where $t = \frac{(x_p - x_i^c)^T (x_j^c - x_i^c)}{(x_j^c - x_i^c)^T (x_j^c - x_i^c)}$.

The distance between the test image x and the feature line $\overline{x_i^c x_j^c}$ can be calculated by

$$d(x, \overline{x_i^c x_j^c}) = \left\|x - x_p\right\|_2,$$

where $\left\|\cdot\right\|_2$ is the Euclidean distance.

The classification decision called the nearest feature distance can be defined as follow:

$$C_{\mathrm{NFL}} = \arg\min_{c} \left\{ \min_{1 \le i < j \le m} d(x, \overline{x_j^c x_i^c}) \right\}$$

for $c = 1,2,3, \ldots, N_c$ and C_{NFL} is the class of the tested image.

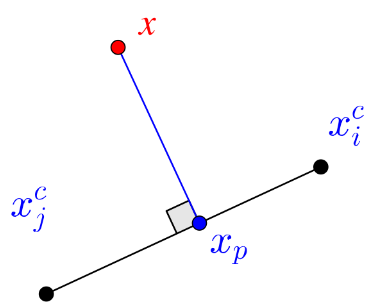

Fig. 2 The nearest feature line

C. *The shortest feature line segment (SFLS)*

This method produces the shortest feature line segment which is classified by the tested image inside a circle, created by two training points x_i^c and x_j^c of the class C_{SFLS} as shown in Fig. 3. The distance metric of SFLS is calculated by

$$d(x, x_i^c x_j^c) = \left\|x_i^c - x_j^c\right\|.$$

The classification decision can be defined as follow:

$$C_{\mathrm{SFLS}} = \arg\min_{c} \left\{ \min_{i,j} d(x, x_i^c x_j^c) \right\},$$

for $c = 1,2,3, \ldots, N_c$, where C_{SFLS} is the class of the tested image.

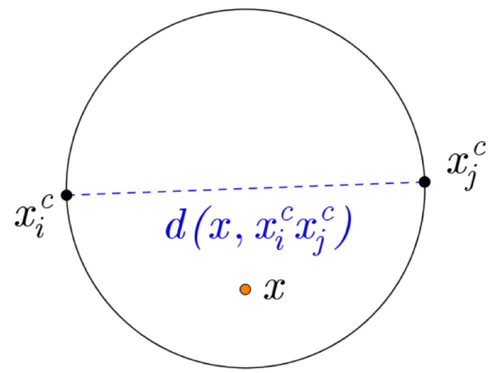

Fig. 3 The metric of the shortest feature line segment

D. *The restricted nearest feature line with ellipse (RNFLE)*

The main idea of this method, which uses ellipse to restrict the feature line. Define x_i^c, x_j^c as foci of any ellipse like Fig. 4, and a_0 as the ratio between the length of ellipse major axis and the length of the center to either focus.

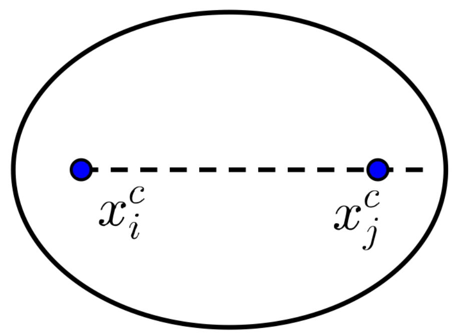

Fig. 4 x_i^c, x_j^c be foci of the ellipse

The classification decision can be defined as follows:

1. If the test (x) is inside the ellipse which is shown in Fig. 5 , the distance between the tested image and the feature line $\overline{z_i^c z_j^c}$ is as follow:

$$\mathrm{d}(\mathrm{x}, \overline{\mathrm{x}_i^c \mathrm{x}_j^c}) = \left\|\mathrm{x} - \mathrm{x}_{ijp}^c\right\|,$$

where x_{ijp}^c represents the projection of x on the feature line $\overline{\mathrm{x}_i^c \mathrm{x}_j^c}$.

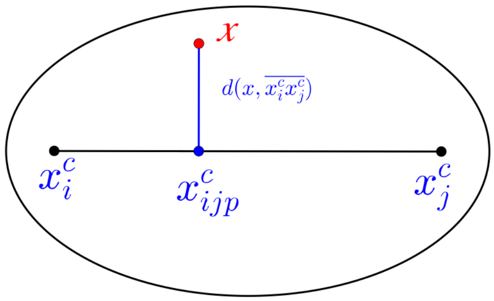

Fig. 5 The tested image inside the ellipse

2. If the test (x) is outside the ellipse which is shown in Fig. 6, the distance between the test image and the feature line is as follow:

$$d(x, \overline{x_i^c x_j^c}) = \min\{||x - x_i^c||, ||x - x_j^c||\},$$

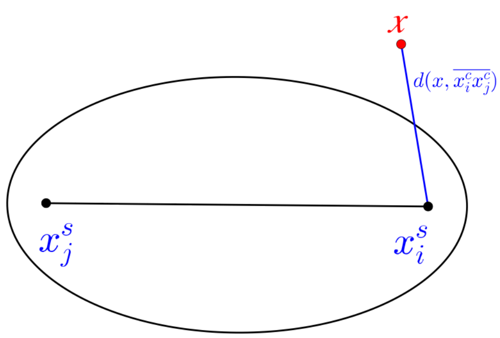

Fig. 6 The tested image is outside the ellipse

The test image is classified into class C_{RNFLE}

$$C_{RNFLE} = \arg\min_{c} \left\{ \min_{1 \le i < j \le m} d(x, x_i^c x_j^c) \right\},$$

where $c = 1,2,3, \ldots, N_c$.

E. The nearest and the center of ellipse (NCE)

This method uses the center of ellipse to find the distance between the tested image and the center.

The ellipse is produced by the three points, z_i^c, z_j^c and z_l^c, which are obtained from the combinations of m distinct points taken from the same class. Without loss of generality, set

$$d(z_i^c, z_j^c) = \operatorname{diam}(\{z_i^c, z_j^c, z_l^c\}),$$

and

$$\operatorname{diam}(A) = \max\{d(u, v) | u, v \in A\}.$$

The center (h, k) is the midpoint of the z_i^c and z_j^c,

$$a_{c_{ijl}} = \frac{1}{2} d(z_i^c, z_j^c)$$

and

$$b_{c_{ijl}} = \sqrt{\frac{(-(x_l - h)\sin\theta + (y_l - k)\cos\theta)^2}{1 - \frac{\left((x_l - h)\cos\theta + (y_j - k)\sin\theta\right)^2}{a_{c_{ijl}}^2}}},$$

where $z_l = (x_l, y_l)$, $d(u, v) = ||u - v||_2$, and $||\quad||_2$ is Euclidean distance.

The classification rule can be defined as follows:
If the tested image (z) is inside an ellipse which is shown in Fig. 7 and $z_c = [h\ k]^T$ is the center of the ellipse, the distance between the tested image and the center of the c_{ijk} ellipse is as follows:

$$d(z, \overline{z_i^c z_j^c z_k^c}) = ||z - z_c||_2.$$

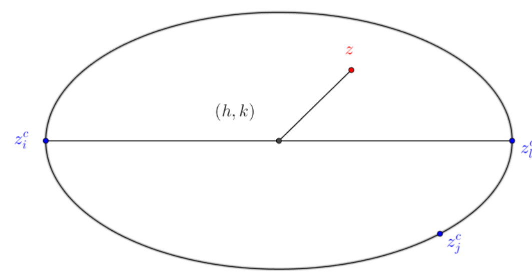

Fig. 7 The tested image inside the ellipse

If the tested image (z) is outside any ellipse which is shown in Fig. 8, or three points lie on the same line, the distance between the tested image and each point in the three points of the ellipse is as follows:

$$d(z, \overline{z_i^c z_j^c z_k^c}) = \min\left\{||z - z_i^c||_2, ||z - z_j^c||_2, ||z - z_k^c||_2\right\}.$$

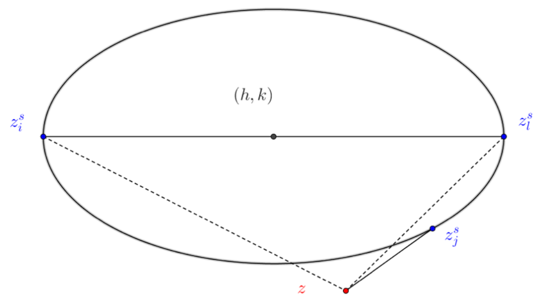

Fig. 8 The tested image outside the ellipse

The tested image is classified into the class C_{NCE},

$$C_{NCE} = \underset{c}{arg\,min}\left\{\underset{1\le i<j<k\le m}{min} d\left(z, \overline{z_i^c z_j^c z_k^c}\right) \mid c = 1,2,3,\ldots,N_c\right\}.$$

III. The Proposed Algorithm

In this section, we improve a classification based on NCE. Firstly, we use the principal component analysis (PCA) [17] to find a subset of the principle component in a set of training faces. Then we project the faces into the principal components space which can be gathered the feature vectors (z_i). Finally, we use the classification rule for face classification as follows:

The center and foci of an ellipse (CFE)

For this algorithm, each class needs at least three points. The z_i^c, z_j^c and z_l^c are the feature vectors from PCA, which are also the three training points from the same class c, $1 \le i < j < l < m, 1 \le c \le N_c$: N_c and m are the number of the class and the number of images per class, respectively.

The propose algorithm or CFE based on PCA can be elucidated as follows:

Step 1: Input the training image set and read the tested image.

Step 2: We use principal component analysis (PCA) to find a subset of the principle component in a set of training faces; then we project faces into the principal components space which can be gathered the feature vectors z_i^c.

Step 3: Produce an ellipse from the three points and the same class. Without loss of generality, set

$$d(z_i^c, z_j^c) = diam(\{z_i^c, z_j^c, z_l^c\}) \tag{1}$$

and

$$diam(A) = \max\{d(u,v) \mid u, v \in A\}. \tag{2}$$

The point (h,k) is the center of the z_i^c and z_j^c,

$$a_{c_{ijl}} = \frac{1}{2} d(z_i^c, z_j^c) \tag{3}$$

and

$$b_{c_{ijl}} = \sqrt{\frac{(-(x_l-h)\sin\theta + (y_l-k)\cos\theta)^2}{1 - \frac{\left((x_l-h)\cos\theta + (y_j-k)\sin\theta\right)^2}{a_{c_{ijl}}^2}}},$$

where $z_l = (x_l, y_l)$, $d(u,v) = ||u - v||_2$, and $||\ \ ||_2$ is Euclidean distance.

The foci of the ellipse are calculated by

$$F_1 = \begin{bmatrix} h \\ k \end{bmatrix} - C \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix},$$

$$F_2 = \begin{bmatrix} h \\ k \end{bmatrix} + C \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \tag{4}$$

where $C = \sqrt{a_{c_{ijl}}^2 - b_{c_{ijl}}^2}$ and $\theta = \arctan\left(\frac{y_i - y_j}{x_i - x_j}\right)$.

Step 4: Let z be the tested image and Z_c be the center of the ellipse. Without loss of generality, let $a_{c_{ijl}} > b_{c_{ijl}}$ and

$$d(z, \text{ellipse } ijl) = d(z, F_1) + d(z, F_2)\,. \tag{5}$$

We define the classification rule as follows:

1). If the tested image is inside the ellipse, that is $d(z, \text{ellipse } ijl) \le 2a_{c_{ijl}}$, (as shown in figure 8), the distance between the tested image and the center of the ellipse is as follows:

$$D(z, ellipse\ ijl) = \frac{d(z, F_1) + d(z, F_2) + d(z, Z_c)}{3}. \tag{6}$$

2). If the tested image is outside the ellipse, that is $d(z, \text{ellipse } ijl) > 2a_{c_{ijl}}$, (as shown in figure 9), or three points lie on the same line, the distance between the tested image and each point in the three points of the ellipse is as follows:

$$D(z, ellipse\ ijl) = \min\{d(z, z_i^c), d(z, z_j^c), d(z, z_l^c)\}. \tag{7}$$

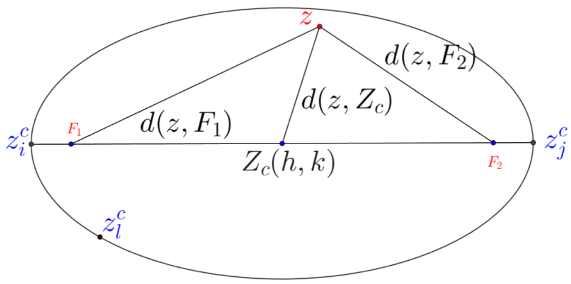

Fig.9 Finding the distance as (6).

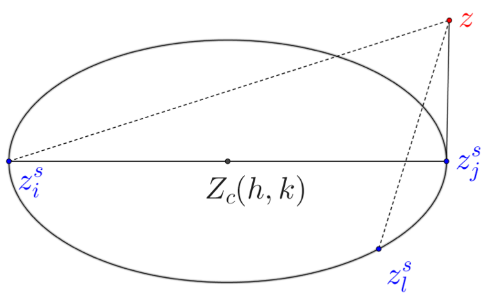

Fig.10 Finding the distance as (7).

The tested image is classified into class C_{CFE}

$$C_{\text{CFE}} = \arg\min_{c} \left\{ \min_{1 \le i < j \le l \le m} D\left(z, ellipse\ ijl\right) \right\}, \tag{8}$$

where C is the class of the training image.

The recognition rate is calculated by

$$\text{The recognition rate} = \frac{N_{cr}}{N_{test}} \times 100\% \tag{9}$$

where N_{cr} the is number of correct recognition of the tested face images and N_{test} is the total number of tested images.

IV. Experimental Results

In this section, we present the experimental results of the SFLS, NCE, and the proposed algorithm (CFE). The databases are used by Faces94 (female and male staff) and Grimace as in [18]. The training image and tested image are transformed as column vector and we use PCA for feature extraction. Each database is divided into two sets, the training set and the set of tested image. The training set is divided into 95%, 90%, 85%, …, 60% proportion and the remaining is the set of tested set. The recognition rate is shown in Table 1, 2, and 3.

Table 1 shows the recognition rate of SFLS, NCE, and CFE. The recognition rate of CFE is better than others except 0.60 proportion, NCE is better than the others.

Table 2 shows the recognition rate of SFLS, NCE, and CFE. The recognition rate of SFLS is better than the others except 0.85,0.75,0.70 and 0.65 proportions, NCE and CFE are better than the SFLS.

Table 3 shows the recognition rate of SFLS, NCE and CFE. The recognition rate of CFE algorithm is better than the others except 0.70 proportion, NCE is better than the others.

Table 1. The recognition rate of Faces94 (female) database with various algorithms

Faces94 (female)	**The recognition rate (%)**		
Proportion	**SFLS**	**NCE**	***CFE***
0.95	94.74	94.74	94.74
0.90	92.11	97.37	97.37
0.85	92.98	94.74	94.74
0.80	94.74	94.74	96.05
0.75	93.68	95.79	95.79
0.70	93.86	94.74	94.74
0.65	93.23	92.48	94.74
0.60	92.76	95.39	94.74

V. Conclusions

In this paper, we introduce an improved algorithm for face recognition system based on PCA by using the center and foci of an ellipse which is produced by three points of the same class, then the tested image is in class C_{CFE} when the average distance between the tested image and each point in the three points of the foci and the center of the ellipse of class C_{CFE} is minimum distance. The proposed method (CFE) tests on the faces94 and Grimace databases. The recognition rate is compared with the shortest feature line segment (SFLS), and the nearest and the center of ellipse (NCE). It is evident that the recognition rate of CFE is better than the SFLS and NCE. Moreover, CFE shows high performance and it has average recognition rate over 96.88%, NCE has average recognition rate over 96.61% and the SFLS has average recognition rate over 96.21%.

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Table 2. The recognition rate of Faces94 (male staff) database with various algorithms

Faces94 (male staff)	**The recognition rate (%)**		
Proportion	**SFLS**	**NCE**	***CFE***
0.95	95.00	100	100
0.90	97.50	97.50	97.50
0.85	98.33	96.67	96.67
0.80	98.75	98.75	98.75
0.75	99.00	97.00	98.00
0.70	95.00	97.50	96.67
0.65	94.29	95.71	95.00
0.60	95.00	97.50	97.50

Table 3. The recognition rate of Grimace database with various algorithms

Grimace	**The recognition rate**		
Proportion	**SFLS**	**NCE**	***CFE***
0.95	94.44	94.44	94.44
0.90	94.44	97.22	97.22
0.85	92.59	94.44	96.3
0.80	98.61	100	100
0.75	96.67	98.89	98.89
0.70	96.3	99.07	98.15
0.65	95.24	96.03	98.41
0.60	95.83	97.92	98.61

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