$$R_{\Gamma,1}(s) = \prod_{q=0}^{4} Z_{P,(\bigwedge^q \bar{\mathfrak{n}})}\left(s+\frac{q}{4}\right)^{(-1)^q}$$
$$= Z_{P,1}(s)\cdot \prod_{q=1}^{4} Z_{P,(\bigwedge^q \bar{\mathfrak{n}})}\left(s+\frac{q}{4}\right)^{(-1)^q}.$$

Hence,

$$\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} = \frac{Z'_{P,1}(s)}{Z_{P,1}(s)} + \sum_{q=1}^{4}(-1)^q \frac{Z'_{P,(\bigwedge^q \bar{\mathfrak{n}})}\left(s+\frac{q}{4}\right)}{Z_{P,(\bigwedge^q \bar{\mathfrak{n}})}\left(s+\frac{q}{4}\right)}.$$

This equation, the equation (4), and the derivation of (5), yield

$$\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} = O\left(t^{J-1+\varepsilon}\right) + \sum_{|t-\gamma_{P,1}|\le 1} \frac{1}{s-\rho_{P,1}} + \sum_{q=1}^{4} O\left(\frac{1}{\frac{q}{4}} t^{J-1+\varepsilon}\right) \quad (6)$$
$$= O\left(t^{J-1+\varepsilon}\right) + \sum_{|t-\gamma_{P,1}|\le 1} \frac{1}{s-\rho_{P,1}}$$

for $s=\sigma^1+\mathrm{i}\,t$, $\frac{1}{2}\le\sigma^1<\frac{1}{4}t-\frac{1}{2}$.

(ii) Reasoning in the same way as in the proof of (a) (i), we obtain that

$$\sum_{|t-\gamma_{P,1}|\le 1} \frac{1}{s-\rho_{P,1}} = O\left(\frac{1}{\eta}t^{D}\right)$$

for $s=\sigma^1+\mathrm{i}\,t$, $\frac{1}{2}+\eta\le\sigma^1<\frac{1}{4}t-\frac{1}{2}$.

Now, (6) implies that

$$\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} = O\left(\frac{1}{\eta}t^{J-1+\varepsilon}\right)$$

for $s=\sigma^1+\mathrm{i}\,t$, $\frac{1}{2}+\eta\le\sigma^1<\frac{1}{4}t-\frac{1}{2}$.

This completes the proof. ■

III. Applications

Note that analogous formulas of the formulas derived in this paper are already very well applied in literature for various settings of locally symmetric spaces.

Such formula [9, p. 102, Th. 6.4.] is applied to obtain a prime geodesic theorem (see, [9, p. 113, Th. 6.19.]) in the case of compact Riemann surfaces.

In the same setting, it is used in the proof of Lemma 4 in [12, p. 213].

In the case of compact locally symmetric spaces of real rank one, such formulas are first derived in [1, p. 314, Th. 4.1.] (even-dimensional case), [7, pp. 177-178, Th. 1.] (odd-dimensional case), and then applied in proofs of the corresponding prime geodesic theorems [6, p. 190, Th. 1.] (even-dimensional case), [8, p. 216, Th. 1.] (odd-dimensional case), respectively.

In [10, p. 99], the author also applied a variant of the formula in the proof of the prime geodesic theorem for d-dimensional real hyperbolic manifolds with cusps.

References

[1] M. Avdispahić and Dž. Gušić, "On the logarithmic derivative of zeta functions for compact even-dimensional locally symmetric spaces of real rank one," *Mathematica Slovaca*, vol. 69, pp. 311–320, 2019.
[2] R. P. Boas, *Entire functions.* New York: Academic Press Inc. Publishers, 1954.
[3] J. B. Conway, *Functions of one complex variable, second edition, vol. 11 of Graduate Texts in Mathematics.* New York: Springer–Verlag, 1978.
[4] A. Deitmar and M. Pavey, "A prime geodesic theorem for SL_4," *Ann. Anal. Glob. Geom.*, vol. 33, pp. 161–205, 2008.
[5] D. Fried, "The zeta functions of Ruelle and Selberg. I," *Ann. Sci. Ec. Norm. Sup.*, vol. 19, pp. 491–517, 1986.
[6] Dž. Gušić, "Prime geodesic theorem for compact even-dimensional locally symmetric Riemannian manifolds of strictly negative sectional curvature," *WSEAS Trans. on Math.*, vol. 17, pp. 188–196, 2018.
[7] Dž. Gušić, "On the Logarithmic Derivative of Zeta Functions for Compact, Odd-dimensional Hyperbolic Spaces," *WSEAS Trans. on Math.*, vol. 18, pp. 176–184, 2019.
[8] Dž. Gušić, "On the Length Spectrum for Compact, Odd-dimensional, Real Hyperbolic Spaces," *WSEAS Trans. on Math.*, vol. 18, pp. 211–222, 2019.
[9] D. Hejhal, *The Selberg trace formula for $PSL(2,\mathbb{R})$. Vol. I. Lecture Notes in Mathematics 548.* Berlin–Heidelberg: Springer–Verlag, 1976.
[10] J. Park, "Ruelle zeta function and prime geodesic theorem for hyperbolic manifolds with cusps," in *Casimir force, Casimir operators and Riemann hypothesis*, Berlin 2010, pp. 89–104.
[11] M. Pavey, "Class Numbers of Orders in Wuartic Fields," Ph.D. dissertation, University of Tubingen, 2006.
[12] B. Randol, "The Riemann hypothesis for Selberg's zeta-function and the asymptotic
behaviour of eigenvalues of the Laplace operator," *Trans. Amer. Soc.*, vol. 236, pp. 209–233, 1978.
[13] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function.* Oxford: Oxford Science Publications, Clarendon Press, 1986.