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One-point spectrum Nordsieck almost collocation methods

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Abstract- A family of multivalue collocation methods for the numerical solution of differential problems is proposed. These methods are developed in order to be suitable for the solution of stiff problems, since they are highly stable and do not suffer from order reduction, as they have uniform order of convergence in the whole integration interval. In addition, they permits to have an efficient implementation, due to the fact that the coefficient matrix of the nonlinear system for the computation of the internal stages has a lower triangular structure with one-point spectrum. The uniform order of convergence is numerically computed in order to experimentally verify theoretical results.

Keywords- Multivalue methods, Collocation, General Linear Methods.

I. Introduction

Consider the following system of Ordinary Differential Equations (ODEs):

$$\begin{cases} y'(t) = f(y(t)), & t \in [t_0, T], \\ y(t_0) = y_0, \end{cases} \tag{1}$$

where $f : \mathbb{R}^k \to \mathbb{R}^k$. In this paper we propose a multivalue collocation method for the solution of (1). Multivalue methods are a generalization of classical methods for the solution of ODEs, such as Runge-Kutta and linear multistep methods, which are special cases of these ones. For details about multivalue methods see [3, 4, 5, 90].

Collocation methods choose a finite-dimensional space of candidate solutions, a number of collocation points, and impose that solution satisfies (1) at the collocation points. Collocation is widely treated in literature. In classical one-step collocation methods, studied by Guillou and Soulé [63] and Wright [89] for Runge-Kutta methods, the collocation function is a polynomial, which exactly interpolates the numerical solution in t_n and satisfies the system

$$P_n'(t_n + c_i h) = f(P_n(t_n + c_i h)), \quad i = 1, 2, ..., m,$$

where $c_1, c_2, .., c_m$ are the collocation points. The solution in the next time step is computed from the function evaluation $y_{n+1} = P_n(t_{n+1})$. Perturbed and discontinuous collocation methods were introduced by Norsett and Wanner [78, 79] in order to extend the collocation principle to a wide range of methods, and not only implicit Runge-Kutta methods.

Multistep collocation methods, presented by Lie and Norsett [74, 75], extend the collocation technique to the family of multistep Runge-Kutta methods. Two-step collocation, introduced by Jackiewicz and Tracogna [71], extends the collocation idea to the class of two-step Runge-Kutta methods, pursuing the aim of deriving highly stable collocation-based methods which do not suffer from order reduction. Almost collocation, treated in [42, 46, 47, 52], relaxes some order conditions in order to improve the balance between order and stability properties.

Multivalue collocation methods, introduced in [58], are able to avoid the order reduction phenomenon, typically arising when collocation based Runge-Kutta methods are applied to stiff systems [65, 72, 84]. As a matter of fact, these methods have uniform order of convergence on the whole integration interval together with high stability properties. Stiffness is very common in mathematical models, for instance in multiscale models, which are very usual in contexts like medicine, population dynamics, chemistry, biology, physiology [67, 77, 85]. Moreover, stiff problems typically arise in the spatial discretization of time-dependent partial differential equations by the method of lines through finite elements or finite differences, see [1, 16, 17, 59] for some applications in physics, continuum mechanics and medicine.

Multivalue collocation methods require the solution of mk simultaneous nonlinear equations at each time step, where k is the dimension of system (1) and m is the number of stages. The coefficient matrix of such system is typically a full matrix.

With the aim of reducing the computational effort, in [21, 22] multivalue almost collocation methods with a lower triangular coefficient matrix have been introduced. A lower triangular matrix allows to solve the equations in m successive stages, with only a k-dimensional system to be solved at each stage.

It is the purpose of this work to construct one-point spectrum multivalue collocation based methods, i.e. methods for which the coefficient matrix is lower triangular and all the elements on the diagonal are equal. This structure allows to further decrease the computational cost because, in solving the nonlinear systems by means of Newton-type iterations, it is possible to use repeatedly the stored LU factorization of the Jacobian. This approach has been expoited in [23] and [52] in the context of two-step almost collocation for ODEs and Volterra Integral Equations, respectively.

Two-step almost collocation methods have been introduced in [42] for the numerical solution of ODEs and are derived from collocation methods by relaxing some interpolation/collocation conditions in order to achieve A-stability together with high uniform order of convergence. Such ideas have been further investigated in [46, 47] for ODEs, in [9, 32, 33] for Volterra integral equations and in [10, 11, 27, 62] for Volterra integro-differential equations. Two-step collocation methods have also been extended to the numerical solution of fractional differential equations [12].

As announced, the paper is focused on the highly stable multivalue extension of the collocation principle. Multivalue methods are commonly represented as General linear methods (GLMs) (see [6, 70] for a general theory, [39, 40, 53] for General Nystrom methods, [41, 55, 81] for second order differential equations), whose linear and nonlinear stability properties have been widely investigated [26, 28, 45, 68]. These methods have also been employed as geometric numerical integrators, as discussed in [7, 38, 43, 44].

The paper is organized as follows. In Section II. a general theory about multivalue collocation methods is presented, recalling some important results on the order of the methods. In Section III. methods with a triangular coefficient matrix with one-point spectrum are constructed and examples of methods with two and three stages are presented. Finally, in Section IV., the order of these methods is experimentally verified.

II. Multivalue collocation methods

Consider the uniform grid $t_n = t_0 + nh$, $n = 0, 1, ..., N$, with $Nh = T - t_0$. Collocation methods approximate the solution of (1) by means of a piecewise collocation polynomial:

$$y(t_n + \theta h) \approx P_n(t_n + \theta h), \quad \theta \in [0, 1],$$

with

$$P_n(t_n + \theta h) = \sum_{i=1}^{r} \alpha_i(\theta) y_i^{[n]} + h \sum_{i=1}^{m} \beta_i(\theta) f(P_n(t_n + c_i h)), \tag{2}$$

where $\{\alpha_i(\theta), \beta_j(\theta), i = 1, ..., r, j = 1, ..., m\}$ are algebraic polynomials of degree less or equal to r. We impose the following interpolation conditions:

$$P_n(t_n) = y_1^{[n]},\ hP_n'(t_n) = y_2^{[n]}, \dots,\ h^{r-1} P_n^{(r-1)}(t_n) = y_{r-1}^{[n]}$$

and collocation conditions

$$P_n'(t_n + c_i h) = f(P_n(t_n + c_i h)), \quad i = 1, 2, .., m.$$

As a consequence, by using the Nordsieck form for the external stages

$$y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix} \approx \begin{bmatrix} y(x_n) \\ hy'(x_n) \\ \vdots \\ h^{r-1} y^{r-1}(x_n) \end{bmatrix}, \tag{3}$$

multivalue collocation methods can be expressed in the GLM form:

$$\begin{aligned} Y_i^{[n]} &= h \sum_{j=1}^{m} a_{ij} f\left(Y_j^{[n]}\right) + \sum_{j=1}^{r} u_{ij} y_j^{[n-1]}, \quad i = 1, 2, ..., m, \\ y_i^{[n]} &= h \sum_{j=1}^{m} b_{ij} f\left(Y_j^{[n]}\right) + \sum_{j=1}^{r} v_{ij} y_j^{[n-1]}, \quad i = 1, 2, ..., r, \end{aligned} \tag{4}$$

$n = 0, ..., N$, where m is the number of internal stages, r is the number of external stages, $\mathbf{c} = [c_1, c_2, ..., c_m]^T$ is the abscissa vector and the coefficient matrices are

$$\begin{aligned} \mathbf{A} &= [\beta_j(c_i)]_{i,j=1,\dots,m} \in \mathbb{R}^{m \times m}, \\ \mathbf{U} &= [\alpha_j(c_i)]_{i=1,\dots,m, j=1,\dots,r} \in \mathbb{R}^{m \times r}, \\ \mathbf{B} &= \left[\beta_j^{(i-1)}(1)\right]_{i=1,\dots,m, j=1,\dots,r} \in \mathbb{R}^{r \times m}, \\ \mathbf{V} &= \left[\alpha_j^{(i-1)}(1)\right]_{i,j=1,\dots,r} \in \mathbb{R}^{r \times r}, \end{aligned}$$

usually collected in the Butcher tableau

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{U} \\ \hline \mathbf{B} & \mathbf{V} \end{array}\right] \tag{5}$$

As observed in [58], the polynomial (2) has globally class C^{r-1} while most interpolants based on Runge-Kutta methods only have global C^1 continuity [60, 61]. Highly continuous interpolants are very useful in many different situations already shown in the existing literature such as scientific visualization [76], functional differential equations with state-dependent delay [66], numerical solution of differential-algebraic equations and nonlinear equations [73, 87, 88], optimal control problems [83], discontinuous initial value problems [86] or, more in general, whenever a smooth dense output is needed [80].

We now recall some important results about the order of convergence of methods (2)-(4) which have been presented in [58].

Theorem 1 *A multivalue collocation method given by $P_n(t_n + \theta h)$ in (2), is an approximation of uniform order p to the solution of problem (1) if and only if the*

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polynomials $\alpha_i(\theta)$ and $\beta_j(\theta)$ in (2) are computed according to the following conditions:

$$\alpha_1(\theta) = 1 \tag{6}$$

$$\frac{\theta^\nu}{\nu!} - \alpha_{\nu+1}(\theta) - \sum_{i=1}^{m} \frac{c_i^{\nu-1}}{(\nu-1)!}\beta_i(\theta) = 0, \quad \nu = 1, ..., r-1, \tag{7}$$

$$\frac{\theta^\nu}{\nu!} - \sum_{i=1}^{m} \frac{c_i^{\nu-1}}{(\nu-1)!}\beta_i(\theta) = 0, \quad \nu = r, ..., p. \tag{8}$$

Corollary 2 *The uniform order of convergence for a multivalue collocation method (2) is $m + r - 1$.*

Theorem 3 *For an A-stable multivalue collocation method (2) the constraint $r \leq m + 1$ must be fulfilled.*

Theorem 4 *Order conditions (6)–(8) are equivalent to the following discrete conditions:*

$$\alpha_j(0) = \delta_{j1}, \quad \alpha_j^{(\nu)}(0) = \delta_{j,\nu+1},$$

with $j = 1, 2, ..., r$, $\nu = 1, 2, ..., r-1$,

$$\beta_j(0) = \beta_j^{(\nu)}(0) = 0,$$

with $j = 1, 2, ..., m$, $\nu = 1, 2, ..., r-1$,

$$\alpha_j'(c_i) = 0, \quad i = 1, 2, ..., r, \quad j = 1, 2, ..., m,$$

and

$$\beta_j'(c_i) = \delta_{ij}, \quad i, j = 1, 2, ..., m,$$

being δ_{ij} the usual Kronecker delta.

By imposing the order conditions (6)-(8), the matrix $\mathbf{A}$, whose elements appear in (4), results to be typically full. Our purpose to derive methods depending on a structured coefficient matrix, requires relaxing some order conditions, as explained in the remainder.

III. Construction of methods

With the aforementioned aim of constructing methods with lower triangular coefficient matrix $\mathbf{A}$ having a one-point spectrum, we follow the lines discussed in [21, 22] and derive multivalue almost collocation methods, by relaxing some of the order conditions (6)-(8). In order to build the matrices of the method, the functions $\{\alpha_i(\theta), \beta_j(\theta)\}, i = 1, ..., r, j = 1, ..., m$ have to be determined. The functions $\beta_j(\theta)$ are written according to Theorem 5.

Theorem 5 *A multivalue collocation method has a lower triangular coefficient matrix $\mathbf{A}$ with a one-point spectrum if and only if the functional basis $\{\beta_j(\theta), j = 1, ..., m\}$ satisfy $\beta_1(c_1) = \beta_2(c_2)$ and $\beta_j(c_i) = 0$ for $i > j$, so:*

$$\beta_j(\theta) = \omega_j(\theta) \prod_{k=1}^{j-1} (\theta - c_k), \quad j = 1, ..., m,$$

where $\omega_j(\theta)$ is a polynomial of degree $r - m + 1$:

$$\omega_j(\theta) = \sum_{k=0}^{r-m+1} \mu_k^{(j)} \theta^k.$$

The parameters $\mu_k^{(j)}$ are taken as degrees of freedom to eventually fulfill some of the conditions (8) and A-stability. The functions $\alpha_j(\theta)$, instead are computed by imposing all the conditions (6)-(7).In the remainder, we always assume $r = m + 1$. In order to study A-stability, we recall that the stability matrix of method (4) is

$$\mathbf{M}(z) = \mathbf{V} + z\mathbf{B}(I - z\mathbf{A})^{-1}\mathbf{U},$$

where I is the identity matrix in $\mathbb{R}^{m\times m}$. The method is A-stable if the roots of the stability function

$$p(\omega, z) = \det(\omega I - \mathbf{M}(z)). \tag{9}$$

are in the unit circle for all $z \in \mathbb{C}$ such that $Re(z) < 0$. By the maximum principle, this happens if the denominator of $p(\omega, z)$ does not have poles in the negative half plane $\mathbb{C}^-$ and if the roots of the $p(\omega, iy)$ are in the unit circle for all $y \in \mathbb{R}$. The last condition can be verified using the Schur criterion [21].

A. A-stable methods with two stages

We present an example of A-stable methods with $m = 2$ and $r = 3$, so $\omega_j(\theta)$ is a polynomial of degree 2. The order of those methods is 3. The collocation polynomial assumes the form

$$\begin{aligned} P_n(t_n + \vartheta h) &= y_1^{[n]} + \alpha_2(\vartheta) y_2^{[n]} + \alpha_3(\vartheta) y_3^{[n]} \\ &\quad + h\left(\beta_1(\vartheta) f(P(t_n + c_1 h)) + \beta_2(\vartheta) f(P(t_n + c_2 h))\right) \end{aligned}$$

and the corresponding Butcher tableau of is given by

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{U} \\ \hline \mathbf{B} & \mathbf{V} \end{array}\right] = \left[\begin{array}{cc|ccc} \beta_1(c_1) & 0 & 1 & \alpha_2(c_1) & \alpha_3(c_1) \\ \beta_2(c_1) & \beta_2(c_2) & 1 & \alpha_2(c_2) & \alpha_3(c_2) \\ \hline \beta_1(1) & \beta_2(1) & 1 & \alpha_2(1) & \alpha_3(1) \\ \beta_1'(1) & \beta_2'(1) & 0 & \alpha_2'(1) & \alpha_3'(1) \\ \beta_1''(1) & \beta_2''(1) & 0 & \alpha_2''(1) & \alpha_3''(1) \end{array}\right].$$

Some values for the free parameters $\mu_k^{(j)}$ have been chosen by imposing the condition (8) for $\nu = r$ and $\beta_1(c_1) = \beta_2(c_2)$

$$\begin{aligned} \mu_1^{(1)} &= -\frac{\mu_0^{(1)}}{c_1} + \frac{c_2^2 \mu_1^{(2)}}{c_1}, \\ \mu_2^{(1)} &= -\frac{1}{3(c_1 - c_2)} - \frac{\mu_0^{(1)}}{c_2^2} - \frac{(c_1 c_2 + c_2^2)\mu_1^{(2)}}{c_1^2}, \\ \mu_3^{(1)} &= -\frac{-2c_1 + c_2}{3c_1^2(c_1 - c_2)} + \frac{\mu_0^{(1)}}{c_1 c_2^2} + \frac{c_2}{c_1^2}\mu_1^{(2)}, \\ \mu_0^{(2)} &= \frac{c_1}{c_2^2}\mu_0^{(1)}, \quad \mu_2^{(2)} = -\frac{c_1}{3(c_1 - c_2)c_2^2} - \frac{c_1}{c_2^4}\mu_0^{(1)} - \frac{\mu_1^{(2)}}{c_2}. \end{aligned}$$

The remaining ones have been chosen by performing the Schur analysis of the stability function (9), with

$\mu_0^{(1)} = 0$ and $\mu_1^{(2)} = 0$, leading to

$$\alpha_2(\vartheta) = \frac{\vartheta^3(c_1^2 + c_1c_2 - c_2^2) - \vartheta^2 c_1^2(c_1 + c_2) + 3\vartheta c_1^2 c_2^2}{3c_1^2 c_2^2},$$

$$\alpha_3(\vartheta) = \frac{2\vartheta^3(c_1 - c_2) + \vartheta^2 c_1(3c_2 - 2c_1)}{6c_1c_2)},$$

$$\beta_1(\vartheta) = \frac{\vartheta^2(\vartheta(2c_1 - c_2) - c_1^2)}{3c_1^2(c_1 - c_2)},$$

$$\beta_2(\vartheta) = \frac{\vartheta^2 c_1(c_1 - \vartheta)}{3c_2^2(c_1 - c_2)}.$$

Figure 1 shows the region of A-stability in the (c_1, c_2)-plane.

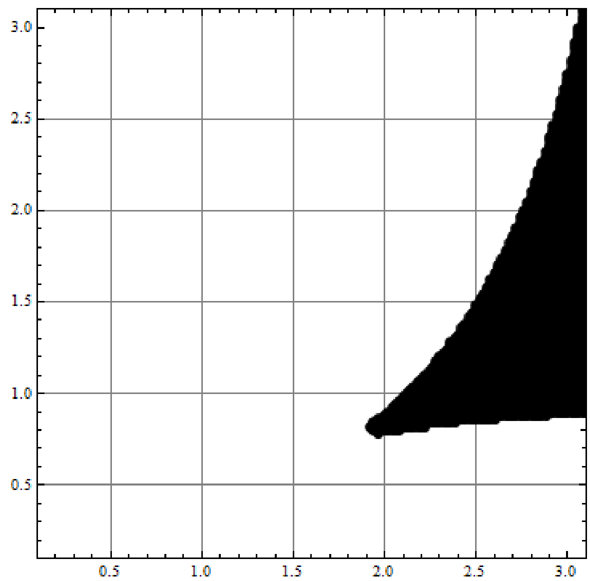

Fig. 1: Region of A-stability in the (c_1, c_2)-plane.

As an example, we choose $c_1 = 22/10$ and $c_2 = 9/10$, obtaining

$$\alpha_2(\vartheta) = \frac{\vartheta\left(15025\vartheta^2 - 37510\vartheta + 29403\right)}{29403},$$

$$\alpha_3(\vartheta) = \frac{\vartheta^2\left(130\vartheta - 187\right)}{594},$$

$$\beta_1(\vartheta) = \frac{5}{4719}\vartheta^2\left(175\vartheta - 242\right),$$

$$\beta_2(\vartheta) = -\frac{440}{3159}\vartheta^2\left(5\vartheta - 11\right),$$

which is the continuous C^2 extension of uniform order $p = 3$ of the A-stable multivalue method:

$$\left[\begin{array}{cc|ccc} \frac{11}{15} & 0 & 1 & \frac{22}{15} & \frac{121}{150} \\ -\frac{351}{4840} & \frac{11}{15} & 1 & \frac{3473}{14520} & -\frac{21}{220} \\ \hline -\frac{335}{4719} & \frac{880}{1053} & 1 & \frac{2306}{9801} & -\frac{19}{198} \\ \frac{205}{4719} & \frac{3080}{3159} & 0 & -\frac{542}{29403} & \frac{8}{297} \\ \frac{2830}{4719} & -\frac{3520}{3159} & 0 & \frac{15130}{29403} & \frac{203}{297} \end{array}\right].$$

B. A-stable methods with three stages

We also show an example of A-stable methods with $m = 3$ and $r = 4$, so $\omega_j(\theta)$ is a polynomial of degree 2. The corresponding methods have order 3. The collocation polynomial is given by

$$\begin{aligned} P_n(t_n + \vartheta h) &= y_1^{[n]} + \alpha_2(\vartheta)y_2^{[n]} + \alpha_3(\vartheta)y_3^{[n]} + \alpha_4(\vartheta)y_4^{[n]} \\ &\quad + h\Big(\beta_1(\vartheta)f(P(t_n + c_1h)) + \beta_2(\vartheta)f(P(t_n + c_2h)) \\ &\quad + \beta_3(\vartheta)f(P(t_n + c_3h))\Big) \end{aligned}$$

and the Butcher tableau (5) of the considered methods depends on the matrices

$$A = \begin{bmatrix} \beta_1(c_1) & 0 & 0 \\ \beta_2(c_1) & \beta_2(c_2) & 0 \\ \beta_3(c_1) & \beta_3(c_2) & \beta_3(c_3) \end{bmatrix}$$

$$B = \begin{bmatrix} \beta_1(1) & \beta_2(1) & \beta_3(1) \\ \beta_1'(1) & \beta_2'(1) & \beta_3'(1) \\ \beta_1''(1) & \beta_2''(1) & \beta_3''(1) \\ \beta_1'''(1) & \beta_2'''(1) & \beta_3'''(1) \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & \alpha_2(c_1) & \alpha_3(c_1) & \alpha_4(c_1) \\ 1 & \alpha_2(c_2) & \alpha_3(c_2) & \alpha_4(c_2) \\ 1 & \alpha_2(c_3) & \alpha_3(c_3) & \alpha_4(c_3) \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & \alpha_2(1) & \alpha_3(1) & \alpha_4(1) \\ 0 & \alpha_2'(1) & \alpha_3'(1) & \alpha_4'(1) \\ 0 & \alpha_2''(1) & \alpha_3''(1) & \alpha_4''(1) \\ 0 & \alpha_2'''(1) & \alpha_3'''(1) & \alpha_4'''(1) \end{bmatrix}.$$

The values of the free parameters $\mu_k^{(j)}$ have been chosen by imposing the condition $\beta_1(c_1) = \beta_2(c_2) = \beta_3(c_3)$ and by performing the Schur analysis of the stability function (9), leading to

$$\alpha_2(\vartheta) = -\frac{\vartheta^4}{4c_1^3} + \frac{\vartheta^3}{(c_1^3+c_2^2c_3^3)}(-c_1^2(c_2-c_3)^2(c_2^2+ c_3^4) - c_1c_2(c_2-c_3)^2(c_2^2+c_3^4) + c_1^3(-c_2^3-c_3^3- c_2c_3^4+c_3^5+c_2^2(c_3+c_3^3)) + c_2^2c_3(c_2^3-c_2^2c_3-2c_3^5+ c_2(c_3^2+c_3^4))) + \frac{\vartheta^2}{4c_1^3c_2^2c_3^3}(4c_1c_2^2(c_1c_2-c_1-c_2+c_3)(c_1^2+c_1c_2+2c_2^2) + (4c_1^4+c_1^2(1-4c_1^2)c_2^2-4c_1(1+c_1^2)c_2^3-4c_2^4)c_3^3+4c_1c_3^4(c_1c_2-c_1c_3-c_2c_3)(c_1^2+c_1c_2+2c_2^2)+4c_3^6(c_1^3+c_1^2c_2+3c_1c_2^2+c_2^3)) + \frac{\vartheta}{c_1^2c_3^3}(2c_1^2c_3^3+c_2(-c_1^2c_2(c_1+c_2)+c_3^3(c_1^3+c_2)- c_3^6)) + \frac{c_1}{c_3^2}(c_2^3-c_3^3),$$

$$\alpha_3(\vartheta) = -\frac{\vartheta^4}{4c_1^2} + \frac{\vartheta^3}{c_1^2c_2c_3^3}(-c_1(c_2-c_3)^2(c_2^2+c_3^4)+ c_1^2(-c_2^3+c_2^2c_3-c_3^3+c_3^5)+c_2c_3(c_2^3-c_2^2c_3-2c_3^5+ c_2(c_3^2+c_3^4))) + \frac{\vartheta^2}{4c_1^2c_2c_3^3}(4c_1^2c_2^3(c_1+2c_2)-4c_1c_2^2 (c_1^2+2c_1c_2+2c_2^2)c_3+4c_1c_2^2(c_1+2c_2)c_3^2+ (4c_1^3+3c_1^2c_2-4c_1c_2^2-4c_2^3)c_3^3+4c_1^2c_2^2c_3^4- 4c_1(c_1+c_2)(c_1+2c_2)c_3^5+4(c_1^2+3c_1c_2+c_2^2)c_3^6)+ \frac{\vartheta}{c_1c_3^2}(c_1c_3^2+c_2c_3^2-c_1c_2(c_1+c_2)+c_1^2c_3^3-c_3^5)+ \frac{c_1c_2}{c_3^2}c_2^2-c_3^2,$$

$$\alpha_4(\vartheta) = -\frac{\vartheta^4}{8c_1} + \frac{\vartheta^3}{6c_1c_3^3}(-3c_1c_2^3+3c_2^2(c_1+c_2)c_3- 3c_2^2c_3^2+(-2c_1+3c_2)c_3^3-3c_1c_2c_3^4+3(2c_1+c_2)c_3^5- 6c_3^6) + \frac{\vartheta^2}{8c_1c_3^3}(4c_2c_3^3(-c_2+c_3^3)-4c_1(c_2-c_3)c_3(2c_2^2- c_2c_3+3c_3^4)+c_1^2(8c_2^3-8c_2^2c_3+5c_3^3+8c_2c_3^4-12c_3^5))- \frac{\vartheta}{2c_3}(c_2(c_2(c_1+c_2)-2c_2c_3-c_1c_3^3+c_3^4))+ \frac{c_1c_2^2}{2c_3}(c_2-c_3),$$

$$\beta_1(\vartheta) = \frac{\vartheta^4}{4c_1^3} + \frac{\vartheta^3}{c_1^3(c_1-c_2)c_3^3}(-c_1c_2^4+c_2^3(c_1+c_2)c_3- c_2^3c_3^2+c_2(-c_1+c_2)c_3^3-c_1c_2^2c_3^4+c_2(c_1+c_2)c_3^5+ (c_1-2c_2)c_3^6) - \frac{\vartheta^2}{4c_1^3(c_1-c_2)c_3^3}(4c_2^2c_3^3(c_2-c_3^3)+ 8c_1c_2(c_2-c_3)c_3(c_2^2+c_3^4)+c_1^3(4c_2^3-4c_2^2c_3+c_3^3+ 4c_2c_3^4-4c_3^5)+c_1^2(-8c_2^4+4c_2^3c_3+8c_3^6+c_2c_3^3 (-5+4c_3^2)+c_2^2(4c_3^2-8c_3^4))) + \frac{c_2\vartheta}{c_1^2}(c_3^3-c_2),$$

$$\beta_2(\vartheta) = \frac{\vartheta^3}{(c_1-c_2)c_2^2c_3^3}(c_1c_2^3-c_2^2(c_1+c_2)c_3+c_2^2c_3^2+ (c_1-c_2)c_3^3+c_1c_2c_3^4-(c_1+c_2)c_3^5+c_3^6)+ \frac{\vartheta^2}{(c_1-c_2)c_2^2c_3^3}(c_1(-c_1c_2^3+c_2^2(c_1+c_2)c_3-c_2^2c_3^2+ (-c_1+c_2)c_3^3-c_1c_2c_3^4+(c_1+c_2)c_3^5-c_3^6))-\vartheta+c_1,$$

$$\beta_3(\vartheta) = -\vartheta^3 + \frac{\vartheta^2}{c_3^3}(c_1c_3^3+c_2c_3^3-c_2^2)+ \frac{c_2\vartheta}{c_3^3}(-c_1c_3^3+c_2(c_1+c_2)) - \frac{c_1c_2^3}{c_3^3}.$$

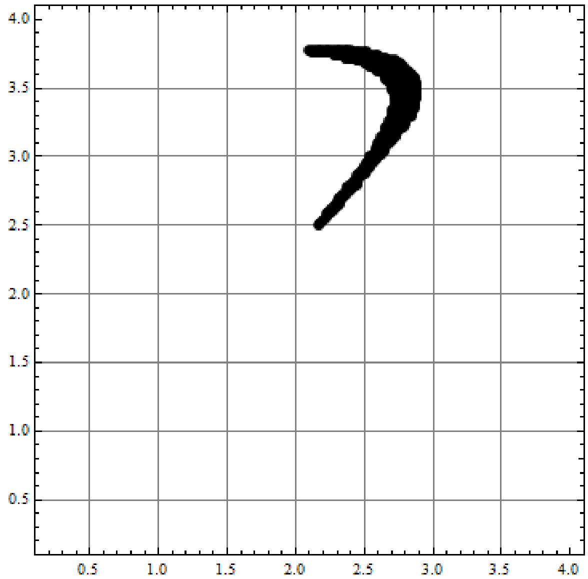

Fig. 2: Region of A-stability in the (c_2, c_3) plane for $c_1 = 4$.

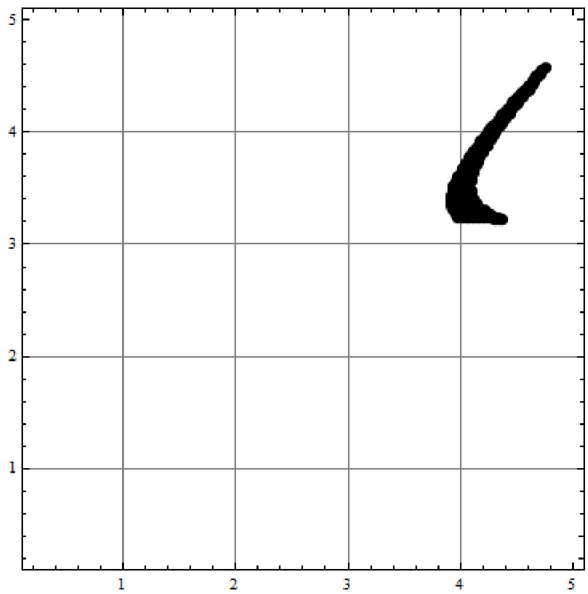

Fig. 3: Region of A-stability in the (c_1, c_3) plane for $c_2 = 28/10$.

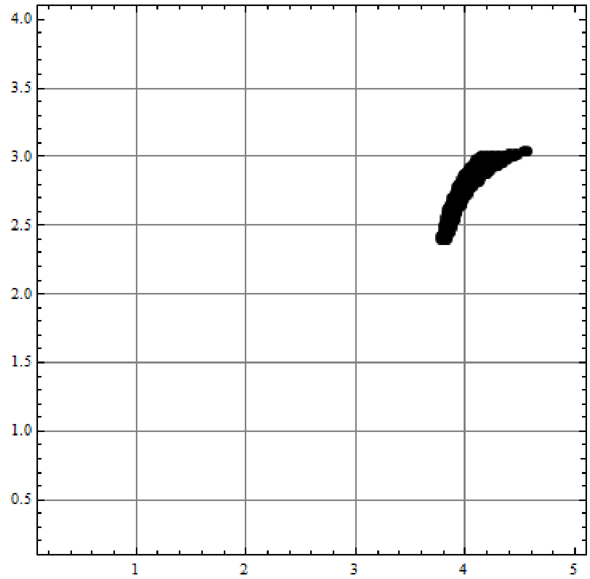

Fig. 4: Region of A-stability in the (c_1, c_2) plane for $c_3 = 35/10$.

Figures 2-4 show regions of A-stability arising from fixing one collocation parameter. As an example, we choose $c_1 = 4$, $c_2 = 28/10$ and $c_3 = 35/10$, obtaining

$$\alpha_2(\vartheta) = -\frac{\vartheta^4}{256} + \frac{131827}{392000}\vartheta^3 - \frac{3492089}{1568000}\vartheta^2 + \frac{276833}{56000}\vartheta - \frac{244}{125},$$

$$\alpha_3(\vartheta) = -\frac{\vartheta^4}{64} + \frac{23347}{28000}\vartheta^3 - \frac{283047}{56000}\vartheta^2 + \frac{19191}{2000}\vartheta - \frac{504}{125},$$

$$\alpha_4(\vartheta) = -\frac{\vartheta^4}{32} + \frac{11321}{12000}\vartheta^3 - \frac{21021}{4000}\vartheta^2 + \frac{8799}{1000}\vartheta - \frac{392}{125},$$

$$\beta_1(\vartheta) = \frac{\vartheta^4}{256} + \frac{4039}{6000}\vartheta^3 - \frac{425077}{96000}\vartheta^2 + \frac{11221}{1600}\vartheta,$$

$$\beta_2(\vartheta) = -\frac{89}{9408}\vartheta^3 - \frac{89}{2352}\vartheta^2 - \vartheta + 4,$$

$$\beta_3(\vartheta) = -\vartheta^3 + \frac{1158}{175}\vartheta^2 - \frac{8712}{875}\vartheta - \frac{256}{125},$$

which is the continuous C^2 extension of uniform order $p = 3$ of the A-stable multivalue method with Butcher tableau depending on

$$A = \begin{bmatrix} \frac{1289}{1000} & 0 & 0 \\ -\frac{60417}{1000000} & \frac{1289}{1000} & 0 \\ -\frac{380093}{1536000} & \frac{857}{1536} & \frac{1289}{1000} \end{bmatrix},$$

$$U = \begin{bmatrix} 1 & \frac{2711}{1000} & \frac{711}{250} & \frac{133}{375} \\ 1 & \frac{1571417}{1000000} & \frac{138117}{250000} & -\frac{341579}{375000} \\ 1 & \frac{973063}{512000} & \frac{133259}{128000} & -\frac{183701}{192000} \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{52197}{16000} & \frac{9497}{3136} & -\frac{5589}{875} \\ \frac{9239}{48000} & -\frac{8963}{9408} & \frac{243}{875} \\ -\frac{45791}{9600} & \frac{89}{4704} & \frac{1266}{175} \\ \frac{16531}{4000} & -\frac{89}{1568} & -6 \end{bmatrix},$$

$$V = \begin{bmatrix} 1 & \frac{859841}{784000} & \frac{9291}{7000} & \frac{15839}{12000} \\ 0 & \frac{232457}{156800} & \frac{26959}{14000} & \frac{159}{160} \\ 0 & -\frac{389383}{156800} & -\frac{29643}{5600} & -\frac{209}{40} \\ 0 & \frac{188553}{98000} & \frac{64791}{14000} & \frac{9821}{2000} \end{bmatrix}.$$

IV. Numerical evidence

In this section we experimentally compute the order of the two methods in Section III.A and III.B, in order to confirm the theoretical result. We consider the Prothero-Robinson problem [6, 64]

$$\begin{cases} y'(t) = \lambda(y(t) - \sin(t)) + \cos(t), & t \in [0, 10], \\ y(0) = 0, \end{cases} \tag{10}$$

with $Re(\lambda) < 0$ which is stiff when $\lambda \ll 0$. We compare the results of the aforementioned methods with those obtained by the two-stage Gaussian Runge-Kutta method

$$\begin{array}{c|cc} \frac{1}{2} - \frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{2} + \frac{\sqrt{3}}{6} & \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \tag{11}$$

which has order 4 and uniform order 2, therefore it suffers from order reduction when applied to a stiff problem.

Tables 1-3 show the error in the final step point for different values of the step size and the experimental order of the methods for different values of λ in (10). The order is computed according to the formula:

$$p = \frac{cd(h) - cd(2h)}{log_{10}(2)},$$

where $cd = -log_{10}||err||_\infty$, with $err = y(t_{end}) - y_{end}$.

Table 1: Absolute errors (in the final step point) and effective orders of convergence of the method in Section III.A applied to (10).

h	$\lambda = -10^3$		$\lambda = -10^6$	
	Error	p	Error	p
1/10	$4.9008\ 10^{-5}$		$4.1930\ 10^{-6}$	
1/20	$3.0606\ 10^{-6}$	4.0011	$2.6733\ 10^{-7}$	3.9713
1/40	$1.9182\ 10^{-7}$	3.9960	$1.7166\ 10^{-8}$	3.9610
1/80	$1.2089\ 10^{-8}$	3.9880	$1.1240\ 10^{-9}$	3.9328

Table 2: Absolute errors (in the final step point) and effective orders of convergence of the method in Section III.B applied to (10).

h	$\lambda = -10^3$		$\lambda = -10^6$	
	Error	p	Error	p
1/10	$3.2132\ 10^{-5}$		$3.1531\ 10^{-5}$	
1/20	$1.7551\ 10^{-6}$	4.1944	$1.6645\ 10^{-6}$	4.2436
1/40	$1.0647\ 10^{-7}$	4.0430	$9.4344\ 10^{-8}$	4.1410
1/80	$7.1312\ 10^{-9}$	3.9002	$5.5944\ 10^{-9}$	4.0759

Table 3: Absolute errors (in the final step point) and effective orders of convergence of (11) applied to (10).

h	$\lambda = -10^3$		$\lambda = -10^6$	
	Error	p	Error	p
1/10	$1.77\ 10^{-4}$		$1.52\ 10^{-4}$	
1/20	$1.32\ 10^{-5}$	3.75	$3.84\ 10^{-5}$	1.98
1/40	$7.82\ 10^{-7}$	4.08	$9.99\ 10^{-6}$	1.94
1/80	$4.78\ 10^{-8}$	4.03	$2.78\ 10^{-6}$	1.85

We can observe that the experimental order is consistent with the theoretical one and, even in the case of stiffness, the methods derived in Sections III.A and III.B do not suffer from order reduction, which is evident in Gaussian Runge-Kutta method. In fact the observed order of Gassian Runge-Kutta method is 2 when $\lambda = -10^6$, (Table 3). This is due to the fact that for such value of the parameter λ, the problem (10) is stiff. Instead methods derived in this paper mantain the uniform order (Tables 1-2).

So, developed methods have high stability properties, together with uniform order of convergence and a structured coefficient matrix which permits a reduction of the computational effort.

V. Conclusions

In this paper we have derived diagonally implicit multivalue almost collocation methods with one-point spectrum coefficient matrix for the numerical solution of differential problems. Like in [58], these methods do not suffer from order reduction since they have uniform order of convergence on the whole integration interval, but they permit to reduce the computational effort. Clearly, requesting a structured coefficient matrix, requires relaxing some order conditions, so it slightly reduces the uniform order of convergence. We have provided examples of A-stable methods with two and three stages having order 3 and experimentally verified their orders. Future work will address the application of the same technique to other types of problems, such as partial differential equations (e.g. advection-diffusion problems [13, 48] or reaction-diffusion problems [50, 56, 57]), integral equations [8, 14, 34], fractional equations [2, 12, 15, 20, 24, 36, 37], stochastic differential equations [18, 19, 30, 49], and as an alternative to exponential fitting for oscillatory problems [25, 29, 31, 35, 51, 54, 69, 82].

VI. Aknowledgmets

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