathcal{A})>\frac{1}{2}$ for $\mathcal{A}\in\mathcal{X}$, $v_\theta^r(\mathcal{B})>\frac{1}{2}$ for $\mathcal{B}\in\mathcal{Y}$. Now,

$$v_\theta^r\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)=min\left\{v_\theta^r(\mathcal{A})\mid\mathcal{A}\in\mathcal{X}\right\}>\frac{1}{2},$$

$$v_\theta^r\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)=min\left\{v_\theta^r(\mathcal{B})\mid\mathcal{B}\in\mathcal{Y}\right\}>\frac{1}{2}.$$

We obtain,

$$v_\theta^r(\mathcal{K})=v_\theta^r\left(\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\to\left(\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)\|\left(\underset{\mathcal{C}\in\mathcal{Z}}{\&}\mathcal{C}\right)\right)\right)$$

$$=v_\theta^r\left(\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)\|\left(\underset{\mathcal{C}\in\mathcal{Z}}{\&}\mathcal{C}\right)\right)^{v_\theta^r(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})}$$

$$=max\left(v_\theta^r\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right),v_\theta^r\left(\underset{\mathcal{C}\in\mathcal{Z}}{\&}\mathcal{C}\right)\right)^{v_\theta^r(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})}.$$

Denote $a=v_\theta^r(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})$, $b=max\left(v_\theta^r(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}),v_\theta^r(\&_{\mathcal{C}\in\mathcal{Z}}\mathcal{C})\right)$.

Since $v_\theta^r(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})>\frac{1}{2}$ and $v_\theta^r(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})>\frac{1}{2}$, we have that $a>\frac{1}{2},b>\frac{1}{2}$.

Now, $v_\theta^r(\mathcal{K})>\frac{1}{2}$ if and only if $b^a>\frac{1}{2}$.

If $b=1$, then $b^a>\frac{1}{2}$ holds true and hence $v_\theta^r(\mathcal{K})>\frac{1}{2}$.

Let $\frac{1}{2}<b<1$. Now, $b^a>\frac{1}{2}$ if and only if $a<log_b\frac{1}{2}$. The last inequality is true since $log_b\frac{1}{2}>1$. Therefore, $v_\theta^r(\mathcal{K})>\frac{1}{2}$.

Similarly, if $\mathcal{X}^{t_1,t_2}\geq\theta$, $\mathcal{Z}^{t_1,t_2}\geq\theta$ then, $v_\theta^r(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})>\frac{1}{2}$ and $v_\theta^r(\&_{\mathcal{C}\in\mathcal{Z}}\mathcal{C})>\frac{1}{2}$. Now, reasoning as in the previous case, we conclude that $a>\frac{1}{2},b>\frac{1}{2}$ and hence $v_\theta^r(\mathcal{K})>\frac{1}{2}$.

$(\Leftarrow)$ Suppose that $\mathcal{X}^{t_1,t_2}\geq\theta$ and $v_\theta^r(\mathcal{K})>\frac{1}{2}$. We have $a>\frac{1}{2}$ and then $b^a>\frac{1}{2}$.

If $b=0$, then $0>\frac{1}{2}$, i.e., a contradiction. Hence, $0<b\leq1$.

If $b=1$, then $b^a>\frac{1}{2}$ holds true.

Let $0<b<1$. We have $b^a>\frac{1}{2}$ if and only if $a<log_b\frac{1}{2}$. The last inequality is satisfied for $\frac{1}{2}<b<1$. We conclude, $b>\frac{1}{2}$.

If $b=v_\theta^r(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})$, then $v_\theta^r(\mathcal{B})>\frac{1}{2}$ for all $\mathcal{B}\in\mathcal{Y}$. Hence, $\mathcal{B}^{t_1,t_2}\geq\theta$ for $\mathcal{B}\in\mathcal{Y}$. Now, $\mathcal{Y}^{t_1,t_2}\geq\theta$. Therefore, $\mathcal{X}^{t_1,t_2}\geq\theta$ and $\mathcal{Y}^{t_1,t_2}\geq\theta$ yield the result.

Analogously, if $b=\mathsf{v}_\theta^{\mathsf{r}}\left(\&_{\mathcal{C}\in\mathcal{Z}}\mathcal{C}\right)$, then $\mathcal{Z}^{t_1,t_2}\geq\theta$. Now, $\mathcal{X}^{t_1,t_2}\geq\theta$, $\mathcal{Z}^{t_1,t_2}\geq\theta$ yield the result. This completes the proof. □

Proof: (for R)

$(\Rightarrow)$ Assume that r satisfies the dependency $\mathcal{X}\to\xrightarrow{\theta}_F\mathcal{Y}$, θ-actively.

Now, $\mathcal{X}^{t_1,t_2}\geq\theta$, $\mathcal{Y}^{t_1,t_2}\geq\theta$ or $\mathcal{X}^{t_1,t_2}\geq\theta$, $\mathcal{Z}^{t_1,t_2}\geq\theta$.

Let $\mathcal{X}^{t_1,t_2}\geq\theta$ and $\mathcal{Y}^{t_1,t_2}\geq\theta$ hold true. We have,

$$\min_{\mathcal{A}\in\mathcal{X}}\left\{\mathcal{A}^{t_1,t_2}\right\}=\mathcal{X}^{t_1,t_2}\geq\theta$$

and

$$\min_{\mathcal{B}\in\mathcal{Y}}\left\{\mathcal{B}^{t_1,t_2}\right\}=\mathcal{Y}^{t_1,t_2}\geq\theta .$$

Therefore, $\mathcal{A}^{t_1,t_2}\geq\theta,\mathcal{A}\in\mathcal{X}$ and $\mathcal{B}^{t_1,t_2}\geq\theta,\mathcal{B}\in\mathcal{Y}$. Consequently, $\mathsf{v}_\theta^{\mathsf{r}}(\mathcal{A})>\frac{1}{2}$, $\mathcal{A}\in\mathcal{X}$ and $\mathsf{v}_\theta^{\mathsf{r}}(\mathcal{B})>\frac{1}{2}$, $\mathcal{B}\in\mathcal{Y}$. We obtain,

$$\mathsf{v}_\theta^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)=\min\left\{\mathsf{v}_\theta^{\mathsf{r}}(\mathcal{A})\mid\mathcal{A}\in\mathcal{X}\right\}>\frac{1}{2}$$

and

$$\mathsf{v}_\theta^{\mathsf{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)=\min\left\{\mathsf{v}_\theta^{\mathsf{r}}(\mathcal{B})\mid\mathcal{B}\in\mathcal{Y}\right\}>\frac{1}{2}.$$

We have,

$$\mathsf{v}_\theta^{\mathsf{r}}(\mathcal{K})=\mathsf{v}_\theta^{\mathsf{r}}\left(\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\to\left(\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)\|\left(\underset{\mathcal{C}\in\mathcal{Z}}{\&}\mathcal{C}\right)\right)\right)$$

$$=1-\mathsf{v}_\theta^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)+\mathsf{v}_\theta^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\cdot \mathsf{v}_\theta^{\mathsf{r}}\left(\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)\|\left(\underset{\mathcal{C}\in\mathcal{Z}}{\&}\mathcal{C}\right)\right)$$

$$=1-\mathsf{v}_\theta^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)+\mathsf{v}_\theta^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\cdot \max\left(\mathsf{v}_\theta^{\mathsf{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right),\mathsf{v}_\theta^{\mathsf{r}}\left(\underset{\mathcal{C}\in\mathcal{Z}}{\&}\mathcal{C}\right)\right).$$

Put $a=\mathsf{v}_\theta^{\mathsf{r}}\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)$, $b=\max\left(\mathsf{v}_\theta^{\mathsf{r}}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right),\mathsf{v}_\theta^{\mathsf{r}}\left(\&_{\mathcal{C}\in\mathcal{Z}}\mathcal{C}\right)\right)$.

Now, $\mathsf{v}_\theta^{\mathsf{r}}\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)>\frac{1}{2}$, $\mathsf{v}_\theta^{\mathsf{r}}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)>\frac{1}{2}$, yield $a>\frac{1}{2},b>\frac{1}{2}$. Hence, $\frac{1}{2}<a\leq1$ and then $2a\leq2$. We obtain, $\frac{1}{2a}\geq\frac{1}{2}$. Since $a>\frac{1}{2}>0$, we have that $\mathsf{v}_\theta^{\mathsf{r}}(\mathcal{K})>\frac{1}{2}$ if and only if $1-a+ab>\frac{1}{2}$ if and only if $\frac{1}{2}+ab>a$ if and only if $\frac{1}{2a}+b>1$, which is true. Hence, $\mathsf{v}_\theta^{\mathsf{r}}(\mathcal{K})>\frac{1}{2}$.

Similarly, if $\mathcal{X}^{t_1,t_2}\geq\theta$ and $\mathcal{Z}^{t_1,t_2}\geq\theta$, then $\mathsf{v}_\theta^{\mathsf{r}}\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)>\frac{1}{2}$ and $\mathsf{v}_\theta^{\mathsf{r}}\left(\&_{\mathcal{C}\in\mathcal{Z}}\mathcal{C}\right)>\frac{1}{2}$. Now, $1\geq a>\frac{1}{2}>0,b>\frac{1}{2}$ and hence $\frac{1}{2a}+b>1$ holds true. Therefore, $\mathsf{v}_\theta^{\mathsf{r}}(\mathcal{K})>\frac{1}{2}$.

$(\Leftarrow)$ Let $\mathcal{X}^{t_1,t_2}\geq\theta$, $\mathsf{v}_\theta^{\mathsf{r}}(\mathcal{K})>\frac{1}{2}$. Now, $0<\frac{1}{2}<a\leq1$. hence, $\mathsf{v}_\theta^{\mathsf{r}}(\mathcal{K})>\frac{1}{2}$ if and only if $\frac{1}{2a}+b>1$. Therefore, $\frac{1}{2a}\geq\frac{1}{2}$ implies that $b>\frac{1}{2}$.

If $b=\mathsf{v}_\theta^{\mathsf{r}}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)$, then $\mathsf{v}_\theta^{\mathsf{r}}(\mathcal{B})>\frac{1}{2}$, $\mathcal{B}\in\mathcal{Y}$. Consequently, $\mathcal{B}^{t_1,t_2}\geq\theta,\mathcal{B}\in\mathcal{Y}$. Now, $\mathcal{Y}^{t_1,t_2}\geq\theta$. Hence, $\mathcal{X}^{t_1,t_2}\geq\theta$, $\mathcal{Y}^{t_1,t_2}\geq\theta$ yield the result.

Similarly, if we assume that $b=\mathsf{v}_\theta^{\mathsf{r}}\left(\&_{\mathcal{C}\in\mathcal{Z}}\mathcal{C}\right)$, we obtain that $\mathcal{Z}^{t_1,t_2}\geq\theta$. Since $\mathcal{X}^{t_1,t_2}\geq\theta$, the theorem follows. This completes the proof. □

Theorem 2. Let $f\notin\mathsf{F}$ be a fuzzy functional or a fuzzy multivalued dependency on a set of attributes U, where F is a set of fuzzy functional and fuzzy multivalued dependencies on U. Let F' resp. f' be the set of fuzzy formulas resp. the fuzzy formula related to F resp. f. The following two conditions are equivalent:

(a) Any two-element fuzzy relation instance on scheme $\mathsf{R}(\mathsf{U})$ which satisfies all dependencies from the set F satisfies also the dependency f.

(b) $\mathsf{v}_\varepsilon^{\mathsf{r}}(f')>\frac{1}{2}$ for every $\mathsf{v}_\varepsilon^{\mathsf{r}}$ suct that $\mathsf{v}_\varepsilon^{\mathsf{r}}(\mathcal{L})>\frac{1}{2}$ for all $\mathcal{L}\in\mathsf{F}'$.

Proof: (for Y) We denote f by $\mathcal{X}\xrightarrow{\theta_1}_F\mathcal{Y}$ when f is a fuzzy functional dependency and by $\mathcal{X}\to\xrightarrow{\theta_1}_F\mathcal{Y}$ when f is a fuzzy multivalued dependency. Therefore, $\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)\to\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)$ and $\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)\to\left(\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)\|\left(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D}\right)\right)$ will denote f' in the first and the second case, respectively, where $\mathcal{Z}=\mathsf{U}-\mathcal{X}\mathcal{Y}$.

We may assume that the set $\{p,q\}$ is the domain of each of the attributes in U.

Fix some $\theta''\in[0,\theta')$, where θ' is the minimum of the strengths of all dependencies that appear in $\mathsf{F}\cup\{f\}$. Suppose that $\theta'<1$. Namely, if $\theta'=1$, then every dependency $f_1\in\mathsf{F}\cup\{f\}$ is of the strength 1. This case is not interesting however.

Define $\mathsf{s}(p,q)=\mathsf{s}(q,p)=\theta''$ to be a similarity relation on $\{p,q\}$.

$(a)\Rightarrow(b)$ Suppose that (b) is not valid.

Now, there is some v_ε^r such that $v_\varepsilon^r(\mathcal{L})>\frac{1}{2}$ for all $\mathcal{L}\in F'$ and $v_\varepsilon^r(f')\le\frac{1}{2}$. Here, v_ε^r is joined to some two-element fuzzy relation instance $r=\{t_1,t_2\}$ on $R(U)$ and some $0\le\varepsilon\le 1$.

Define $W=\left\{\mathcal{A}\in U \middle| v_\varepsilon^r(\mathcal{A})>\frac{1}{2}\right\}$.

Assume that $W=\varnothing$. In this case, $v_\varepsilon^r(\mathcal{A})\le\frac{1}{2}$ for all $\mathcal{A}\in U$. Hence, $v_\varepsilon^r(\&_{\mathcal{A}\in\mathcal{M}}\mathcal{A})\le\frac{1}{2}<1$ for any $\mathcal{M}\subseteq U$.

If $v_\varepsilon^r(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})=0$, $v_\varepsilon^r(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})=0$ resp. $v_\varepsilon^r(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})=0$, $v_\varepsilon^r((\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})\|(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D}))=0$, then $v_\varepsilon^r(f')\le\frac{1}{2}$ yields $1\le\frac{1}{2}$, i.e., a contradiction. Hence, $v_\varepsilon^r(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})\ne 0$ or $v_\varepsilon^r(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})\ne 0$ resp. $v_\varepsilon^r(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})\ne 0$ or $max\left(v_\varepsilon^r(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}),v_\varepsilon^r(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right)\ne 0$. We may assume that $v_\varepsilon^r(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})\ne 0$ resp. $max\left(v_\varepsilon^r(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}),v_\varepsilon^r(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right)\ne 0$. Now, $v_\varepsilon^r(f')\le\frac{1}{2}$ implies

$$v_\varepsilon^r\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)^{v_\varepsilon^r(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})}\le\frac{1}{2} \quad (2)$$

resp.

$$max\left(v_\varepsilon^r\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right),v_\varepsilon^r\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)^{v_\varepsilon^r(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})}\le\frac{1}{2}, \quad (3)$$

i.e.,

$$v_\varepsilon^r\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\ge log_{v_\varepsilon^r(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})}\frac{1}{2} \quad (4)$$

resp.

$$v_\varepsilon^r\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\ge log_{max\left(v_\varepsilon^r(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}),v_\varepsilon^r(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right)}\frac{1}{2}. \quad (5)$$

Therefore, $0<v_\varepsilon^r(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})\le\frac{1}{2}$ resp. $0<max\left(v_\varepsilon^r(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}),v_\varepsilon^r(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right)\le\frac{1}{2}$ yields $v_\varepsilon^r(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})=1$. This is contradiction. Hence, $W\ne\varnothing$.

Assume that $W=U$. In this case, $v_\varepsilon^r(\mathcal{A})>\frac{1}{2}$ for all $\mathcal{A}\in U$. Consequently, $v_\varepsilon^r(\&_{\mathcal{A}\in\mathcal{M}}\mathcal{A})>\frac{1}{2}$ for all $\mathcal{M}\in U$.

Now, (2) resp. (3) holds true.

If $v_\varepsilon^r(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})=1$ resp. $max\left(v_\varepsilon^r(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}),v_\varepsilon^r(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right)=1$, then $1\le\frac{1}{2}$, i.e., a contradiction. Hence, (4) resp. (5) holds true. Therefore, $\frac{1}{2}<v_\varepsilon^r(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})<1$ resp. $\frac{1}{2}<max\left(v_\varepsilon^r(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}),v_\varepsilon^r(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right)<1$ yields $v_\varepsilon^r(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})>1$. This is a contradicton. We obtain, $W\ne U$.

Define $r'=\{t',t''\}$ by Table 1 below.

r' is a two-element fuzzy relation instance on $R(U)$.

We shall prove that this instance satisfies all dependencies from the set F, but violates the dependency f.

Let $\mathcal{K}\xrightarrow{\theta_2}_F\mathcal{L}$ be any fuzzy functional dependency from te set F.

Table 1:

	attributes of W	other attributes
t'	$p,p,...,p$	$p,p,...,p$
t''	$p,p,...,p$	$q,q,...,q$

Assume that $v_\varepsilon^r(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})\le\frac{1}{2}$. Then, there exists $\mathcal{A}_0\in\mathcal{K}$ such that

$$v_\varepsilon^r(\mathcal{A}_0)=min\left\{v_\varepsilon^r(\mathcal{A})\middle|\mathcal{A}\in\mathcal{K}\right\}=v_\varepsilon^r\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\le\frac{1}{2},$$

i.e., $\mathcal{A}_0\notin W$. We have $\mathcal{A}_0^{t',t''}=\theta''$ and hence

$$\mathcal{K}^{t',t''}=\min_{\mathcal{A}\in\mathcal{K}}\left\{\mathcal{A}^{t',t''}\right\}=\theta''.$$

Since $s(p,q)=\theta''$, we know that $\mathcal{M}^{t',t''}\ge\theta''$ for any set of attributes $\mathcal{M}\in U$. Therefore, $\mathcal{L}^{t',t''}\ge\theta''$. We obtain,

$$\mathcal{L}^{t',t''}\ge\theta''=min\left(\theta_2,\mathcal{K}^{t',t''}\right),$$

i.e., r' satisties $\mathcal{K}\xrightarrow{\theta_2}_F\mathcal{L}$.

Assume that $v_\varepsilon^r(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A})>\frac{1}{2}$. Now,

$$v_\varepsilon^r\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right)^{v_\varepsilon^r(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})}=v_\varepsilon^r\left(\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\to\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right)\right)>\frac{1}{2}.$$

The last inequality is satisfied if $v_\varepsilon^r(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B})=1$. If $v_\varepsilon^r(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B})=0$, then $0>\frac{1}{2}$, i.e., a contradiction.

Let $0<v_\varepsilon^r(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B})<1$. We have,

$$v_\varepsilon^r\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)<log_{v_\varepsilon^r(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B})}\frac{1}{2}.$$

Therefore, $v_\varepsilon^r(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B})>\frac{1}{2}$. Now, $v_\varepsilon^r(\mathcal{B})>\frac{1}{2}$ for all $\mathcal{B}\in\mathcal{L}$ and then $\mathcal{B}\in W$ for $\mathcal{B}\in\mathcal{L}$. We obtain, $\mathcal{L}\subseteq\mathcal{W}$. Hence, $\mathcal{L}^{t',t''}=1$. We have,

$$\mathcal{L}^{t',t''}=1\geq min\left(\theta_2,\mathcal{K}^{t',t''}\right),$$

i.e., r' satisfies the dependency $\mathcal{K}\xrightarrow{\theta_2}_F\mathcal{L}$.

Let $\mathcal{K}\to\xrightarrow{\theta_2}_F\mathcal{L}$ by any fuzzy multivalued dependency from te set F.

Suppose that $\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A}\right)\leq\frac{1}{2}$. Then, reasoning as in the previous case, we obtain that $\mathcal{K}^{t',t''}=\theta''$.
Hence, there is $t'''\in\mathsf{r}'$, $t'''=t'$ such that

$$\mathcal{K}^{t''',t'}=1\geq min\left(\theta_2,\mathcal{K}^{t',t''}\right),$$
$$\mathcal{L}^{t''',t'}=1\geq min\left(\theta_2,\mathcal{K}^{t',t''}\right),\qquad(6)$$
$$\mathcal{M}^{t''',t''}\geq\theta''=min\left(\theta_2,\mathcal{K}^{t',t''}\right),$$

where $\mathcal{M}=\mathsf{U}-\mathcal{K}\mathcal{L}$. Therefore, r' satisfies $\mathcal{K}\to\xrightarrow{\theta_2}_F\mathcal{L}$.

Let $\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A}\right)>\frac{1}{2}$. Now,

$$max\left(\mathsf{v}_\varepsilon^\mathsf{r}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right),\mathsf{v}_\varepsilon^\mathsf{r}\left(\underset{\mathcal{D}\in\mathcal{M}}{\&}\mathcal{D}\right)\right)^{\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A}\right)}$$

$$=\left(\mathsf{v}_\varepsilon^\mathsf{r}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right)\|\left(\underset{\mathcal{D}\in\mathcal{M}}{\&}\mathcal{D}\right)\right)^{\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A}\right)}$$

$$=\mathsf{v}_\varepsilon^\mathsf{r}\left(\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\to\left(\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right)\|\left(\underset{\mathcal{D}\in\mathcal{M}}{\&}\mathcal{D}\right)\right)\right)>\frac{1}{2}.$$

This inequality is satisfied if $max\left(\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right),\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D}\right)\right)=1$. If $max\left(\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right),\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D}\right)\right)=0$, then $0>\frac{1}{2}$, i.e., a contradiction.

If $0<max\left(\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right),\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D}\right)\right)<1$, then

$$\mathsf{v}_\varepsilon^\mathsf{r}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)<log_{max\left(\mathsf{v}_\varepsilon^\mathsf{r}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right),\mathsf{v}_\varepsilon^\mathsf{r}\left(\underset{\mathcal{D}\in\mathcal{M}}{\&}\mathcal{D}\right)\right)}\frac{1}{2}.$$

Therefore, $max\left(\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right),\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D}\right)\right)>\frac{1}{2}$. Hence, $\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right)>\frac{1}{2}$ or $\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D}\right)>\frac{1}{2}$.

If $\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right)>\frac{1}{2}$, then $\mathcal{L}\subseteq\mathcal{W}$ and hence $\mathcal{L}^{t',t''}=1$. Similarly, since $\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A}\right)>\frac{1}{2}$, we conclude that $\mathcal{K}^{t',t''}=1$. Now, there is $t'''\in\mathsf{r}'$, $t'''=t''$ such that

$$\mathcal{K}^{t''',t'}=1\geq min\left(\theta_2,\mathcal{K}^{t',t''}\right),$$
$$\mathcal{L}^{t''',t'}=1\geq min\left(\theta_2,\mathcal{K}^{t',t''}\right),\qquad(7)$$
$$\mathcal{M}^{t''',t''}=1\geq min\left(\theta_2,\mathcal{K}^{t',t''}\right).$$

Hence, r' satisfies $\mathcal{K}\to\xrightarrow{\theta_2}_F\mathcal{L}$.

If $\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D}\right)>\frac{1}{2}$, then $\mathcal{M}^{t',t''}=1$. In this case, there is $t'''\in\mathsf{r}'$, $t'''=t'$ such that (7) holds true. In other words, r' satisfies the dependency $\mathcal{K}\to\xrightarrow{\theta_2}_F\mathcal{L}$.

It remains to prove that the instance r' violates $\mathcal{X}\xrightarrow{\theta_1}_F\mathcal{Y}$ resp. $\mathcal{X}\to\xrightarrow{\theta_1}_F\mathcal{Y}$.

Let
$$\mathsf{v}_\varepsilon^\mathsf{r}\left(\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\to\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)\right)=\mathsf{v}_\varepsilon^\mathsf{r}\left(f'\right)\leq\frac{1}{2}.$$

If $\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)=0$ and $\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)=0$, then $1\leq\frac{1}{2}$, i.e., a contradiction. Hence, $\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)\neq0$ or $\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)\neq0$. We may assume that $\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)\neq0$. Now,

$$\mathsf{v}_\varepsilon^\mathsf{r}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)^{\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)}\leq\frac{1}{2}.$$

If $\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)=1$, then $1\leq\frac{1}{2}$, i.e., a contradiction. Therefore, $0<\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)<1$. We obtain,

$$\mathsf{v}_\varepsilon^\mathsf{r}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\geq log_{\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)}\frac{1}{2}.$$

If $\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)>\frac{1}{2}$, then $\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)>1$, i.e., a contradiction. Hence, $0<\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)\leq\frac{1}{2}$ and then $\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A}\right)=1$. Now, as before, we conclude that $\mathcal{Y}^{t',t''}=\theta''$ and $\mathcal{X}^{t',t''}=1$. Therefore,

$$\mathcal{Y}^{t',t''}=\theta''<\theta'\leq\theta_1=min\left(\theta_1,\mathcal{X}^{t',t''}\right).$$

This means that r' violates $\mathcal{X}\xrightarrow{\theta_1}_F\mathcal{Y}$.

Now, let
$$\mathsf{v}_\varepsilon^\mathsf{r}\left(\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\to\left(\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)\|\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)\right)=\mathsf{v}_\varepsilon^\mathsf{r}\left(f'\right)\leq\frac{1}{2}.$$

Reasoning as in the previous case, we conclude that $\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)\neq0$ or $max\left(\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right),\mathsf{v}_\varepsilon^\mathsf{r}\left(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D}\right)\right)\neq0$. We have,

$$max\left(\mathsf{v}_{\varepsilon}^{\mathsf{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right),\mathsf{v}_{\varepsilon}^{\mathsf{r}}\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)^{\mathsf{v}_{\varepsilon}^{\mathsf{r}}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})}\leq\frac{1}{2}.$$

Then, $0<max\left(\mathsf{v}_{\varepsilon}^{\mathsf{r}}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right),\mathsf{v}_{\varepsilon}^{\mathsf{r}}\left(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D}\right)\right)\leq\frac{1}{2}$ and $\mathsf{v}_{\varepsilon}^{\mathsf{r}}\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)=1$, i.e., $\mathsf{v}_{\varepsilon}^{\mathsf{r}}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)\leq\frac{1}{2}$, $\mathsf{v}_{\varepsilon}^{\mathsf{r}}\left(\&_{\mathcal{D}\in\mathcal{Y}}\mathcal{D}\right)\leq\frac{1}{2}$, $\mathsf{v}_{\varepsilon}^{\mathsf{r}}\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)=1$. We obtain, $\mathcal{Y}^{t',t''}=\theta'',\mathcal{Z}^{t',t''}=\theta'',\quad \mathcal{X}^{t',t''}=1$.

If $t'''\in\mathsf{r}'$, $t'''=t'$, then

$$\begin{aligned}&\mathcal{X}^{t''',t'}=1\geq min\left(\theta_1,\mathcal{X}^{t',t''}\right),\\&\mathcal{Y}^{t''',t'}=1\geq min\left(\theta_1,\mathcal{X}^{t',t''}\right),\\&\mathcal{Z}^{t''',t''}=\theta''<\theta'\leq\theta_1=min\left(\theta_1,\mathcal{X}^{t',t''}\right).\end{aligned}\tag{8}$$

If $t'''\in\mathsf{r}'$, $t'''=t''$, then

$$\begin{aligned}&\mathcal{X}^{t''',t'}=1\geq min\left(\theta_1,\mathcal{X}^{t',t''}\right),\\&\mathcal{Y}^{t''',t'}=\theta''<\theta'\leq\theta_1=min\left(\theta_1,\mathcal{X}^{t',t''}\right),\\&\mathcal{Z}^{t''',t''}=1\geq min\left(\theta_1,\mathcal{X}^{t',t''}\right).\end{aligned}\tag{9}$$

In other words, the instance r' violates $\mathcal{X}\rightarrow\xrightarrow{\theta_1}_F\mathcal{Y}$.

$(\mathrm{b})\Rightarrow(\mathrm{a})$ Suppose that (a) is not valid.

Now, there is a two-element fuzzy relation instance $\mathsf{r}'=\{t',t''\}$ on scheme $\mathsf{R}(\mathsf{U})$, such that r' satisfies all dependencies in F and r' does not satisfy f. Therefore, r' does not satisfy $\mathcal{X}\xrightarrow{\theta_1}_F\mathcal{Y}$ resp. $\mathcal{X}\rightarrow\xrightarrow{\theta_1}_F\mathcal{Y}$.

Define $W=\left\{\mathcal{A}\in\mathsf{U}\;\middle|\;\mathcal{A}^{t',t''}=1\right\}$.

Assume that $W=\varnothing$. Now, $\mathcal{A}^{t',t''}=\theta''$ for all $\mathcal{A}\in\mathsf{U}$. Therefore, $\mathcal{M}^{t',t''}=\theta''$ for all $\mathcal{M}\subseteq\mathsf{U}$.

In the case when r' does not satisfy $\mathcal{X}\xrightarrow{\theta_1}_F\mathcal{Y}$, we obtain

$$\mathcal{Y}^{t',t''}<min\left(\theta',\mathcal{X}^{t',t''}\right),$$

i.e., $\theta''<min\left(\theta_1,\theta''\right)=\theta''$. This is a contradiction.

Similarly, in the case when r' does not satisfy $\mathcal{X}\rightarrow\xrightarrow{\theta_1}_F\mathcal{Y}$, we have that the conditions

$$\begin{aligned}&\mathcal{X}^{t',t'}\geq min\left(\theta_1,\mathcal{X}^{t',t''}\right),\\&\mathcal{Y}^{t',t'}\geq min\left(\theta_1,\mathcal{X}^{t',t''}\right),\\&\mathcal{Z}^{t',t''}\geq min\left(\theta_1,\mathcal{X}^{t',t''}\right)\end{aligned}\tag{10}$$

don't hold simultaneously. Since the first and the second condition in (10) hold obviously true, we obtain

$$\theta''=\mathcal{Z}^{t',t''}<min\left(\theta_1,\mathcal{X}^{t',t''}\right)=min\left(\theta_1,\theta''\right)=\theta'',$$

which is a contradiction. Therefore, $W\neq\varnothing$.

Assume that $W=\mathsf{U}$. Now, $\mathcal{A}^{t',t''}=1$ for every $\mathcal{A}\in\mathsf{U}$. Therefore, $\mathcal{M}^{t',t''}=1$ for every $\mathcal{M}\subseteq\mathsf{U}$.

In the case when r' does not satisfy $\mathcal{X}\xrightarrow{\theta_1}_F\mathcal{Y}$, we have that

$$1=\mathcal{Y}^{t',t''}<min\left(\theta_1,\mathcal{X}^{t',t''}\right)=min\left(\theta_1,1\right)=\theta_1.$$

This is a contradiction.

In the case when r' does not satisfy $\mathcal{X}\rightarrow\xrightarrow{\theta_1}_F\mathcal{Y}$, the conditions given by (10) don't hold simultaneously. The first and the second condition in (10) are always satisfied, hence

$$1=\mathcal{Z}^{t',t''}<min\left(\theta_1,\mathcal{X}^{t',t''}\right)=min\left(\theta_1,1\right)=\theta_1.$$

This is a contradiction. We conclude, $W\neq\mathsf{U}$.

Now, we define $\mathsf{v}_1^{\mathsf{r}'}$ in the following way. Let

$$\frac{1}{2}<\mathsf{v}_1^{\mathsf{r}'}\left(\mathcal{A}\right)\leq 1 \text{ if } \mathcal{A}\in W,$$

$$0\leq\mathsf{v}_1^{\mathsf{r}'}\left(\mathcal{A}\right)\leq\frac{1}{2} \text{ if } \mathcal{A}\in\mathsf{U}-W.$$

We shall prove that $\mathsf{v}_1^{\mathsf{r}'}\left(\mathcal{L}\right)>\frac{1}{2}$ for every $\mathcal{L}\in\mathsf{F}'$ and $\mathsf{v}_1^{\mathsf{r}'}\left(f'\right)\leq\frac{1}{2}$.

Suppose that $\mathcal{L}\in\mathsf{F}'$ is of the form

$$\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\rightarrow\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right).$$

This fuzzy formula corresponds to some fuzzy functional dependency $\mathcal{K}\xrightarrow{\theta_2}_F\mathcal{L}$ from the set F.

Suppose, that $\mathsf{v}_1^{\mathsf{r}'}\left(\mathcal{L}\right)\leq\frac{1}{2}$. Then, as earlier, it follows that $\mathsf{v}_1^{\mathsf{r}'}\left(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A}\right)\neq 0$ or $\mathsf{v}_1^{\mathsf{r}'}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right)\neq 0$.

Assume that $\mathsf{v}_1^{\mathsf{r}'}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right)\neq 0$. We have ,

$$\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right)^{\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A})}\leq\frac{1}{2}.$$

Then, $\mathsf{v}_1^{\mathsf{r}'}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right)<1$. We obtain,

$$\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\geq log_{\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B})}\frac{1}{2}.$$

Therefore, $0<\mathsf{v}_1^{\mathsf{r}'}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right)\leq\frac{1}{2}$ and $\mathsf{v}_1^{\mathsf{r}'}\left(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A}\right)=1$, i.e., $\mathcal{L}^{t',t''}=\theta''$ and $\mathcal{K}^{t',t''}=1$.

We obtain,

$$\mathcal{L}^{t',t''}=\theta''<\theta'\le\theta_2=min(\theta_2,1)=min(\theta_2,\mathcal{K}^{t',t''}),$$

which contradicts the fact that r' satisfies $\mathcal{K}\xrightarrow{\theta_2}_F\mathcal{L}$. Therefore, $\mathrm{v}_1^{\mathrm{r}'}(\mathcal{L})>\frac{1}{2}$.

Suppose that $\mathcal{L}\in\mathrm{F}'$ is of the form

$$\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\to\left(\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right)\|\left(\underset{\mathcal{D}\in\mathcal{M}}{\&}\mathcal{D}\right)\right),$$

where $\mathcal{M}=\mathrm{U}-\mathcal{KL}$. This fuzzy formula corresponds to some fuzzy multivalued dependency $\mathcal{K}\to\xrightarrow{\theta_2}_F\mathcal{L}$ from the set F.

Assume that $\mathrm{v}_1^{\mathrm{r}'}(\mathcal{L})\le\frac{1}{2}$.

As before, we have that $\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A})\neq 0$ or $max\left(\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}),\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D})\right)\neq 0$.

Suppose that $max\left(\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}),\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D})\right)\neq 0$. We have,

$$max\left(\mathrm{v}_1^{\mathrm{r}'}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right),\mathrm{v}_1^{\mathrm{r}'}\left(\underset{\mathcal{D}\in\mathcal{M}}{\&}\mathcal{D}\right)\right)^{\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A})}\le\frac{1}{2}.$$

Then, $max\left(\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}),\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D})\right)<1$. We obtain,

$$\mathrm{v}_1^{\mathrm{r}'}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\ge log_{max\left(\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}),\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D})\right)}\frac{1}{2}.$$

Therefore, $0<max\left(\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}),\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D})\right)\le\frac{1}{2}$ and $\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A})=1$, i.e., $\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B})\le\frac{1}{2}$, $\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D})\le\frac{1}{2}$, $\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A})=1$. Hence, $\mathcal{L}^{t',t''}=\theta''$, $\mathcal{M}^{t',t''}=\theta''$, $\mathcal{K}^{t',t''}=1$. In the case, the third condition of the conditions

$$\mathcal{K}^{t',t'}\ge min(\theta_2,\mathcal{K}^{t',t''}),$$
$$\mathcal{L}^{t',t'}\ge min(\theta_2,\mathcal{K}^{t',t''}),\qquad(11)$$
$$\mathcal{M}^{t',t''}\ge min(\theta_2,\mathcal{K}^{t',t''})$$

does not hold. Furthermore, the second condition of the conditions

$$\mathcal{K}^{t'',t'}\ge min(\theta_2,\mathcal{K}^{t',t''}),$$
$$\mathcal{L}^{t'',t'}\ge min(\theta_2,\mathcal{K}^{t',t''}),\qquad(12)$$
$$\mathcal{M}^{t'',t''}\ge min(\theta_2,\mathcal{K}^{t',t''})$$

does not hold. This contradicts the fact that r' satisfies the dependency $\mathcal{K}\to\xrightarrow{\theta_2}_F\mathcal{L}$. Hence, $\mathrm{v}_1^{\mathrm{r}'}(\mathcal{L})>\frac{1}{2}$.

It remains to prove that $\mathrm{v}_1^{\mathrm{r}'}(f')\le\frac{1}{2}$.

Suppose that the instance r' does not satisfies the dependency $\mathcal{X}\xrightarrow{\theta_1}_F\mathcal{Y}$.

Assume that $\mathrm{v}_1^{\mathrm{r}'}(f')>\frac{1}{2}$.

If $\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})\le\frac{1}{2}$, then $\mathcal{X}^{t',t''}=\theta''$. Hence,

$$\mathcal{Y}^{t',t''}\ge\theta''=min(\theta_1,\theta'')=min(\theta_1,\mathcal{X}^{t',t''}).$$

This contradicts the fact that r' violates $\mathcal{X}\xrightarrow{\theta_1}_F\mathcal{Y}$.

If $\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})>\frac{1}{2}$, then

$$\mathrm{v}_1^{\mathrm{r}'}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)^{\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})}>\frac{1}{2}.$$

This inequality is satistied if $\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})=1$.

If $\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})=0$, then $0>\frac{1}{2}$, i.e., a contradiction.

If $0<\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})<1$, then

$$\mathrm{v}_1^{\mathrm{r}'}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)<log_{\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})}\frac{1}{2}.$$

Therefore, $\mathrm{v}_1^{\mathrm{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})>\frac{1}{2}$, i.e., $\mathcal{Y}^{t',t''}=1$. Now,

$$\mathcal{Y}^{t',t''}\ge min(\theta_1,\mathcal{X}^{t',t''}),$$

which is a contradiction. We conclude, $\mathrm{v}_1^{\mathrm{r}'}(f')\le\frac{1}{2}$.

Suppose that r' does not satisfy $\mathcal{X}\to\xrightarrow{\theta_1}_F\mathcal{Y}$.

Now, the third condition of the conditions given by (10) does not hold, i.e.,

$$\mathcal{Z}^{t',t''}<min(\theta_1,\mathcal{X}^{t',t''}).\qquad(13)$$

Moreover, the first and the second condition of the conditions

$$\mathcal{X}^{t'',t'}\ge min(\theta_1,\mathcal{X}^{t',t''}),$$
$$\mathcal{Y}^{t'',t'}\ge min(\theta_1,\mathcal{X}^{t',t''}),\qquad(14)$$
$$\mathcal{Z}^{t'',t''}\ge min(\theta_1,\mathcal{X}^{t',t''})$$

don’t hold simultaneously.

Assume that $\mathrm{v}_1^{\mathrm{r}'}(f')>\frac{1}{2}$.

If $\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})\le\frac{1}{2}$, then $\mathcal{X}^{t',t''}=\theta''$. Hence,

$$\mathcal{Z}^{t',t''}\ge\theta''=min(\theta_1,\theta'')=min(\theta_1,\mathcal{X}^{t',t''}),$$

which contradicts (13).

If $\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})>\frac{1}{2}$, then $\mathcal{X}^{t',t''}=1$ and

$$max\left(\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right),\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)^{\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})}>\frac{1}{2}.$$

The last inequality is satisfied if $max\left(\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}),\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right)=1$.

If $max\left(\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}),\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right)=0$, then $0>\frac{1}{2}$, i.e., a contradiction.

If $0<max\left(\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}),\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right)<1$, then

$$\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\ge log_{max\left(\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}),\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right)}\frac{1}{2}.$$

Therefore, $max\left(\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}),\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right)>\frac{1}{2}$, i.e., $\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})>\frac{1}{2}$ or $\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})>\frac{1}{2}$. Hence, $\mathcal{Y}^{t',t''}=1$ or $\mathcal{Z}^{t',t''}=1$.

In the first case, the conditions given by (14) are satisfied simultaneously, while in the second case, the condition (13) does not hold. Hence, a contradiction. We conclude, $\mathsf{v}_1^{\mathsf{r}'}(f')\le\frac{1}{2}$.

This completes the proof. □

Proof: (for R)

We write $\mathcal{X}\xrightarrow{\theta_1}_F\mathcal{Y}$ resp. $\mathcal{X}\to\xrightarrow{\theta_1}_F\mathcal{Y}$ instead of f if f is a fuzzy functional resp. a fuzzy multivalued dependency. Consequently, we write

$$\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\to\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)$$

resp.

$$\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\to\left(\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)|||\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)$$

instead of f', where $\mathcal{Z}=\mathsf{U}-\mathcal{X}\mathcal{Y}$.

As in the case (Y), choose the set $\{p,q\}$ to be the domain of each of the attributes in U.

We fix some $\theta''\in[0,\theta')$, where θ' is the minimum of the strengths of all dependencies that appear in $\mathsf{F}\cup\{f\}$. Assume that $\theta'<1$ and put $\mathsf{s}=(p,q)=\mathsf{s}(q,p)=\theta''$ to be a similarity relation on $\{p,q\}$.

$(a)\Rightarrow(b)$ Assume that (b) does not hold.

Then, there exists some $\mathsf{v}_\varepsilon^{\mathsf{r}}$ such that $\mathsf{v}_\varepsilon^{\mathsf{r}}(\mathcal{L})>\frac{1}{2}$, $\mathcal{L}\in\mathsf{F}'$ and $\mathsf{v}_\varepsilon^{\mathsf{r}}(f')\le\frac{1}{2}$. As in the case (Y), $\mathsf{v}_\varepsilon^{\mathsf{r}}$ is joined to some two-element fuzzy relation instance $\mathsf{r}=\{t_1,t_2\}$ on $\mathsf{R}(\mathsf{U})$ and some ε, $0\le\varepsilon\le1$.

Denote $W=\left\{\mathcal{A}\in\mathsf{U}\;\middle|\;\mathsf{v}_\varepsilon^{\mathsf{r}}(\mathcal{A})>\frac{1}{2}\right\}$.

Suppose that $W=\varnothing$. Then, $\mathsf{v}_\varepsilon^{\mathsf{r}}(\mathcal{A})\le\frac{1}{2}$, $\mathcal{A}\in\mathsf{U}$.

Since $\mathsf{v}_\varepsilon^{\mathsf{r}}(f')\le\frac{1}{2}$, we have that

$$1-\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)+\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)$$

$$=\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\to\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)\right)=\mathsf{v}_\varepsilon^{\mathsf{r}}(f')\le\frac{1}{2}$$

resp.

$$1-\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)+\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\cdot max\left(\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right),\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)$$

$$=1-\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)+\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\cdot\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)|||\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)$$

$$=\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\to\left(\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)|||\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)\right)$$

$$=\mathsf{v}_\varepsilon^{\mathsf{r}}(f')\le\frac{1}{2}.$$

Hence,

$$\frac{1}{2}+\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)\le\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)$$

resp.

$$\frac{1}{2}+\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\cdot max\left(\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right),\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)\le\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right).$$

Therefore, the fact that $\mathsf{v}_\varepsilon^{\mathsf{r}}(\mathcal{A})\ge0$ for all $\mathcal{A}\in\mathsf{U}$ yields that $\mathsf{v}_\varepsilon^{\mathsf{r}}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})\ge\frac{1}{2}$ holds always true. Now, there exist some $\mathcal{A}_0\in\mathcal{X}\subseteq\mathsf{U}$, such that

$$\mathsf{v}_\varepsilon^{\mathsf{r}}(\mathcal{A}_0)=min\left\{\mathsf{v}_\varepsilon^{\mathsf{r}}(\mathcal{A})\,|\mathcal{A}\in\mathcal{X}\right\}=\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\ge\frac{1}{2}.$$

This is a contradiction. We conclude, $W\ne\varnothing$.

Suppoes that $W=\mathrm{U}$. Then, $\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\mathcal{A})>\frac{1}{2}$, $\mathcal{A}\in\mathrm{U}$. Therefore, $\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{A}\in\mathcal{M}}\mathcal{A})>\frac{1}{2}$ for any $\mathcal{M}\subseteq\mathrm{U}$. Since $\mathrm{v}_{\varepsilon}^{\mathrm{r}}(f')\leq\frac{1}{2}$, we have that

$$\frac{1}{2\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})}+\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)\leq 1$$

resp.

$$\frac{1}{2\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})}+max\left(\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right),\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)\leq 1 .$$

Now, $\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})\leq 1$ yields, $\frac{1}{2\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})}\geq\frac{1}{2}$. Hence, $\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})\leq\frac{1}{2}$ resp. $max\left(\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right),\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)\leq\frac{1}{2}$. This means that we always have that $\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})\leq\frac{1}{2}$. This is a contradiction. We conclude, $W\neq\mathrm{U}$.

As in the case (Y), let $\mathrm{r}'=\{t',t''\}$ be the two element fuzzy relation instance on $\mathrm{R}(\mathrm{U})$, given by Table 1.

We shall prove that r' satisfies all dependencies in F and violates the dependency f.

Let $\mathcal{K}\xrightarrow{\theta_2}_F\mathcal{L}$ be any fuzzy functional dependency from the set F. We have,

$$1-\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)+\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right)$$

$$=\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\rightarrow\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right)\right)>\frac{1}{2},$$

i.e.,

$$\frac{1}{2}+\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right)>\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right).$$

Suppose that $\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A})\leq\frac{1}{2}$. Reasoning as in the case (Y), we conclude that there is $\mathcal{A}_0\in\mathcal{K}$ such that $\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\mathcal{A}_0)\leq\frac{1}{2}$. i.e., that $\mathcal{A}_0\notin W$. Hence, $\mathcal{A}_0^{t',t''}=\theta''$ and then $\mathcal{K}^{t',t''}=\theta''$. As before, the fact that $\mathrm{s}(p,q)=\theta''$ yields that $\mathcal{M}^{t',t''}\geq\theta''$ for every $\mathcal{M}\subseteq\mathrm{U}$. Now, $\mathcal{L}^{t',t''}\geq\theta''$ and then

$$\mathcal{L}^{t',t''}\geq\theta''=min\left(\theta_2,\mathcal{K}^{t',t''}\right).$$

This means that r' satisfies the dependency $\mathcal{K}\xrightarrow{\theta_2}_F\mathcal{L}$.

Now, suppose that $\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A})>\frac{1}{2}$. We obtain,

$$\frac{1}{2\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A})}+\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right)>1 .$$

Since $\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A})\leq 1$, we have that $\frac{1}{2\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A})}\geq\frac{1}{2}$. Hence, $\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B})>\frac{1}{2}$. Now, $\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\mathcal{B})>\frac{1}{2},\mathcal{B}\in\mathcal{L}$ and then $\mathcal{B}\in W$, $\mathcal{B}\in\mathcal{L}$. In other words, $\mathcal{L}\subseteq W$. Therefore, $\mathcal{L}^{t',t''}=1$. We obtain,

$$\mathcal{L}^{t',t''}=1\geq min\left(\theta_2,\mathcal{K}^{t',t''}\right).$$

This means that r' satisfies $\mathcal{K}\xrightarrow{\theta_2}_F\mathcal{L}$.

Let $\mathcal{K}\rightarrow\xrightarrow{\theta_2}_F\mathcal{L}$ be any fuzzy multivalued dependency from the set F. We have,

$$1-\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)+\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\cdot max\left(\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right),\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{D}\in\mathcal{M}}{\&}\mathcal{D}\right)\right)$$

$$=\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\rightarrow\left(\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right)\|\left(\underset{\mathcal{D}\in\mathcal{M}}{\&}\mathcal{D}\right)\right)\right)>\frac{1}{2},$$

i.e.,

$$\frac{1}{2}+\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\cdot max\left(\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right),\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{D}\in\mathcal{M}}{\&}\mathcal{D}\right)\right)>\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right),$$

where $\mathcal{M}=\mathrm{U}$- $\mathcal{KL}$.

Assume that $\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A})\leq\frac{1}{2}$. Reasoning as in the previous case, we conclude that $\mathcal{K}^{t',t''}=\theta''$. Hence, there exists $t'''\in\mathrm{r}'$, $t'''=t'$ such that (6) holds true. This means that the fuzzy relation instance r' satisfies the dependency $\mathcal{K}\rightarrow\xrightarrow{\theta_2}_F\mathcal{L}$.

Suppose that $\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A})>\frac{1}{2}$. We obtain,

$$\frac{1}{2\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A})}+max\left(\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right),\mathrm{v}_{\varepsilon}^{\mathrm{r}}\left(\underset{\mathcal{D}\in\mathcal{M}}{\&}\mathcal{D}\right)\right)>1 .$$

Since $\frac{1}{2\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A})}\geq\frac{1}{2}$, we conclude that $max\left(\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}),\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{D}\in M}\mathcal{D})\right)>\frac{1}{2}$. Therefore, $\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B})>\frac{1}{2}$ or $\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{D}\in M}\mathcal{D})>\frac{1}{2}$.

If $\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B})>\frac{1}{2}$, then, reasoning as in the case (Y), we obtain that $\mathcal{L}^{t',t''}=1$, $\mathcal{K}^{t',t''}=1$. Therefore, there exists $t'''\in\mathrm{r}'$, $t'''=t''$, such that (7) holds true. In other words, r' satisfies the dependency $\mathcal{K}\rightarrow\xrightarrow{\theta_2}_F\mathcal{L}$.

Similarly, if $\mathrm{v}_{\varepsilon}^{\mathrm{r}}(\&_{\mathcal{D}\in M}\mathcal{D})>\frac{1}{2}$, we have that $\mathcal{M}^{t',t''}=1$. Now, there exists $t'''\in\mathrm{r}'$, $t'''=t'$, such that (7) is valid, i.e., r' satisfies $\mathcal{K}\rightarrow\xrightarrow{\theta_2}_F\mathcal{L}$.

In remans to prove that r' violates the dependency $\mathcal{X}\xrightarrow{\theta_1}_F\mathcal{Y}$ resp. $\mathcal{X}\rightarrow\xrightarrow{\theta_1}_F\mathcal{Y}$. Let

$$1-\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)+\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)$$

$$=\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\to\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)\right)=\mathsf{v}_\varepsilon^{\mathsf{r}}(f')\le\frac{1}{2}.$$

Hence,

$$\frac{1}{2}+\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)\le\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right).$$

Since $\mathsf{v}_\varepsilon^{\mathsf{r}}(\mathcal{A})\ge 0$ for $\mathcal{A}\in\mathsf{U}$, we conclude that $\mathsf{v}_\varepsilon^{\mathsf{r}}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})\ge\frac{1}{2}$. Therefore, $\mathcal{X}\subseteq W$ and then $\mathcal{X}^{t',t''}=1$. Now,

$$\frac{1}{2\mathsf{v}_\varepsilon^{\mathsf{r}}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})}+\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)\le 1.$$

As, before, we obtain that $\mathsf{v}_\varepsilon^{\mathsf{r}}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})\le\frac{1}{2}$, i.e., $\mathcal{Y}^{t',t''}=\theta''$. We have,

$$\mathcal{Y}^{t',t''}=\theta''<\theta'\le\theta_1=\min\left(\theta_1,\mathcal{X}^{t',t''}\right).$$

Hence, r' does not satisfy the dependency $\mathcal{X}\xrightarrow{\theta_1}_F\mathcal{Y}$. Now, suppose that

$$1-\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)+\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\cdot\max\left(\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right);\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)$$

$$=\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\to\left(\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)\|\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)\right)=\mathsf{v}_\varepsilon^{\mathsf{r}}(f')\le\frac{1}{2}.$$

We have,

$$\frac{1}{2}+\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\cdot\max\left(\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right);\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)\le\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right).$$

Now, $\mathsf{v}_\varepsilon^{\mathsf{r}}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})\ge\frac{1}{2}$ and then $\mathcal{X}\subseteq W$, i.e., $\mathcal{X}^{t',t''}=1$. Moreover,

$$\frac{1}{2\mathsf{v}_\varepsilon^{\mathsf{r}}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})}+\max\left(\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right);\mathsf{v}_\varepsilon^{\mathsf{r}}\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)\le 1.$$

Therefore, $\max\left(\mathsf{v}_\varepsilon^{\mathsf{r}}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B});\mathsf{v}_\varepsilon^{\mathsf{r}}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right)\le\frac{1}{2}$ and hence $\mathsf{v}_\varepsilon^{\mathsf{r}}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})\le\frac{1}{2}$, $\mathsf{v}_\varepsilon^{\mathsf{r}}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\le\frac{1}{2}$. Consequently $\mathcal{Y}^{t',t''}=\theta''$, $\mathcal{Z}^{t',t''}=\theta''$.

If $t'''\in\mathsf{r}'$ and $t'''=t'$, that reaspning in the same way as in the cas (Y), we conclude that (8) holds true.

Similarly, if $t'''\in\mathsf{r}'$ and $t'''=t''$, we obtain that (9) holds true.

Therefore, the instance r' does not satisfy the dependency $\mathcal{X}\to\xrightarrow{\theta_1}_F\mathcal{Y}$.

$(b)\Rightarrow(a)$ Assume that (a) does not hold.

Now, as in the case (Y), there exists a two-element fuzzy relation instance $\mathsf{r}'=\{t_1,t_2\}$ on $\mathsf{R}(\mathsf{U})$, such that r' satisfies all dependencies from the set F but violates the dependency f. Hence, r' violates $\mathcal{X}\xrightarrow{\theta_1}_F\mathcal{Y}$ resp. $\mathcal{X}\to\xrightarrow{\theta_1}_F\mathcal{Y}$.

Reasoning in exactly the same way as in the case (Y), we conclude that $W\ne\varnothing$ and $W\ne\mathsf{U}$, where $W=\left\{\mathcal{A}\in\mathsf{U}\,\middle|\,\mathcal{A}^{t',t''}=1\right\}$.

Now, we prove that $\mathsf{v}_1^{\mathsf{r}'}(\mathcal{L})>\frac{1}{2}$, $\mathcal{L}\in\mathsf{F}'$ and $\mathsf{v}_1^{\mathsf{r}'}(f')\le\frac{1}{2}$, where $\mathsf{v}_1^{\mathsf{r}'}$ is defined by

$$\mathsf{v}_1^{\mathsf{r}'}(\mathcal{A})\in\left(\frac{1}{2},1\right],\ \mathcal{A}\in W,$$

$$\mathsf{v}_1^{\mathsf{r}'}(\mathcal{A})\in\left[0,\frac{1}{2}\right],\ \mathcal{A}\in\mathsf{U}-W.$$

Let $\mathcal{L}\in\mathsf{F}'$ be of the form

$$\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\to\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right).$$

The fuzzy formula corresponds to some fuzzy functional dependency $\mathcal{K}\xrightarrow{\theta_2}_F\mathcal{L}$ from the set F.

If $\mathsf{v}_1^{\mathsf{r}'}(\mathcal{L})\le\frac{1}{2}$, then

$$1-\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)+\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right)\le\frac{1}{2},$$

i.e.,

$$\frac{1}{2}+\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right)\le\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right).$$

Therefore, $\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A})\ge\frac{1}{2}$ and then $\mathcal{K}^{t',t''}=1$. Moreover,

$$\frac{1}{2\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A})}+\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right)\le 1.$$

Now, as before, we conclude that $\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B})\le\frac{1}{2}$ and hence $\mathcal{L}^{t',t''}=\theta''$. We have,

$$\mathcal{L}^{t',t''}=\theta''<\theta'\le\theta_2=\min(\theta_2,1)=\min\left(\theta_2,\mathcal{K}^{t',t''}\right).$$

This, however, contradictions the fact that the instance r' satisfies the dependency $\mathcal{K}\xrightarrow{\theta_2}_F\mathcal{L}$. Hence, $\mathsf{v}_1^{\mathsf{r}'}(\mathcal{L})>\frac{1}{2}$.

Now, let $\mathcal{L}\in\mathsf{F}'$ be of the form

$$\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\to\left(\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right)\|\left(\underset{\mathcal{D}\in\mathcal{M}}{\&}\mathcal{D}\right)\right),$$

where $\mathcal{M}=\mathsf{U}\text{-}\ \mathcal{KL}$. This fuzzy formula corresponds to some fuzzy multivalued dependency $\mathcal{K}\rightarrow\xrightarrow{\theta_2}_F \mathcal{L}$ from the set F.

If $\mathsf{v}_1^{\mathsf{r}'}(\mathcal{L})\le\frac{1}{2}$, then

$$1-\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)+\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\cdot max\left(\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right),\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{D}\in\mathcal{M}}{\&}\mathcal{D}\right)\right)\le\frac{1}{2},$$

i.e.,

$$\frac{1}{2}+\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\cdot max\left(\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right),\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{D}\in\mathcal{M}}{\&}\mathcal{D}\right)\right)\le\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right).$$

Now, $\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A})\ge\frac{1}{2}$ and then $\mathcal{K}^{t',t''}=1$. Furtermore,

$$\frac{1}{2\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A})}+max\left(\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right),\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{D}\in\mathcal{M}}{\&}\mathcal{D}\right)\right)\le 1.$$

We obtain, $max\left(\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B});\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D})\right)\le\frac{1}{2}$ and then $\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B})\le\frac{1}{2}$, $\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D})\le\frac{1}{2}$. Therefore, $\mathcal{L}^{t',t''}=\theta''$ and $\mathcal{M}^{t',t''}=\theta''$. Consequently, the third condition of the conditions (11) is not valid and the second condition of the conditions (12) is not valid as well. This contradicts the fact that the instance instance r' satisfies $\mathcal{K}\rightarrow\xrightarrow{\theta_2}_F \mathcal{L}$. Hence, $\mathsf{v}_1^{\mathsf{r}'}(\mathcal{L})>\frac{1}{2}$.

It remains to prove that $\mathsf{v}_1^{\mathsf{r}'}(f')\le\frac{1}{2}$.

Assume that r' violates $\mathcal{X}\xrightarrow{\theta_1}_F \mathcal{Y}$.

Suppose that $\mathsf{v}_1^{\mathsf{r}'}(f')>\frac{1}{2}$. We have,

$$\frac{1}{2}+\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)>\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right).$$

If $\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})\le\frac{1}{2}$, then $\mathcal{X}^{t',t''}=\theta''$ and hence, reasoning in the same way as in (Y), we obtain that $\mathcal{Y}^{t',t''}\ge min\left(\theta_1,\mathcal{X}^{t',t''}\right)$. This contradicts the fact that r' does not satisfy $\mathcal{X}\xrightarrow{\theta_1}_F \mathcal{Y}$.

Let $\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})>\frac{1}{2}$. Now,

$$\frac{1}{2\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})}+\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)>1.$$

We obtain $\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})>\frac{1}{2}$ and then $\mathcal{Y}^{t',t''}=1$. Now, as in the case (Y), $\mathcal{Y}^{t',t''}\ge min\left(\theta_1,\mathcal{X}^{t',t''}\right)$. This is a contradiction. Therefore, $\mathsf{v}_1^{\mathsf{r}'}(f')\le\frac{1}{2}$.

Assume that r' violates the dependency $\mathcal{X}\rightarrow\xrightarrow{\theta_1}_F \mathcal{Y}$. Reasoning in exactly the same way as in the case (Y), we obtain that (13) holds true and that first and the second condition of the conditions (14) don't hold at the same time.

Let $\mathsf{v}_1^{\mathsf{r}'}(f')>\frac{1}{2}$. We have,

$$\frac{1}{2}+\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\cdot max\left(\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right);\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)>\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right).$$

If $\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})\le\frac{1}{2}$, then $\mathcal{X}^{t',t''}=\theta''$. Hence, as in the case (Y), $\mathcal{Z}^{t',t''}\ge min\left(\theta_1,\mathcal{X}^{t',t''}\right)$. This contradicts (13).

Let $\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})>\frac{1}{2}$. Now, $\mathcal{X}^{t',t''}=1$ and

$$\frac{1}{2\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})}+max\left(\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right),\mathsf{v}_1^{\mathsf{r}'}\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)>1.$$

We obtain, $max\left(\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B});\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right)>\frac{1}{2}$. Hence, $\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})>\frac{1}{2}$ or $\mathsf{v}_1^{\mathsf{r}'}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})>\frac{1}{2}$, i.e., $\mathcal{Y}^{t',t''}=1$ or $\mathcal{Z}^{t',t''}=1$. In the first case, the conditions (14) are satisfied. In the second case, the condition (13) is not valid. Hence, a contradiction. We conclude, $\mathsf{v}_1^{\mathsf{r}'}(f')\le\frac{1}{2}$. This completes the proof. □

IV. Conclusion

The results presented in this paper can be similarly verified for many other individual fuzzy implication operators. Such operators may be residuated (R) as well. One could try to vary t-norms as well as t-conorms. In particular, it would be nice to determine the degree of generality to which our results may be applied.

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