Fuzzy Numerical Solution to Horizontal Infiltration

N. Samarinas, C. Tzimopoulos and C. Evangelides

***Abstract*—**In this paper we examine the fuzzy numerical solution to a second order partial differential equation, called absorption equation, which in general describes the water movement. The uncertainties that appear, from either human or machine errors, in this equation greatly affect the results and for this reason they should be taken into account. The solution in this problem is to use the fuzzy set theory. Here, we present an implicit finite difference scheme in combination with fuzzy logic. Since the problem refers to a partial differential equation, the Generalized Hukuhara (gH) derivative was used in order to find the correct form of the linear system of equations.

***Keywords*—**Absorption equation, finite difference, fuzzy logic, infiltration, numerical solution.

I. Introduction

FUZZY logic is derived from the development of fuzzy sets theory of Lofti Zadeh [1]; it is well structured and performs well in ambiguous or uncertain environments. It is commonly accepted that the techniques based on classical logic have proved unsuccessful to approximate the procedures of common sense, learning from experience, etc. [2]. Classical (two-valued) logic deals with propositions that are either true or false. In many-valued logic, a generalization of the classical logic, the propositions have more than two truth values. Fuzzy logic is an extension of the many-valued logic in the sense of incorporating fuzzy sets and fuzzy relations as tools into the system of many-valued logic [3].

Many important dynamical systems in the real word can be described by partial differential equations. It is known that both analytical and numerical methods have been developed to solve problems of partial differential equations. However, calculation for solving partial differential equations is very difficult where the exact solution of these problems can be found only in some special cases. When we are studying in fields of physics and engineering, we often meet problems of fuzzy partial differential equations which have to be solved as numerical methods [4]. This case for numerical methods for solving fuzzy differential equations have been rapidly growing in recent years.

Since the above problem concerns differential equations regarding fuzzy logic, we should mention that a number of studies were carried out in that field. Initially, the concept of fuzzy derivative was introduced by Chang and Zadeh [5], followed by Dubois and Prade [6] who used extension principle in their approach. Fuzzy differential functions were studied by Puri and Ralescu [7], who extended Hukuhara derivative (H-derivative) of a set of values appearing in fuzzy sets. Kaleva and Seikkala in [8], [9], [10] developed the fuzzy initial value problem. But this method has presented certain drawbacks, and in many cases this solution was not a good generalization of the classic case. The generalized Hukuhara differentiability (gH – differentiability) was introduced by [11] and [12] which overcomes this drawback. This new derivative is defined for a larger class of fuzzy functions than Hukuhara derivative. Allahviranloo in [13], introduced the (gH-p) differentiability for partial derivatives as an extension of the above theory. Tzimopoulos et al. in [14] used the above method and gave a fuzzy analytical solution to a parabolic differential equation.

The numerical method for solving fuzzy differential equations was introduced by Ma [15]. Subsequently, numerical solutions of fuzzy differential equations were examined by Friedman [16], Bede [17] and Abbasbandy [18]. The existence of solution for fuzzy partial differential equations were investigated also by Buckley and Feuring in [19]; their proposed method works only for elementary partial differential equations. Based on Seikkala derivative Allahviranloo in [20], and Kermani et al. in [21] use a numerical method which is an explicit difference method to solve partial differential equations. Farajzadeh in [22] gives an explicit method for solving fuzzy partial differential equation. More recently, Uthirasamy in [23] gives studies on numerical solutions of fuzzy boundary value problems and fuzzy partial differential equations.

In this article, a fuzzy numerical solution of a linear one dimensional infiltration equation, with initial and boundary conditions, is presented applying an implicit finite difference scheme. This equation is a parabolic partial differential equation, describing the water horizontal movement in porous medium. The calculation of water flow in the unsaturated zone requires the knowledge of the initial and boundaries conditions as well as the various soil parameters. Until today, these conditions and the parameters were assumed well

N. Samarinas is with the Department of Rural and Surveying Engineering, Aristotle University of Thessaloniki, Greece, 546 23 (e-mail: samnikiforos1@gmail.com).

C. Tzimopoulos is with Department of Rural and Surveying Engineering, Aristotle University of Thessaloniki, Greece, 546 23 (e-mail: ctzimop@gmail.com).

C. Evangelides is with Department of Rural and Surveying Engineering, Aristotle University of Thessaloniki, Greece, 546 23 (corresponding author to provide phone: +30 2310996147; fax:+30 2130996147 e-mail evan@vergina.eng.auth.gr:).

defined and this assumption is based principally in measurements. But in reality they are subject to different kinds of uncertainty due to human and machine imprecision.

II. Fuzzy Sets

Definition 1. A fuzzy set u on a universe set X is a mapping $u: X \to [0,1]$ assigning to each element $x \in X$ a degree of membership $0 \le u(x) \le 1$. The membership function *u(x)* is also defined as $\mu_u(x)$.

Definition 2. We denote by $\mathbb{R}_F$ the class of fuzzy subsets $u: \mathbb{R} \to [0,1]$, satisfying the following properties[7], [8]:

1. u is normal, that is, there exists, $x_0 \in \mathbb{R}$ with $u(x_0) = 1$
2. u is a convex fuzzy set. that is,
$u(\lambda x + (1-\lambda)y \ \ge min\{u(x), u(y)\}, \forall x, y \in \mathbb{R}, \forall \lambda \in [0,1]$
3. u is upper semi-continuous on $\mathbb{R}$.
4. $\overline{\{x \in \mathbb{R} | u(x) > 0\}}$ is compact, where $\bar{A}$ denotes the closure of A

Then $\mathbb{R}_F$ is called the space of fuzzy numbers.

Definition 3. Let u be a fuzzy number. The α-level set $[u]^a$ is a non-empty compact interval for all $0 \le a \le 1$. Denote $u^-(a) = min[u]^a$ and $u^+(a) = max[u]^a$. In other words, $u^-(a)$ denotes the left hand side function $u^-(a): [0,1] \to \mathbb{R}$ and is monotonic increasing-lower semi-continuous and $u^+(a)$ denotes the right side $u^+(a): [0,1] \to \mathbb{R}$ and is monotonic decreasing and upper semi-continuous. We use the notation $[u]^a = [u^-(a), u^+(a)]$.

Definition 4. The necessary and sufficient condition for $(u^-(a), u^+(a))$ to define a fuzzy number are as follows:

1. $u^-(a)$ is a bounded monotonic increasing (non-decreasing) left-continuous function for all $a \in [0,1]$ and right-continuous for $a = 0$.
2. $u^+(a)$ is a bounded monotonic decreasing (non-increasing) left-continuous function for all $a \in [0,1]$ and right-continuous for $a = 0$.
3. $u^-(a) \le u^+(a), \ \ 0 \le a \le 1$

Definition 5. The metric structure is given by the transformed Hausdorff distance, in fuzzy sets, $D: \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\}$, by

$$D(u,v) = \sup_{a \in [0,1]} max\{|u^-(a) - v^-(a)|, |u^+(a) - v^+(a)|\} \qquad (1)$$

where $[u]^a = [u^-(a), u^+(a)], [v]^a = [v^-(a), v^+(a)]$

Then it is easy to see that D is a metric in $\mathbb{R}_F$ and has the following properties:

1. $D(u+w, v+w) = D(u,v)$
2. $D(\lambda u, \lambda v) = |\lambda| D(u,v)$
3. $D(u+v, w+e) \le D(u,w) + D(v,e)$ and $(\mathbb{R}_F, D)$ is a complete metric space, for all $u, v, w, e \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}$.

Definition 6. A fuzzy set u is called triangular fuzzy number with center z, left width $z_1 > 0$ and right width $z_2 > 0$, if its membership function has the following form:

$$u(x) = \begin{cases} 1 - \dfrac{z-x}{z_1} & if\ z - z_1 \le x \le z \\ 1 - \dfrac{x-z}{z_2} & if\ z \le x \le z + z_2 \\ 0 & otherwise \end{cases}$$

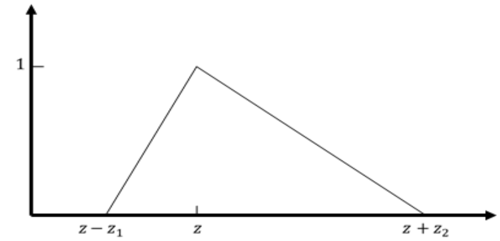


Fig.1 fuzzy triangular number

It can easily be verified that

$$[u]^a = [z - (1-a)z_1, z + (1-a)z_2]$$

for all $a \in [0,1]$.

III. Physical Problem

A. Absorption equation

Infiltration is a common physical phenomenon of water movement in porous media which is of the great interest in much earth and plant sciences. Historically [24] presented two basic ideas in the development of soil water movement: The capillary potential and capillary conductivity. In case of one dimensional horizontal flow the phenomenon is described by the equation:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(D \frac{\partial \theta}{\partial x}\right) \qquad (2)$$

where, θ=the moisture content(cm^3/cm^3) and
D=diffusivity (cm^2/s)

This equation represents the water movement in a horizontal column and is called *absorption equation* by [25], because it describes the wetting up for the column under tension. In case when the column is semi-infinite, have initial moisture content θ_0 and the initial and boundaries conditions are:

$$\begin{aligned} &t = 0, \ \ \theta(x,0) = \theta_0 \\ &t > 0, \ \ \theta(0,t) = \theta_1, \ \ \underset{x\to\infty}{\theta}(x,t) = \theta_0 \end{aligned} \qquad (3)$$

For $\theta_1 > \theta_0$, (2) with conditions (3) describes the horizontal infiltration(absorption) of water by application of a constant moisture content at x=0. Analytical solutions of (2) are available under several simplifications as were referred in [25].

B. Crisp Model

The linearized form of (2) subject to conditions (3) is:

$$\frac{\partial \theta}{\partial t} = D_0 \frac{\partial^2 \theta}{\partial x^2} \qquad (4)$$

Introducing now non-dimensional variables Θ, X, T,

$$\Theta = \frac{\theta - \theta_0}{\theta_1 - \theta_0}, \ \ X = \frac{x}{L}, \ \ \tau = \frac{D_0 t}{L^2}, \qquad (5)$$

then (4) becomes:

$$\frac{\partial \Theta}{\partial \tau} = \frac{\partial^2 \Theta}{\partial X^2} \qquad (6)$$

subject to the following initial and boundary non-dimensional conditions:

$$\begin{aligned} &\tau = 0,\ \ \Theta(X,0) = 0 \\ &\tau > 0,\ \ \Theta(0,T) = \Theta_1 = 1,\ \ \underset{X\to\infty}{\Theta}(X,\tau) = 0 \end{aligned} \qquad (7)$$

The solution of (5) is [14]:

$$\Theta = erfc\left(\frac{X}{\sqrt{4\tau}}\right) = erfc\left(\frac{X}{2\sqrt{D_0 t}}\right) = erfc(z), \qquad (8)$$

where $z = \frac{X}{\sqrt{4\tau}}$

C. Fuzzy Model

The boundary condition is assumed fuzzy number:

$$\Theta_1|_{X=0} = [1-\varepsilon+\varepsilon\cdot\alpha, 1+\varepsilon-\varepsilon\cdot\alpha] \qquad (9)$$

where ε = the fuzziness and α = the α-level cut
Then the following expression has been proved in [14] to be a solution to the fuzzy equation:

$$\Theta|_a = \{\Theta_1\cdot erfc(z)\}_a = [\Theta_a^-, \Theta_a^+] \qquad (10)$$

$$\begin{aligned} &\Theta^- = (1-\varepsilon+\varepsilon\cdot\alpha)\cdot erfc(z), \\ &\Theta^+ = (1+\varepsilon-\varepsilon\cdot\alpha)\cdot erfc(z) \end{aligned} \qquad (10a)$$

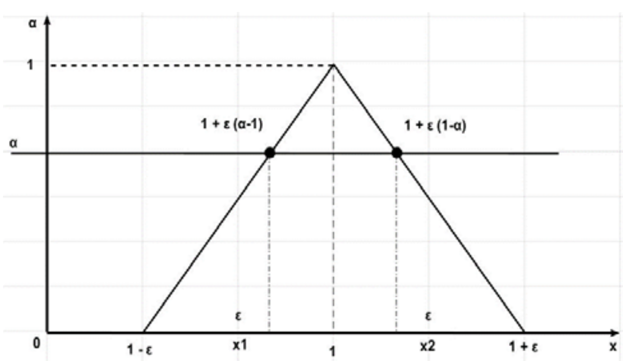


Fig. 2 Θ_1 fuzzy boundary condition

IV. Hukuhara General Partial Derivative

Definition 7. Let $(x_0,t_0)\in D$ (D is a crisp set), then the first generalized Hukuhara partial derivative ([gH-p]) of fuzzy valued function $f(x,t): D\to\mathbb{R}_F$ at (x_0,t_0) with respect to variables x, t are the functions $\frac{\partial f_a(x_0,t_0)}{\partial x_{i,gH}}$ and $\frac{\partial f_a(x_0,t_0)}{\partial t_{i,gH}}$ given by:

$$\frac{\partial f_a(x_0,t_0)}{\partial x_{i,gH}} = \lim_{h\to 0}\frac{f(x_0+h,t_0)\Theta_{gH} f(x_0,t_0)}{h}$$

$$\frac{\partial f_a(x_0,t_0)}{\partial t_{i,gH}} = \lim_{k\to 0}\frac{f(x_0+h,t_0)\Theta_{gH} f(x_0,t_0)}{k}$$

provided that $\left\{\frac{\partial f_a(x_0,t_0)}{\partial x_{i,gH}}, \frac{\partial f_a(x_0,t_0)}{\partial t_{i,gH}}\right\}\in\mathbb{R}_F$

Definition 8.

A. First order

A fuzzy-valued function f of two variables is a rule that assigns to each ordered pair of real numbers (x, t), in a set D a unique fuzzy number denoted by *f(x,t)*. Let $f(x,t): D\to\mathbb{R}_F$, $(x_0,t_0)\in D$ and $f_a^-(x,t), f_a^+(x,t)$ be real valued functions and partial differentiable with respect to *x* and *t*. We say that [26], [27], [13]:

- $f(x,t)$ is [(i)-p]-differentiable w.r.t *x* at (x_0,t_0) if

$$\frac{\partial f_a(x_0,t_0)}{\partial x_{i,gH}} = \left[\frac{\partial f_a^-(x_0,t_0)}{\partial x}, \frac{\partial f_a^+(x_0,t_0)}{\partial x}\right]$$

- $f(x,t)$ is [(ii)-p]-differentiable w.r.t *x* at (x_0,t_0) if

$$\frac{\partial f_a(x_0,t_0)}{\partial x_{i,gH}} = \left[\frac{\partial f_a^+(x_0,t_0)}{\partial x}, \frac{\partial f_a^-(x_0,t_0)}{\partial x}\right]$$

Notation. The same is valid for $\frac{\partial f_a(x_0,t_0)}{\partial t}$

Definition 9.

B. Second order

Let $f(x,t): D\to\mathbb{R}_F$, and $\frac{\partial f_a(x_0,t_0)}{\partial x_{i.gH}}$ be [gH-p]-differentiable at $(x_0,t_0)\in D$ with respect to *x*. We say that [26], [13]:

1. $\frac{\partial f_a(x_0,t_0)}{\partial x_{i,gH}}$ is [(i)-p]- differentiable w.r.t *x* if:

$$\frac{\partial^2 f_a(x_0,t_0)}{\partial x_{i.gH}^2} = \begin{cases} \dfrac{\partial^2 f_a^-(x_0,t_0)}{\partial x^2}, \dfrac{\partial^2 f_a^+(x_0,t_0)}{\partial x^2} \\ if\ f(x,t)\ is\ [(i)-p] differentiable \\ \\ \dfrac{\partial^2 f_a^+(x_0,t_0)}{\partial x^2}, \dfrac{\partial^2 f_a^-(x_0,t_0)}{\partial x^2} \\ if\ f(x,t)\ is\ [(ii)-p] differentiable \end{cases}$$

2. $\frac{\partial f_a(x_0,t_0)}{\partial x_{ii.gH}}$ is [(ii)-p]- differentiable w.r.t *x* if:

$$\frac{\partial^2 f_a(x_0,t_0)}{\partial x_{ii.gH}^2} = \begin{cases} \dfrac{\partial^2 f_a^+(x_0,t_0)}{\partial x^2}, \dfrac{\partial^2 f_a^-(x_0,t_0)}{\partial x^2} \\ if\ f(x,t)\ is\ [(i)-p] differentiable \\ \\ \dfrac{\partial^2 f_a^-(x_0,t_0)}{\partial x^2}, \dfrac{\partial^2 f_a^+(x_0,t_0)}{\partial x^2} \\ if\ f(x,t)\ is\ [(ii)-p] differentiable \end{cases}$$

V. Fuzzy Finite Difference

A. Implicit Scheme

The main idea in the finite difference method is to replace the derivative in a partial differential equation with difference quotients.

Assume that u is a function of the independent crisp variables x and t. Subdivide the x-t plane into sets of equal rectangles of sides h, k, by equally spaced grid lines parallel to O_t, defined by $x_i = ih, i = 0,1,2,\dots,m,$ and equally spaced grid lines parallel to O_x, defined by $t_j = jk, j = 0,1,2,\dots,n,$ where m and n are positive integers with $h = {}^{L}/_{m}$ and $k = {}^{T}/_{n}$.

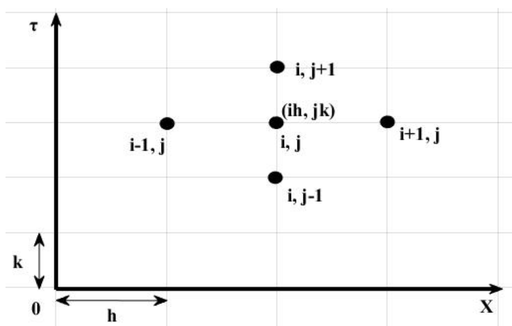


Fig. 3 Grid

Using the above theory of [gH] differentiability we have the following results [12]:

The first derivative with respect to x is equal to

$$\frac{\partial u_a(x_0,t_0)}{\partial x_{i,gH}} = \left[\frac{\partial u_a^+(x_0,t_0)}{\partial x}, \frac{\partial u_a^-(x_0,t_0)}{\partial x}\right]$$

which means that u is [(ii)-p] differentiable with respect to x according to definition 8.

Also second derivative is equal to:

$$\frac{\partial^2 u_a(x_0,t_0)}{\partial x_{i.gH}^2} = \left[\frac{\partial^2 u_a^-(x_0,t_0)}{\partial x^2}, \frac{\partial^2 u_a^+(x_0,t_0)}{\partial x^2}\right]$$

which means that $\frac{\partial u_a(x_0,t_0)}{\partial x_{i,gH}}$ is [(ii)-p] differentiable with respect to x according to definition 9.

The first derivative with respect to t is equal to

$$\frac{\partial u_a(x_0,t_0)}{\partial t_{i,gH}} = \left[\frac{\partial u_a^-(x_0,t_0)}{\partial t}, \frac{\partial u_a^+(x_0,t_0)}{\partial t}\right]$$

which means that u is [(i)-p] differentiable with respect to t according to definition 8 [12].

Finally we conclude that:

$$\frac{\partial u^-}{\partial t} = \frac{\partial^2 u^-}{\partial x^2}, \tag{11a}$$

and

$$\frac{\partial u^+}{\partial t} = \frac{\partial^2 u^+}{\partial x^2}. \tag{11b}$$

By using implicit discretization scheme and applying Taylor's theorem we get the approximate formulae:

$$\frac{\partial u^-}{\partial t}_{i,j} = \frac{u_{i,j+1}^- - u_{i,j}^-}{dt} + O(dt) \tag{12a}$$

$$\frac{\partial u^+}{\partial t}_{i,j} = \frac{u_{i,j+1}^+ - u_{i,j}^+}{dt} + O(dt) \tag{12b}$$

$$\frac{\partial^2 u^-}{\partial x^2}_{i,j} = \frac{u_{i+1,j+1}^- - 2u_{i,j+1}^- + u_{i-1,j+1}^-}{(dx)^2} + O((dx)^2) \tag{13a}$$

$$\frac{\partial^2 u^+}{\partial x^2}_{i,j} = \frac{u_{i+1,j+1}^+ - 2u_{i,j+1}^+ + u_{i-1,j+1}^+}{(dx)^2} + O((dx)^2) \tag{13b}$$

By using the Crank-Nicolson scheme the combination of (12a) and (13a), gives the following equation

$$\begin{aligned} -r\,u_{i-1,j+1}^- + (2+2r)u_{i,j+1}^- - r\,u_{i+1,j+1}^- \\ = ru_{i-1,j}^- + (2-2r)\,u_{i,j}^- + r\,u_{i+1,j}^- \end{aligned} \tag{14a}$$

Similarly, by using the Crank-Nicolson scheme the combination of (12b) and (13b), gives also the following equation:

$$\begin{aligned} -r\,u_{i-1,j+1}^+ + (2+2r)u_{i,j+1}^+ - r\,u_{i+1,j+1}^+ \\ = ru_{i-1,j}^+ + (2-2r)\,u_{i,j}^+ + r\,u_{i+1,j}^+ \end{aligned} \tag{14b}$$

where $r = \frac{dt}{(dx)^2}$

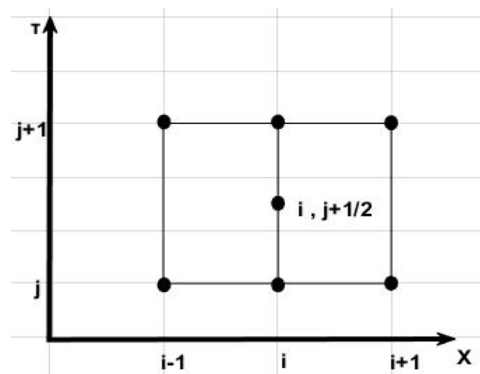


Fig. 4 Crank – Nicolson grid

This fuzzy numerical method called Crank-Nicolson method, which it is an implicit finite difference scheme that replace the derivative in a partial deferential equation with difference quotients.

B. Matrix form of Crank-Nicolson scheme

Implicit finite difference methods include more than one value of the unknown function at the next time step and in order to solve them, a system of equations is required. That is why it is particularly useful for these methods to be expressed through matrices. The triangular matrices used in the Crank-Nicolson equations are usually strong diagonally structured. The determinant of these matrices is not zero, so the system is not impossible, nor does it have an infinite number of solutions.

Matrix form of the difference schemes (14a), (14b):

$$P_1 u_{j+1}^- = Q_1 u_j^- + G_{j+1}, \qquad P_1 u_{j+1}^+ = Q_1 u_j^+ + G_{j+1}$$

where

$$P_1 = \begin{bmatrix} 2+2r & -r & 0 & . & . & . & . \\ -r & 2+2r & -r & & & & . \\ 0 & & . & & & & . \\ . & & & . & & & . \\ . & & & & . & & . \\ . & & & & -r & 2+2r & -r \\ . & & & & & -r & 2+2r \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} 2-2r & r & 0 & . & . & . & . \\ r & 2-2r & r & & & & . \\ 0 & & . & & & & . \\ . & & & . & & & . \\ . & & & & . & & . \\ . & & & & r & 2-2r & r \\ . & & & & & r & 2-2r \end{bmatrix}$$

$$u_{j+1}^- = [u_{1,j+1}^-, \dots, u_{N-1,j+1}^-]^T, \quad u_{j+1}^+ = [u_{1,j+1}^+, \dots, u_{N-1,j+1}^+]^T$$

$$G_{j+1} = [ru_{0,j}, 0, \dots, 0, 0, ru_{N,j}]^T$$

The finite difference schemes are solved as system of equations. One of the most common method of solving such systems is the Thomas method (algorithm) which is a simplified form of Gaussian elimination that can be used to solve tridiagonal systems of equations [28].

VI. Application

We will recall (6):

$$\frac{\partial \Theta}{\partial \tau} = \frac{\partial^2 \Theta}{\partial X^2}$$

with the initial and boundary conditions:

$$\tau = 0, \;\; \Theta(X,0) = 0$$
$$\tau > 0, \;\; \Theta(0,T) = \Theta_1 = 1, \;\; \underset{X\to\infty}{\Theta}(X,\tau) = 0$$

We assume that the water content varies 15%. Thus, a fuzziness ε of 15% is introduced on Θ_1, that is from (9) we have the boundary condition to be assumed as a fuzzy number:

$$\Theta_1|_{X=0} = [0.85 + 0.15\alpha, 1.15 - 0.15\alpha]$$

and then the solution (10) becomes:

$$\Theta|_a = \{\Theta_1 \cdot erfc(z)\}_a = [\Theta_a^-, \Theta_a^+]$$

with

$$\Theta^- = (0.85 + 0.15\alpha) \cdot erfc(z),$$

$$\Theta^+ = (1.15 - 0.15\alpha) \cdot erfc(z)$$

and $z = \frac{X}{\sqrt{4\tau}}$

Fig. 5 presents the membership function of the Θ_1.

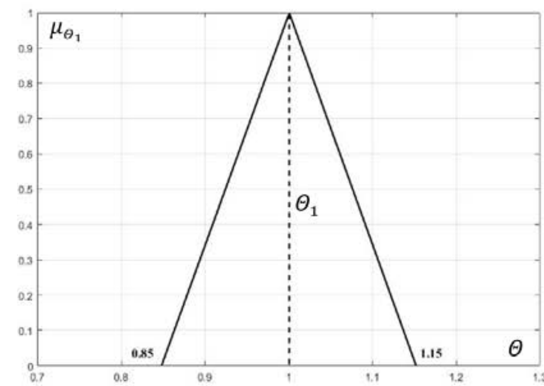


Fig. 5 Θ_1 fuzzy boundary condition

Since examining the first and second derivatives of (6) and found the degree of differentiability, the next step is to construct the numerical model in order to solve the problem with numerical method (finite difference). Thus, we will use the fuzzy implicit numerical scheme Crank – Nicolson and the system of the following equations should be solved in the following scheme and we have replaced the fuzzy function u by Θ:

$$-r\,\Theta_{i-1,j+1}^- + (2+2r)\Theta_{i,j+1}^- - r\,\Theta_{i+1,j+1}^- = r\Theta_{i-1,j}^- + (2-2r)\,\Theta_{i,j}^- + r\,\Theta_{i+1,j}^- \quad (15a)$$

$$-r\,\Theta_{i-1,j+1}^+ + (2+2r)\Theta_{i,j+1}^+ - r\,\Theta_{i+1,j+1}^+ = r\Theta_{i-1,j}^+ + (2-2r)\,\Theta_{i,j}^+ + r\,\Theta_{i+1,j}^+ \quad (15b)$$

where $r = \frac{d\tau}{(dX)^2}$

According with initial and boundary conditions the below grid defined.

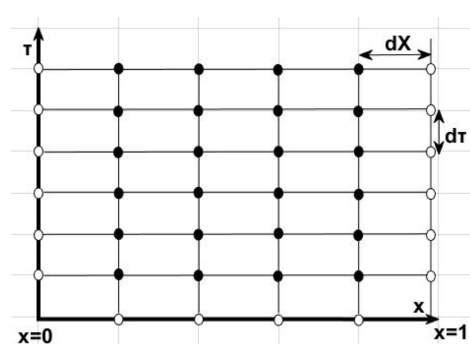


Fig. 6 Grid for initial and boundary conditions

Generally, when numerical methods are applied, comparison with the analytical solution of the problem -when it exists- is a way of verifying the results. For this reason, the finite difference method will be applied below and will be compared with the results of the analytical solution.

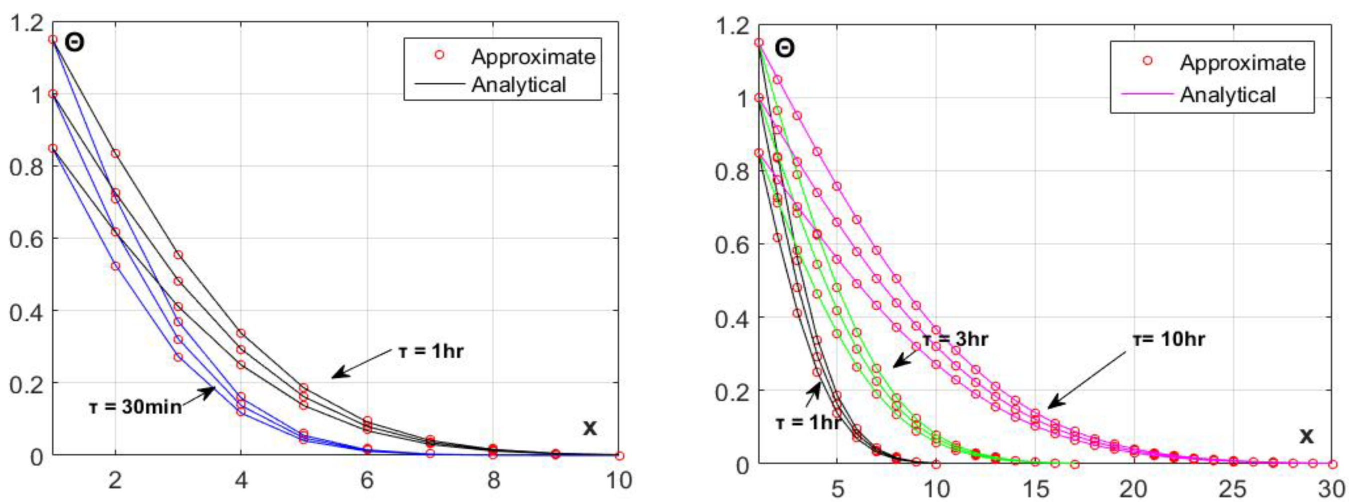
Another very important point to note when using differential equations, it is that it is a well-posed problem according to Jacques Hadamard [29].

We use (15a), (15b) to approximate the analytical solution with dX = 0.035, dτ = 0.0005 therefore r = 0.040816.

Table I shows the analytical (A.S) solutions and approximate solutions from finite difference scheme (F.D), at the points X= 3, 5, 7 and for the times τ = 0.0025, 0.005, 0.015, 0.05 for α-cut=0. Fig. 7 shows the moisture content profiles for different times and Fig. 8 shows the membership function $\Theta(X,\tau)$ at X=3 for finite difference scheme values.

Table I. Analytical – Approximate solution at X=3, 5, 7 and for the times τ=0.0025(30 min), 0.005(1hr), 0.015(3hr), 0.05(10hr)

Crisp												
X	**3**				**5**				**7**			
τ	**F.D**		**A.S**		**F.D**		**A.S**		**F.D**		**A.S**	
0.0025	0.3203		0.3222		0.0531		0.0477		0.0050		0.0030	
0.005	0.4817		0.4839		0.1627		0.1615		0.0384		0.0357	
0.015	0.6854		0.6861		0.4183		0.4189		0.2253		0.2253	
0.05	0.8247		0.8248		0.6578		0.6580		0.5064		0.5066	
Fuzzy												
τ	**F.D**		**A.S**		**F.D**		**A.S**		**F.D**		**A.S**	
	Θ^-	Θ^+	Θ^-	Θ^+	Θ^-	Θ^+	Θ^-	Θ^+	Θ^-	Θ^+	Θ^-	Θ^+
0.0025	0.2722	0.3683	0.2739	0.3705	0.0451	0.0611	0.0406	0.0549	0.0042	0.0057	0.0025	0.0034
0.005	0.4095	0.5540	0.4113	0.5565	0.1383	0.1871	0.1383	0.1857	0.0326	0.0441	0.0304	0.0411
0.015	0.5826	0.7882	0.5832	0.7890	0.3555	0.4810	0.3561	0.4818	0.1951	0.2591	0.1915	0.2591
0.05	0.7010	0.9484	0.7011	0.9485	0.5591	0.7564	0.5593	0.7567	0.4305	0.5824	0.4306	0.5826



(a) (b)

Fig. 7 (a) Analytical – approximate solutions for τ=30min, τ=1hr, (b) Analytical – approximate solutions for τ=1hr, τ=3hr, τ=10hr

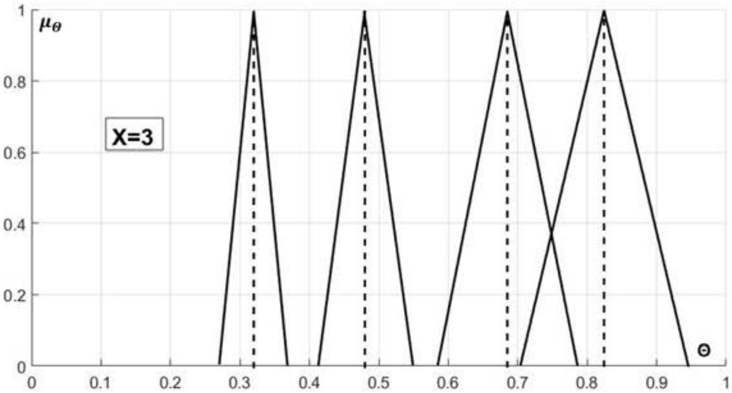


Fig. 8 Membership function of $\Theta(X,\tau)$ at X=3 for finite difference scheme values and for the times τ=0.0025, 0.005, 0.015, 0.05

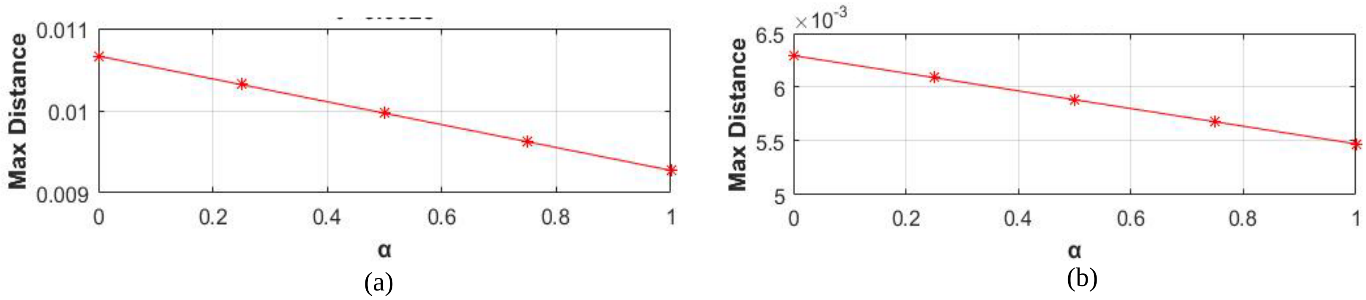


Fig. 9 Transformed Hausdorff distance for α=0, 0.25, 0.5, 0.75, 1 and for the times (a) τ=0.0025, (b) τ=0.005

VII. Conclusion

With the generalized Hukuhara (gH) derivative we can now have reliable results from finite difference schemes in order to solve partial differential equations. According to the degree of differentiability, we found the fuzzy system of equations of implicit Crank-Nicolson scheme. Then the results from numerical model were compared with the analytical solutions. The approaches according to the transformed Hausdorff distance were very good. In engineering it is necessary to handle the fuzziness of the problem. The fuzziness of soil water movement diminishes as flows moves up in horizontal column.

References

[1] L.A. Zadeh, "Fuzzy Sets", Inform. And Control 8, 338-353, 1965.

[2] E. Cox, "*The fuzzy systems handbook, a practitioner's guide to building, using, and maintaining fuzzy systems*", Boston: AP Professional, 1994.

[3] C. Tzimopoulos and B. Papadopoulos, "*Fuzzy logic with application in engineering*", Thessaloniki: ZITI, 2013.

[4] K. Nemati and M. Matinfar, "An implicit method for fuzzy parabolic partial differential equations", *J. Nonlinaer Sci. Appl.*, pp. no. 2, pp. 61-71, 2008.

[5] S. Chang and L. Zadeh, "On fuzzy mapping and control", *IEEE Thrans, System Man Cybernet. no. 2,* 1972, pp. 30-34.

[6] D. Dubois and H. Prade, "Towards fuzzy differential calculus: part 3, differentiation", *Fuzzy Sets and Systems, 8,* 1982pp. 30-34.

[7] M. Puri and D. Ralescu, "Differentials of fuzzy functions", *J. Math. Anal. Appl, 91,* 1983, pp. 321-325.

[8] O. Kaleva, "Fuzzy differential equations", *Fuzzy Sets and Systems, 24,* 1987, pp. 301-307,.

[9] O. Kaleva, "The cauchy problem for fuzzy differential equations", *Fuzzy Sets and Systems, 24,* 1990, pp. 389-396.

[10] S. Seikkala, "On the fuzzy initial value problem", *Fuzzy Sets and Systems, 24,* 1987, pp. 319-330.

[11] B. Bede and S. Gal, "Generalizations of the differentiability of fuzzy number valued functions with applications to fuzzy differential equations", *Fuzzy Sets and Systems, 151,* 2005, pp. 581-599.

[12] L. Stefanini and B. Bede, "Generalized Hukuhara differentiability of interval-valued functions and interval differential equations", *Nonlinear Analysis, 71,* 2009, pp. 1311-1328.

[13] T. Allahviranloo, Z. Gouyandeh, A. Armand και A. Hasanoglu, "On fuzzy solutions for the heat equation based on generalized Hukuhara differentiability", *Fuzzy Sets and Systems, 265,* 2015, pp. 1-23.

[14] C. Tzimopoulos, K. Papadopoulos and C. Evangelides, "Fuzzy analytical solution to horizontal infiltration", to *ICNAAM- International Conference of Numerical Analysis and Applied Mathematics(ICNAAM 2017)*, Thessaloniki.

[15] M. Ma, M. Friedman and K. Kandel, "Numerical solutions of fuzzy differential", *Fuzzy Sets and Systems, 105,* 1999, pp. 133-138.

[16] M. Friedman, M. Ma and A. Kandel, "Numerical solutions of fuzzy differential", *Fuzzy Sets and Systems, 106,* 1999, pp. 35-48.

[17] B. Bede, "A note on two-point boundary value problems associated with nonlinear", *Fuzzy Sets and Systems, 157,* 2006, pp. 986-989.

[18] S. Abbaspandy and T. Allahviranloo, "Numerical solution of fuzzy differential", *Math. Comput. Appl., 7,* 2002, pp. 41-52,.

[19] J. Buckley and T. Feuring, "Introduction to fuzzy partial differential", *Fuzzy Sets and Systems,* 1999, pp. 241-248.

[20] T. Allahviranloo, "Difference methods for fuzzy partial differential equations", *Computational Methods in Applied Mathematics, 2,* 2002, pp. 233-242.

[21] M. Kermani and F. Saburi, "Numerical Method for Fuzzy Partial Differential Equations", Appl. Math. Sc., *1,* 2007, pp. 1299-1309,.

[22] A. Farajzadeh, A. Hossein Pour and M. Amini, "An Explicit Method for Solving Partial Differential Equation", *International Mathematical Forum, 21,* 2010, pp. 1025-1036,.

[23] N. Uthirasamy, "*Studies on numerical solutions of fuzzy boundary value*

problems and fuzzy partial differential equations", Tamil Nadu, India: Department of Mathematics, Periyar University, PhD, 2014, p. 98.

[24] E. Buckingham, "Studies on the movement of soil moisture", U.S Department of Agriculture, Bureau of Soils-*B. 38*, 1907, pp. 61

[25] J. Philip, "Theory of infiltration", *Advan. Hydrosci. 5*, 1987,pp. 215-296.

[26] A. Khastan and J. Nieto, "A boundary value problem for second order fuzzy differential equations", *Nonlinear Anal.*, 72, 2010, pp. 3583-3593

[27] S. P. Modal and T. K Roy, "Second order linear di erential equations with generalized trapezoidal intuitionistic Fuzzy boundary value", *Journal of Linear and Topological Algebra*, vol. 04, no. 02, 2015, pp:115-129

[28] G. Smith, "*Numerical Solution of Partial Differential Equations: finite difference method*", New York: Oxford, 1965, pp. 179.

[29] J. Hadamard, "*Sur les problèmes aux dérivées partielles et leur signification physique*", Princeton University Bulletin, 1902, pp:49-52.