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On wavelet decomposition of the singular splines

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Abstract—One of the approaches to the problem of approximating functions with a singularity is the creation of an approximating apparatus based on splines with the same feature. For the wavelet decomposition of spline spaces it is important that the property of the embedding of these spaces is associated with embedding grids. The purpose of this paper is to consider ways of constructing spaces of splines with a predefined singularity and obtain their wavelet decomposition. Here the concept of generalized smoothness is used, within which the mentioned singularity is generalized smooth. This approach leads to the construction of a system of embedded spaces on embedded grids. A spline-wavelet decomposition of mentioned spaces is presented. Reconstruction formulas are done.

Keywords— generalized smoothness, reconstruction formulas, singular splines, spline-wavelet decomposition

I. INTRODUCTION

In practice, we often deal with functions which have singular points (i.e. with the discontinuation of functions or their derivatives in certain points).

Two approaches exist for an approximation of such functions. The first of them is the extraction (additive or multiplicative) of singularities from the discussed function, and the approximation of the function rest. The second one is an introduction of the mentioned singularities in the approximation apparatus (see [1],[4], [11], [22]). The second approach is usually simpler because it is required less than the priori information about the function in question (it is not required to know the exact function characteristics, its asymptotic behavior, orders of the defining multipliers, etc.)
In [11] the singular splines determined by the kernel of the differential operator in special boundary value problems were used for the collocation method. In the work [22] the singular splines are constructed with an implicit representation of manifold on which coefficient of model elliptic problem are discontinuous. As a result a meshless method of optimal order with the computational advantages of the B-spline calculus was obtained.

The general approach to the construction of splines and finite element approximations is to use approximation relations (in the case of a cubic grid, these relations are called the Strang-Mikhlin relations, see [2] - [3]). The use of the above relations allows one to obtain spline approximations that are asymptotically optimal with respect to the N-width of standard compact sets. To construct singular splines, it is sufficient to include the singularity in the right-hand sides of the approximation relations.

It is very important to have the sequence of embedded spaces for the successive approximation of the mentioned functions. It is also very important for the spline-wavelet theory (see [4],[6],[12]). With regard to splines, it is most convenient to obtain the system of embedded spaces on embedded grids. Simple examples show that the use of the approximative relations does not guarantee the mentioned property.

An in-depth study of the cases when a system of embedded spaces is obtained has shown that in these cases the splines have maximum smoothness (or, which is the same, the minimum defect). It turned out that when considering singular splines, one can use the generalized smoothness in order to achieve the embedding of spaces on the embedded grids. The next step is to construct a wavelet decomposition of the spaces of singular splines.

The notion of generalized continuity was introduced in [20]. In the aforementioned work, necessary and sufficient conditions for the pseudo-continuity of coordinate splines and their derivatives are obtained. Further development of this idea led to the study of the generalized smoothness of spline spaces of the Lagrangian and Hermitian types (see [21], [24] - [25], [27]), as well as method spaces in finite elements (see [23], [26]). In this paper, this idea is applied to the construction of spaces of singular splines and to their wavelet decomposition.

Additional possibilities arise in the wavelet expansion. In this case, in addition to the compressed flow (called in this case the main flow), a refinement (so-called wavelet) flow is also formed. The wavelet flow has a large volume. It is stored at the flow (sender) and can be issued to the receiver in whole or in part, on demand. The wavelet flow, together with the main flow, allows the receiver to reconstruct the exact original flow. This is the value of the wavelet decomposition. However, with the classical approach, the construction of an adaptive wavelet decomposition is not possible.

Let us dwell in more detail on the wavelet expansion. In the classical sense, the wavelet decomposition is usually defined by two functions: the scaling function and a function called the parent wavelet. The rest of the functions, and function spaces considered in the classical approach, are obtained by the similarity transformation and integer shifts of the two mentioned functions. A number of requirements are imposed on these functions, the main of which is the scale ratio that the scaling function must satisfy. The implementation of the multi-scale ratio leads to the possibility of constructing a chain of embedded function spaces. Note right away that the aforementioned chain of embedded spaces is the foundation for obtaining the wavelet decomposition. The main difficulty lies in finding a scaling function that satisfies a number of additional properties (approximation property, property of rapid decay at infinity, etc.), as well as in finding the parent wavelet.

In papers [11], [15] the non-classical approach is based on the construction of a system of embedded spaces, and abandoning the idea of using only two functions (scaling function and parent wavelet). In this case, the multiple-scale ratio is replaced by calibration ratios. These ratios provide a representation of the coordinate splines associated with the embedded grid through a linear combination of coordinate splines associated with the original fine grid. The coordinate splines themselves are obtained from the approximation relations mentioned above. The result of using these relations is a system of embedded spline spaces associated with a system of embedded grids. Projection a spline space into an embedded space gives a wavelet decomposition.

The purpose of this paper is to consider ways of constructing spaces of splines with a prescribed singularity and obtain their wavelet decomposition. Here the concept of generalized smoothness is used, within which the mentioned singularity is generalized smooth. This approach leads to the construction of a system of embedded spaces on embedded grids. The sequence of embedded spaces is necessary for approximating singular functions, for implementing the finite element method in the degenerate problems, for constructing multi-grid methods, and also for the wavelet analysis of the singular numerical flows.

In this paper the approximation relations are considered to determine coordinate splines with a predefined singularity. The concept of generalized smoothness allows us to consider functions with singularity as generalized smooth functions. The linear shells of the singular coordinate splines are spaces with the property of embedding on embedding grids. We discuss the spline-wavelet decomposition of the mentioned spaces. The decomposition and reconstruction formulas are done.

II. INITIAL NOTATION AND AUXILIARY STATEMENTS

Let Z be the set of all integers, and R^1 be the set of all real numbers, and R^3 be the linear space of all three-dimensional vectors with real components. In the future, all vectors are represented as column vectors. The dot $\cdot$ denotes a scalar product of vectors (for example, $\mathbf{a}\cdot\mathbf{b}$ is the scalar product of vectors **a** and **b**), the sign × between the vectors means the vector product of them (for example, $\mathbf{a}\times\mathbf{b}$ is the vector product of vectors **a** and **b**). A square matrix of the third order with columns $\mathbf{a},\mathbf{b},\mathbf{c}\in R^3$ will be denoted by $(\mathbf{a},\mathbf{b},\mathbf{c})$, and its determinant will be denoted by the symbol $\det(\mathbf{a},\mathbf{b},\mathbf{c})$. On the real axis R^1, we introduce a grid

$$X : \ldots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \ldots, \qquad (1)$$

and put

$$\lim_{j\to-\infty} x_j = a, \qquad \lim_{j\to+\infty} x_j = b. \qquad (2)$$

Introduce notation $M = \cup_{j\in Z}(x_j,x_{j+1})$, $S_j = [x_{j-1},x_{j+2}]$, $J_k = \{k-1,k,k+1\}$, $k,j\in Z$. Let $U(x_i,x_{i+1})$ be the linear space of functions defined on the interval (x_i,x_{i+1}). By definition put $U = \ldots\otimes U(x_{-1},x_0)\otimes U(x_0,x_1)\otimes U(x_1,x_2)\otimes\ldots$.

Let (t) be a three-component vector function with the component belonging to the space U.

Discuss a complete chain of vectors $\{\mathbf{b}_j\}_{j\in Z}$ in R^3, i.e. a sequence of the vectors $\mathbf{b}_j$ that

$$\det(\mathbf{b}_{j-1},\mathbf{b}_j,\mathbf{b}_{j+1}) \neq 0 \qquad \forall j\in Z. \qquad (3)$$

Consider the functions $\Omega_j(t)$ defined on M, which is equal to zero on the set $M\backslash S_j$ and satisfy the approximate ratios

$$\sum_{j\in Z} \mathbf{b}_j\,\Omega_j \equiv \varphi(t) \qquad \forall t\in M, \qquad \operatorname{supp}\Omega_j\subset S_j \qquad (4)$$

Taking into account relations (1)–(4) by the Cramer's formulas we have

$$\Omega_j(t)\equiv 0 \qquad \forall t\in M\backslash S_j, \quad \forall j\in Z, \qquad (5)$$

$$\Omega_j(t) = \frac{\det(\mathbf{b}_{j-2},\mathbf{b}_{j-1},\varphi(t))}{\det(\mathbf{b}_{j-2},\mathbf{b}_{j-1},\mathbf{b}_j)} \quad \text{for} \quad t\in(x_{j-1},x_j), \qquad (6)$$

$$\Omega_j(t) = \frac{\det(\mathbf{b}_{j-1},\varphi(t),\mathbf{b}_{j+1})}{\det(\mathbf{b}_{j-1},\mathbf{b}_j,\mathbf{b}_{j+1})} \quad \text{for} \quad t\in(x_j,x_{j+1}), \qquad (7)$$

$$\Omega_j(t) = \frac{\det(\varphi(t),\mathbf{b}_{j+1},\mathbf{b}_{j+2})}{\det(\mathbf{b}_j,\mathbf{b}_{j+1},\mathbf{b}_{j+2})} \quad \text{for } t\in(x_{j+1},x_{j+2}). \qquad (8)$$

The functions $\Omega_j(t)$ are called *coordinate B_φ-splines*, and the vector function $\varphi(t)$ is a *generating function.*

Let $F^{s,-}_{i+1}$ and $F^{s,+}_{i+1}$, $s = 0,1$, be linear functionals in the space $U(x_i,x_{i+1})$ $\forall i\in Z$.

Consider the next condition.

(A) Ratio $F^{s,-}_j\varphi = F^{s,+}_j\varphi$ $\quad\forall j\in Z$, $s=0,1$, are right.

The application of a functional to a vector function means applying it to each component, so we have the numerical vector.

If the condition (A) is true, we say that *the condition of the generalized smoothness of the first order is fulfilled*. In this case we use the notation

$$\mathbf{F}_j^s = F_j^{s,-}\varphi = F_j^{s,+}\varphi, \qquad s=0,1. \tag{9}$$

In what follows we define

$$\mathbf{b}_j = \mathbf{N}_j \times \mathbf{N}_{j+1}, \tag{10}$$

where

$$\mathbf{N}_j = \mathbf{F}_j^0 \times \mathbf{F}_j^1. \tag{11}$$

In the next section, it is proved that if the chain $\{\mathbf{N}_j\}_{j\in Z}$ is complete then the chain $\{\mathbf{b}_j\}_{j\in Z}$ is also complete. To construct the complete chain $\{\mathbf{N}_j\}_{j\in Z}$ we use a parametric family of functionals and its derivative with respect to the parameter.

III. SOME ALGEBRAIC IDENTITIES

Then it will be convenient to use the next two famous formula for a double vector product:

$$(\mathbf{x} \times \mathbf{y}) \times (\mathbf{z} \times \mathbf{w}) = \det(\mathbf{x},\mathbf{z},\mathbf{w})\mathbf{y} - \det(\mathbf{y},\mathbf{z},\mathbf{w})\mathbf{x}, \tag{12}$$

$$(\mathbf{x}\times\mathbf{y})\times(\mathbf{z}\times\mathbf{w}) = \det(\mathbf{x},\mathbf{y},\mathbf{w})\mathbf{z} - \det(\mathbf{x},\mathbf{y},\mathbf{z})\mathbf{w} \ \forall \mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w} \in \mathrm{R}^3. \tag{13}$$

In this section, a series of simple provable lemmas will be stated. By (12)–(13) we get proof of the next assertions.

Lemma 1. *For* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f} \in \mathrm{R}^3$ *the next identity is true*

$$\det(\mathbf{a} \times \mathbf{b}, \mathbf{c} \times \mathbf{d}, \mathbf{e} \times \mathbf{f})$$
$$= \det(\mathbf{a},\mathbf{c},\mathbf{d})\det(\mathbf{b},\mathbf{e},\mathbf{f}) - \det(\mathbf{b},\mathbf{c},\mathbf{d})\det(\mathbf{a},\mathbf{e},\mathbf{f}). \tag{14}$$

Proof. In identity $\det(\mathbf{a} \times \mathbf{b}, \mathbf{c} \times \mathbf{d}, \mathbf{e} \times \mathbf{f}) = [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})] \cdot (\mathbf{e} \times \mathbf{f})$ we put $\mathbf{x} = \mathbf{a}$, $\mathbf{y} = \mathbf{b}$, $\mathbf{x} = \mathbf{c}$, $\mathbf{w} = \mathbf{d}$, and use (12). This proves formula (14).

Corollary 1. *For arbitrary vectors* $\mathbf{a}, \mathbf{b}, \mathbf{d}, \mathbf{f} \in \mathrm{R}^3$ *the next identity is right*

$$\det(\mathbf{a} \times \mathbf{b}, \mathbf{a} \times \mathbf{d}, \mathbf{a} \times \mathbf{f}) = 0. \tag{15}$$

Proof. If we put $\mathbf{a} = \mathbf{c} = \mathbf{e}$ in identity (14) then we obtain relation (15).

Lemma 2. *For arbitrary vectors* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathrm{R}^3$ *we have the relation* $\det(\mathbf{A} \times \mathbf{B}, \mathbf{B} \times \mathbf{C}, \mathbf{C} \times \mathbf{D}) = \det(\mathbf{A}, \mathbf{B}, \mathbf{C})\det(\mathbf{B}, \mathbf{C}, \mathbf{D})$.

Proof. In (14) we discuss $\mathbf{a} = \mathbf{A}$, $\mathbf{b} = \mathbf{c} = \mathbf{B}$, $\mathbf{d} = \mathbf{e} = \mathbf{C}$, $\mathbf{f} = \mathbf{D}$. This completes the proof.

Lemma 3. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{c}$ *be vectors from the space* R^3. *Then* $\det(\mathbf{A} \times \mathbf{B}, \mathbf{B} \times \mathbf{C}, \mathbf{c}) = \det(\mathbf{A},\mathbf{B},\mathbf{C})(\mathbf{B} \cdot \mathbf{c})$.

Proof. The proof follows from (12), if we take $\mathbf{x} = \mathbf{A}$, $\mathbf{y} = \mathbf{z} = \mathbf{B}$, $\mathbf{w} = \mathbf{C}$.

Corollary 2. *If* $\mathbf{B} \cdot \mathbf{c} = 0$, *then* $\det(\mathbf{A}\times\mathbf{B}, \mathbf{B}\times\mathbf{C}, \mathbf{c}) = 0$.

Lemma 4. *Let* $\mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{c}$ *be vectors from the space* R^3. *If* $\mathbf{C} \cdot \mathbf{c} = 0$ *then the identity* $\det(\mathbf{B} \times \mathbf{C}, \mathbf{c}, \mathbf{D} \times \mathbf{E}) = \det(\mathbf{C},\mathbf{D},\mathbf{E})(\mathbf{B} \cdot \mathbf{c})$ *holds.*

Proof. Analogously to the discussion in the proof of Lemma 1 we have

$$\det(\mathbf{D} \times \mathbf{E}, \mathbf{B} \times \mathbf{C}, \mathbf{c}) = [(\mathbf{D} \times \mathbf{E}) \times (\mathbf{B} \times \mathbf{C})] \cdot \mathbf{c}.$$

Consider formula (13) for $\mathbf{x} = \mathbf{D}$, $\mathbf{y} = \mathbf{E}$, $\mathbf{z} = \mathbf{B}$, $\mathbf{w} = \mathbf{C}$. Taking into account $\mathbf{C} \cdot \mathbf{c} = 0$ we complete the proof.

By Lemma 2 and Lemma 3 we get the next statement.

Corollary 3. *If* $\mathbf{B} \cdot \mathbf{c} = 0$ *and* $\mathbf{C} \cdot \mathbf{c} = 0$, *then*

$$\det(\mathbf{B} \times \mathbf{C}, \mathbf{c}, \mathbf{D} \times \mathbf{E}) = 0.$$

Lemma 5. *Let* $\mathbf{B}, \mathbf{c}, \mathbf{c}', \mathbf{D}$ *be arbitrary vectors from the space* R^3. *If* $\mathbf{C} = \mathbf{c} \times \mathbf{c}'$ *then the formula*

$$\det\begin{pmatrix} \mathbf{B}\cdot\mathbf{c}' & \mathbf{B}\cdot\mathbf{c} \\ \mathbf{D}\cdot\mathbf{c}' & \mathbf{D}\cdot\mathbf{c} \end{pmatrix} = \det(\mathbf{B},\mathbf{C},\mathbf{D})$$

is correct.

Proof. In Lagrangue's identity $\det\begin{pmatrix} \mathbf{x}\cdot\mathbf{z} & \mathbf{x}\cdot\mathbf{w} \\ \mathbf{y}\cdot\mathbf{z} & \mathbf{y}\cdot\mathbf{w} \end{pmatrix} = (\mathbf{x}\times\mathbf{y})\cdot(\mathbf{z}\times\mathbf{w})$ we put $\mathbf{x} = \mathbf{B}$, $\mathbf{y} = \mathbf{D}$, $\mathbf{z} = \mathbf{c}'$, $\mathbf{w} = \mathbf{c}$, and take into account the equality $\mathbf{C} = \mathbf{c}\times\mathbf{c}'$, $(\mathbf{B}\times\mathbf{D})\cdot(\mathbf{c}'\times\mathbf{c}) = \det(\mathbf{B},\mathbf{C},\mathbf{D})$. This completes the proof.

Suppose that

$$\mathbf{b}_j = \mathbf{N}_j \times \mathbf{N}_{j+1}. \tag{16}$$

Lemma 6. *If each of the next sets of three vectors* $(\mathbf{N}_{j-1}, \mathbf{N}_j, \mathbf{N}_{j+1})$, $(\mathbf{N}_j, \mathbf{N}_{j+1}, \mathbf{N}_{j+2})$ *is linear independent, then the set of three vectors* $(\mathbf{b}_{j-1}, \mathbf{b}_j, \mathbf{b}_{j+1})$ *is also linear independent.*

Proof. Taking into account formulas (10), (16), and using Lemma 2 for $\mathbf{A} = \mathbf{N}_{j-1}$, $\mathbf{B} = \mathbf{N}_j$, $\mathbf{C} = \mathbf{N}_{j+1}$, $\mathbf{D} = \mathbf{N}_{j+2}$, we obtain the linear independence of the set of three vectors $\mathbf{b}_{j-1}$, $\mathbf{b}_j$, $\mathbf{b}_{j+1}$.

Theorem 1. *If the vector chain* $\{\mathbf{N}_j\}_{j\in Z}$ *is complete then the vector chain* $\{\mathbf{b}_j\}_{j\in Z}$ *is also complete.*

Proof. The proof follows from the definition of completeness of the vector chain by Lemma 6.

IV. PARAMETERIZED FAMILIES

Let $F^-(\tau)$ and $F^+(\tau)$, $\tau \in (a,b)$, be linear nonzero functionals in the space U with the next property

$$\mathrm{supp}F^-(x_i) \subset (x_{i-1},x_i], \ \mathrm{supp}F^+(x_i) \subset [x_i,x_{i+1}).$$

Let U_0 be a linear space of functions u, $u \in U$, with property

$$F^-(t)u = F^+(t)u \ \forall t \in (a,b). \tag{17}$$

Consider the next condition.

(B) *The components of the vector function* $\varphi(\mathrm{t})$ *belong to the space* U_0.

If the condition (B) is right then we use the notation

$$\mathbf{\Phi}(t) = F^-(t)\varphi = F^+(t)\varphi \qquad \forall t \in (a,b). \tag{18}$$

If $\mathbf{\Phi} \in C^1$ we use the notion $\mathbf{\Phi}_j = \mathbf{\Phi}(x_j)$, $\mathbf{\Phi}'_j = \mathbf{\Phi}'(x_j)$,

$$\mathbf{N}(t) = \mathbf{\Phi}(t)\times\mathbf{\Phi}'(t), \qquad \mathbf{N}_j = \mathbf{\Phi}_j \times \mathbf{\Phi}'_j. \tag{19}$$

Expressions (17)–(19) discussed for $t \in (a,b)$ are named by parameterized families.

In what follows that $\mathbf{\Phi} \in C^2(M)$ and ξ is fixed in set M. The certain integer i exists so that $\xi \in (x_i, x_{i+1})$. We discuss a value $\varepsilon > 0$ for which $x_i + \varepsilon < \xi < x_{i+1} - \varepsilon$. Consider vector function $\mathbf{\Phi}_h(t) = h^{-1}\int_{t-h/2}^{t+h/2}\mathbf{\Phi}(\tau)d\tau$, where $h \in (0, \varepsilon)$, $t \in (x_i + \varepsilon, x_{i+1} - \varepsilon)$. Suppose that

$$[c,d] \subset (x_i + \varepsilon, x_{i+1} - \varepsilon). \tag{20}$$

It is evident that $\mathbf{\Phi}_h \in C^3[c,d]$.

By definition, put

$$\mathbf{N}_{(h)}(t) = \mathbf{\Phi}_h(\mathrm{t})\times\mathbf{\Phi}'_h(\mathrm{t}). \tag{21}$$

It is clear that $\lim_{h\to+0}\mathbf{N}_{(h)}(t) = \mathbf{N}(t)$ and the derivatives $\mathbf{N}'_{(h)}(t)$, and $\mathbf{N}''_{(h)}(t)$ exist for all $t \in [c,d]$.

Theorem 2. *If* $\mathbf{\Phi} \in C^2(M)$*, and the vector function* $\mathbf{N}_{(h)}(t)$ *is defined by formula (21) then identity*

$$\lim_{h\to+0} \det(\mathbf{N}_{(h)}, \mathbf{N}'_{(h)}, \mathbf{N}''_{(h)})(t) = \det(\mathbf{\Phi}\times\mathbf{\Phi}', \mathbf{\Phi}\times\mathbf{\Phi}'', \mathbf{\Phi}'\times\mathbf{\Phi}'')(t) \quad (22)$$

is true.

Proof. By relations $\mathbf{N}'_{(h)} = \mathbf{\Phi}'_h\times\mathbf{\Phi}'_h + \mathbf{\Phi}_h\times\mathbf{\Phi}''_h = \mathbf{\Phi}_h\times\mathbf{\Phi}''_h$, $\mathbf{N}''_{(h)} = \mathbf{\Phi}'_h\times\mathbf{\Phi}''_h + \mathbf{\Phi}_h\times\mathbf{\Phi}'''_h$, we have

$$\det(\mathbf{N}_{(h)}, \mathbf{N}'_{(h)}, \mathbf{N}''_{(h)}) = \det(\mathbf{\Phi}_h\times\mathbf{\Phi}'_h, \mathbf{\Phi}_h\times\mathbf{\Phi}''_h, \mathbf{\Phi}'_h\times\mathbf{\Phi}''_h + \mathbf{\Phi}_h\times\mathbf{\Phi}'''_h). \quad (23)$$

Using (15) with $\mathbf{a} = \mathbf{\Phi}_h$, $\mathbf{b} = \mathbf{\Phi}'_h$, $\mathbf{d} = \mathbf{\Phi}''_h$, $\mathbf{f} = \mathbf{\Phi}'''_h$, we obtain

$$\det(\mathbf{\Phi}_h\times\mathbf{\Phi}'_h, \mathbf{\Phi}_h\times\mathbf{\Phi}''_h, \mathbf{\Phi}_h\times\mathbf{\Phi}'''_h) = 0.$$

Therefore the relation (23) can be represented in the form

$$\det(\mathbf{N}_{(h)}, \mathbf{N}'_{(h)}, \mathbf{N}''_{(h)}) = \det(\mathbf{\Phi}_h\times\mathbf{\Phi}'_h, \mathbf{\Phi}_h\times\mathbf{\Phi}''_h, \mathbf{\Phi}'_h\times\mathbf{\Phi}''_h). \quad (24)$$

Let h tend to zero in identity (24). Passage to the limit gives the relation (22). This completes the proof.

Theorem 3. *If* $\mathbf{\Phi} \in C^2(M)$*, and a vector function* $\mathbf{N}_{(h)}(t)$ *is defined by formula (21) then the identity*

$$\lim_{h\to+0} \det(\mathbf{N}_{(h)}, \mathbf{N}'_{(h)}, \mathbf{N}''_{(h)})(t) = [\det(\mathbf{\Phi},\mathbf{\Phi}',\mathbf{\Phi}'')(t)]^2 \quad (25)$$

is true.

Proof. By relation (14) for $\mathbf{a} = \mathbf{c} = \mathbf{\Phi}$, $\mathbf{b} = \mathbf{e} = \mathbf{\Phi}'$, $\mathbf{d} = \mathbf{f} = \mathbf{\Phi}''$, we have

$$\det(\mathbf{\Phi}\times\mathbf{\Phi}', \mathbf{\Phi}\times\mathbf{\Phi}'', \mathbf{\Phi}'\times\mathbf{\Phi}'')(t) = [\det(\mathbf{\Phi},\mathbf{\Phi}',\mathbf{\Phi}'')(t)]^2. \quad (26)$$

It remains to apply formula (22). This completes the proof.

Theorem 4. *Let* $\mathbf{\Phi}(t)$ *be vector function defined on the segment* $[c,d]$ *and* $\mathbf{\Phi} \in C^2[c,d]$*. Suppose the inequality*

$$|\det(\mathbf{\Phi}(t),\mathbf{\Phi}'(t),\mathbf{\Phi}''(t))| \geq \alpha \qquad \forall t \in [c,d] \quad (27)$$

is fulfilled, where $\alpha > 0$*. If* τ *and* η *are positive numbers, and* $t, t+\tau+\eta \in [c,d]$*,then the relation*

$$\det(\mathbf{N}(t),\mathbf{N}(t+\tau),\mathbf{N}(t+\tau+\eta)) \geq \alpha^2\tau\eta(\tau+\eta)/2 \quad (28)$$

is true.

Proof. Firstly, we use the vector function $\mathbf{N}_{(h)}$ defined by formula (21). By well known properties of determinant we have $\det(\mathbf{N}_{(h)}(t),\mathbf{N}_{(h)}(t+\tau),\mathbf{N}_{(h)}(t+\tau+\eta)) = \det(\mathbf{N}_{(h)}(t),\mathbf{N}_{(h)}(t+\tau)-\mathbf{N}_{(h)}(t),\mathbf{N}_{(h)}(t+\tau+\eta)-\mathbf{N}_{(h)}(t+\tau))$. By Newton's formula we get

$$\det(\mathbf{N}_{(h)}(t),\mathbf{N}_{(h)}(t+\tau),\mathbf{N}_{(h)}(t+\tau+\eta)) = \int_0^\tau d\xi \int_\tau^{\tau+\eta} \det(\mathbf{N}_{(h)}(t), \mathbf{N}'_{(h)}(t+\xi),\mathbf{N}'_{(h)}(t+\eta))d\zeta,$$

Again by Newton's formula we have $\mathbf{N}'_{(h)}(t+\zeta) - \mathbf{N}'_{(h)}(t+\xi) = \int_\xi^\zeta \mathbf{N}''_{(h)}(t+\sigma)d\sigma$. Now we obtain

$$\det(\mathbf{N}_{(h)}(t),\mathbf{N}_{(h)}(t+\tau),\mathbf{N}_{(h)}(t+\tau+\eta)) = \int_0^\tau d\xi \int_\tau^{\tau+\eta} d\zeta \int_\xi^\zeta \det(\mathbf{N}_{(h)}(t), \mathbf{N}'_{(h)}(t+\xi),\mathbf{N}''_{(h)}(t+\sigma)d\sigma. \quad (29)$$

Let ε' be a positive value, $\varepsilon' < \alpha^2$. According to (25) and (27) a value $h_0 = h_0(\varepsilon') > 0$ exists such that $\lim_{h\to+0} h_0(\varepsilon') = 0$. For $h \in (0,h_0)$ we have

$$\det(\mathbf{N}_{(h)}(t),\mathbf{N}'_{(h)}(t),\mathbf{N}''_{(h)}(t)) \geq \alpha^2 - \varepsilon'. \quad (30)$$

Taking into account the positivity of the integral kern in (29), we obtain

$$\det(\mathbf{N}_{(h)}(t),\mathbf{N}_{(h)}(t+\tau),\mathbf{N}_{(h)}(t+\tau+\eta)) \geq (\alpha^2 - \varepsilon')\int_0^\tau d\xi \int_\tau^{\tau+\eta} d\zeta \int_\xi^\zeta d\sigma.$$

Let us give formula (28) the integration and passing of the limit under condition $\varepsilon' \to +0$. This completes the proof.

Remark 1. *We would like to emphasize that formulas (22), (25) and (26) are valid for all* $t \in M$ *because we can discuss the segment* $[c,d]$ *(see (20)).*

If $u \in U_0$, we write shortly $F_j^{\pm}u$ instead of F_j^-u and F_j^+u. If $u \in U_0$ and $F^{\pm}(t)\, u$ differentiable function then we write $(F^{\pm})'(t)u$ instead of $\frac{d}{dt}F^{\pm}u$ and $(F^{\pm})'_j u$ instead of $\frac{d}{dt}F^{\pm}(x_j)\, u$. If $F^-(t) = F^+(t)\varphi$ is a differentiable vector function then we write $\mathbf{\Phi}^{(s)}(t)$ instead of $(F^{\pm}(t)\varphi)^{(s)}$. If the discussed function has s derivatives on the interval (a,b), we use the next notations,

$F_j^- = F^-(x_j)$, $F_j^+ = F^+(x_j)$, $\mathbf{N}_j = \mathbf{N}(x_j)$, $\mathbf{\Phi}_j^{(s)} = \mathbf{\Phi}^{(s)}(x_j)$.

Corollary 4. *Suppose* $c = x_{j-1}$, $d = x_{j+2}$, $\mathbf{\Phi} \in C^2[c,d]$ *and condition (27) is fulfilled. Then*

$$\det(\mathbf{N}(x_{j-1}),\mathbf{N}(x_j),\mathbf{N}(x_{j+1})) \geq \alpha^2(x_{j+1}-x_j)(x_j-x_{j-1})(x_{j+1}-x_{j-1})/2. \quad (31)$$

Proof. Implying formula (28), we put $t = x_{j-1}$, $\tau = x_j - x_{j-1}$, $\eta = x_{j+1} - x_j$. As a result we obtain relation (31). This completes the proof.

Corollary 5. *If* $\mathbf{\Phi} \in C^2(a,b)$ *and condition*

$$|\det(\mathbf{\Phi}(t),\mathbf{\Phi}'(t),\mathbf{\Phi}''(t))| > 0 \ \forall t \in (a,b) \quad (32)$$

is fulfilled. Then the vector chain $\{\mathbf{N}_j\}_{j\in Z}$ *is complete.*

Proof. In the discussed case, condition (32) is right for arbitrary $j \in Z$, i.e. the triple $\{\mathbf{N}_{j-1},\mathbf{N}_j,\mathbf{N}_{j+1}\}$ is linear independent for all $j \in Z$. Thus the vector chain $\{\mathbf{N}_j\}_{j\in Z}$ is complete. This concludes the proof.

Consider the next condition.

(D) Functionals $F^-(t)$ and $F^+(t)$ are differentiable (as abstract functions).

Under condition (D) we put

$$F_{i+1}^{0,-}u = F_{i+1}^-u, \qquad F_i^{0,+}u = F_i^+u \qquad \forall i \in Z, \quad (33)$$

$$F_{i+1}^{1,-}u = (F^-)'_{i+1}u, \quad F_i^{1,+}u = (F^+)'_i u \quad \forall i \in Z. \quad (34)$$

Theorem 5. *Suppose* $\varphi \in U_0$*, and* $\mathbf{\Phi}(\tau) = F^{\pm}(\tau)\varphi$ *is a twice continuously differentiable function on the interval* (a,b)*, i.e.* $\mathbf{\Phi} \in C^2(a,b)$*. Then*

1) the functionals (33)–(34) satisfy ~~to~~ the condition (A)*, and the vectors* $\mathbf{N}_j$, $\mathbf{b}_j$, $\mathbf{F}^s{}_j = \mathbf{\Phi}_j^{(s)}$, $s = 0,1$*, are defined,*

2) if besides the relation

$$\det(\mathbf{\Phi}(t),\mathbf{\Phi}'(t),\mathbf{\Phi}''(t)) \neq 0 \ \forall t \in (a,b), \quad (35)$$

is fulfilled, then the vector chain $\{\mathbf{N}_j\}_{j\in Z}$ *is complete.*

Proof. The first part of the proved assertion follows from formulas (9)–(11), (33) and (34). Therefore we prove the second part. We discuss the vector function $\mathbf{N}(t)=\mathbf{\Phi}(t)\times\mathbf{\Phi}'(t)$. By formula (35) according to Theorem 3 we see that the Wronskian $\det(\mathbf{N},\mathbf{N}',\mathbf{N}'')(t)$ is not zero on the interval (a,b). Using its continuity we have a number $\alpha > 0$ such that the condition (27) is fulfilled for all $t \in (a,b)$. Thus by Theorem 4, the inequality (28) is right. Therefore

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$\det(\mathbf{N}(x_{j-1}),\mathbf{N}(x_j),\mathbf{N}(x_{j+1}))\geq\alpha(x_{j+1}-x_j)(x_j-x_{j-1})(x_{j+1}-x_{j-1})/2\ \forall j\in Z$. Using the relations $\mathbf{N}_j=\mathbf{N}(x_j)$, we obtain the completeness of vector chain $\{\mathbf{N}_j\}_{j\in Z}$. This concludes the proof.

Corollary 6. *If the conditions of Theorem 5 are right then vector chain* $\{\mathbf{b}_j\}$ *defined by formulas* $\mathbf{b}_j=\mathbf{N}_j\times\mathbf{N}_{j+1}\ \forall j\in Z$ $\mathbf{N}_i=\mathbf{\Phi}(x_i)\times\mathbf{\Phi}'(x_i)\ \forall i\in Z$, *is complete, and coordinate splines* $\Omega_j(t)$ *exist.*

Proof. The proof of this assertion follows from Theorem 3 and Theorem 5.

V. EXAMPLE OF PARAMETRIZED FAMILY

Consider a vector function $\varphi(t)=(1,t,1/t)^T$. Let $\{F^{\pm}(t)\}$ be a family of linear functionals, $F^{\pm}(t)u=tu(t)$. By definition we put $\mathbf{\Phi}(t)=F^{\pm}(t)\varphi=(t,t^2,1)^T,\mathbf{N}(t)=\mathbf{\Phi}(t)\times\mathbf{\Phi}'(t)$. We have $\mathbf{N}(t)=(-2t,1,t^2)$. Consider a determinant

$\Delta_0=det(\mathbf{N}(x),\mathbf{N}(y),\mathbf{N}(z))=2(z-x)(z-y)(y-x)$.

If x,y,z are different numbers then triple $\{\mathbf{N}(x),\mathbf{N}(y),\mathbf{N}(z)\}$ is a linear independent system of vectors. By formula (10) $\mathbf{b}_j=\mathbf{b}(x_j,x_{j+1}),where\ \mathbf{b}(x,y)=\mathbf{N}(x)\times\mathbf{N}(y)$. We have

$$\mathbf{b}(x,y)=2(y-x)((x+y)/2,xy,1)^T.$$

Let us introduce the expression

$\Delta=det(\mathbf{b}(x,y),\mathbf{b}(y,z),\mathbf{b}(z,v))\quad for\ x,y,z,v\in R^1$.

Simple transformations give

$\Delta=4(y-x)(z-y)(v-z)(z-x)(v-y)(z-y)$.

This shows the linear independence of vectors $\mathbf{b}(x,y)$, $\mathbf{b}(y,z)$, $\mathbf{b}(z,v)$, if the points x,y,z,v are different. Consider a determinant $\Delta_1=det(\mathbf{b}(x,y),\mathbf{b}(y,z),\varphi(t))$. It is clear to see $that$ $\Delta_1=2(y-x)(z-y)(z-x)(y-t)^2/t$. Thus we have

$$\Delta_1/\Delta=\frac{(y-t)^2/t}{2(v-z)(v-y)(z-y)}.$$

The last formula gives the right part of relation (6), if we put $x=x_{j-2},y=x_{j-1},z=x_j,v=x_{j+1}$. Thus we have

$$\Omega_j(t)=\frac{(x_{j-1}-t)^2/t}{2(x_{j+1}-x_j)(x_{j+1}-x_{j-1})(x_j-x_{j-1})}\quad \text{for}\quad t\in(x_{j-1},x_j).$$

Similarly, the spline $\Omega_j(t)$ can be found on the intervals (x_j,x_{j+1}) and (x_{j+1},x_{j+2}).

VI. BIORTHOGONAL SYSTEM

Later in this section we discuss a biorthogonal system of functionals to the coordinate splines. We suppose that j is fixed, and what follows is that afterwards, we try to exclude it off the notation. By definition put

$$\mathbf{A}=N_{j-2},\qquad \mathbf{B}=N_{j-1},\qquad \mathbf{C}=\mathbf{N}_j,\quad (36)$$

$$\mathbf{D}=N_{j+1},\qquad \mathbf{B}=N_{j+2},\qquad \mathbf{F}=\mathbf{N}_{j+3}.\quad (37)$$

By (10) and (36) -- (37) we have $\mathbf{b}_{j-2}=\mathbf{A}\times\mathbf{B},\mathbf{b}_{j-1}=\mathbf{B}\times\mathbf{C},\mathbf{b}_j=\mathbf{C}\times\mathbf{D},\mathbf{b}_{j+1}=\mathbf{D}\times\mathbf{E},\ \mathbf{b}_{j+2}=\mathbf{E}\times\mathbf{F}.$

According to formulas (6) -- (8) and (36) – (37) we obtain

$$\Omega_j(t)=\frac{det(\mathbf{A}\times\mathbf{B},\mathbf{B}\times\mathbf{C},\varphi(t))}{det(\mathbf{A}\times\mathbf{B},\mathbf{B}\times\mathbf{C},\mathbf{C}\times\mathbf{D})}\ \ for\ \ t\in(x_{j-1},x_j),\qquad (38)$$

$$\Omega_j(t)=\frac{det(\mathbf{B}\times\mathbf{C},\varphi(t),\mathbf{D}\times\mathbf{E})}{det(\mathbf{B}\times\mathbf{C},\mathbf{C}\times\mathbf{D},\mathbf{D}\times\mathbf{E})}\ \ for\ \ t\in(x_j,x_{j+1}),\qquad (39)$$

$$\Omega_j(t)=\frac{det(\varphi(t),\mathbf{D}\times\mathbf{E},\mathbf{E}\times\mathbf{F})}{det(\mathbf{C}\times\mathbf{D},\mathbf{D}\times\mathbf{E},\mathbf{E}\times\mathbf{F})}\ \ for\ \ t\in(x_{j+1},x_{j+2}).\qquad (40)$$

Theorem 6. The next assertions are right.

1. *For* $t\in(x_{j-1},x_j)$ *the* B_φ *−spline* $\Omega_j(t)$ *can be represented in the form*
   $\Omega_j(t)=\mathbf{B}\cdot\varphi(t)[\det(\mathbf{B},\mathbf{C},\mathbf{D})]^{-1}$, (41)
2. *For* $t\in(x_j,x_{j+1})$ *the* B_φ *−spline* $\Omega_j(t)$ *can be represented in the form*
   $\Omega_j(t)=\mathbf{B}\cdot\varphi(t)[\det(\mathbf{B},\mathbf{C},\mathbf{D})]^{-1}-\det(\mathbf{B},\mathbf{D},E)\mathbf{B}\cdot\varphi(t)[\det(\mathbf{B},\mathbf{C},\mathbf{D})]^{-1}[\det(\mathbf{C},\mathbf{D},\mathbf{E})]^{-1}$, (42)

   and it can also be written by the formula

   $\Omega_j(t)=\mathbf{E}\cdot\varphi(t)[\det(\mathbf{B},\mathbf{C},\mathbf{D})]^{-1}-\det(\mathbf{B},\mathbf{D},\mathbf{E})\mathbf{B}\cdot\varphi(t)[\det(\mathbf{B},\mathbf{C},\mathbf{D})]^{-1}[\det(\mathbf{C},\mathbf{D},\mathbf{E})]^{-1}$, (43)
3. *For* $t\in(x_{j-1},x_j)$ *the* B_φ *−spline* $\Omega_j(t)$ *can be represented in the form*
   $\Omega_j(t)=\mathbf{E}\cdot\varphi(t)[\det(\mathbf{C},\mathbf{D},\mathbf{E})]^{-1}$. (44)

   Here notations (36) – (37) are used.

Proof. Taking into account formulas (6) – (8), (16) and (36) – (37), we use Lemma 3. We have $det(\mathbf{A}\times\mathbf{B},\mathbf{B}\times\mathbf{C},\varphi(t))=det(\mathbf{A},\mathbf{B},\mathbf{C})(\mathbf{B}\cdot\varphi(t)),det(\varphi(t),\mathbf{D}\times\mathbf{E},\mathbf{E}\times\mathbf{F})=det(\mathbf{D},\mathbf{E},\mathbf{F})(\mathbf{E}\cdot\varphi(t))$. By (12) – (13) we convert the numerator in formula (39). We obtain $det(\mathbf{B}\times\mathbf{C},\varphi(t),\mathbf{D}\times\mathbf{E})=det(\mathbf{C},\mathbf{D},\mathbf{E})\mathbf{B}\cdot\varphi(t)-det(\mathbf{B},\mathbf{D},\mathbf{E})\mathbf{C}\cdot\varphi(t),det(\mathbf{B}\times\mathbf{C},\varphi(t),\mathbf{D}\times\mathbf{E})=det(\mathbf{B},\mathbf{C},\mathbf{D})\mathbf{E}\cdot\varphi(t)-det(\mathbf{B},\mathbf{C},\mathbf{E})\mathbf{D}\cdot\varphi(t).The\ denominators\ of\ relations$ $(38)-(40)$ are converted by Lemma 2 such that we deduce $det(\mathbf{A}\times\mathbf{B},\mathbf{B}\times\mathbf{C},\mathbf{C}\times\mathbf{D})=det(\mathbf{A},\mathbf{B},\mathbf{C})det(\mathbf{B},\mathbf{C},\mathbf{D}),det(\mathbf{B}\times\mathbf{C},\mathbf{C}\times\mathbf{D},\mathbf{D}\times\mathbf{E})=det(\mathbf{B},\mathbf{C},\mathbf{D})det(\mathbf{C},\mathbf{D},\mathbf{E}),det(\mathbf{C}\times\mathbf{D},\mathbf{D}\times\mathbf{E},\mathbf{E}\times\mathbf{F})=det(\mathbf{C},\mathbf{D},\mathbf{E})det(\mathbf{D},\mathbf{E},\mathbf{F})$. Finally we replace the numerators and the denominators in formulas (38) – (40) according to the mentioned transformations. As a result we have relations (42) – (44). This completes the proof.

By definition put

$$\mathbf{a}=\mathbf{\Phi}_{j-2},\mathbf{a}'=\mathbf{\Phi}'_{j-2},\mathbf{b}=\mathbf{\Phi}_{j-1},\mathbf{b}'=\mathbf{\Phi}'_{j-1},\qquad (45)$$

$$\mathbf{c}=\mathbf{\Phi}_j,\mathbf{c}'=\mathbf{\Phi}'_j,\mathbf{d}=\mathbf{\Phi}_{j+1},\mathbf{d}'=\mathbf{\Phi}'_{j+1},\qquad (46)$$

$$\mathbf{c}=\mathbf{\Phi}_{j+2},\mathbf{c}'=\mathbf{\Phi}'_{j+2},\mathbf{d}=\mathbf{\Phi}_{j+3},\mathbf{f}'=\mathbf{\Phi}'_{j+3}.\qquad (47)$$

Lemma 7. *The next formulas are right,*

$$F^{\pm}_{j-1}\Omega_j=\frac{d}{dt}F^{\pm}_{j-1}\Omega_j=0,\qquad (48)$$

$$F_j^{\pm}\Omega_j = [\det(\mathbf{B},\mathbf{C},\mathbf{D})]^{-1}\mathbf{B}\cdot\mathbf{c} \quad (49)$$

$$\frac{d}{dt}F_j^{\pm}\Omega_j = [\det(\mathbf{B},\mathbf{C},\mathbf{D})]^{-1}\mathbf{B}\cdot\mathbf{c}' \quad (50)$$

$$F_{j+1}^{\pm}\Omega_j = [\det(\mathbf{C},\mathbf{D},\mathbf{E})]^{-1}\mathbf{E}\cdot\mathbf{d} \quad (51)$$

$$\frac{d}{dt}F_{j+1}^{\pm}\Omega_j = [\det(\mathbf{C},\mathbf{D},\mathbf{E})]^{-1}\mathbf{E}\cdot\mathbf{d}' \quad (52)$$

$$F_{j+2}^{\pm}\Omega_j = \frac{d}{dt}F_{j+2}^{\pm}\Omega_j = 0, \quad (53)$$

Theorem 7. *The linear functionals* $\{G_j\}_{j\in Z}$ *defined by formula*

$$G_j(u) = \mathbf{D}\cdot(\mathbf{c}\frac{d}{dt}F_j^{\pm}u - \mathbf{c}'F_j^{\pm}u), \quad (54)$$

have the properties

$$G_j(\Omega_i) = \delta_{i,j} \qquad \forall i,j\epsilon Z. \quad (55)$$

Proof. If $j \le j'-1$ or $j'+2 \le j$ then the point x_j is the inner point of the support of the function $\Omega_{j'}$. Therefore the function and its derivative are equal to zero in this point. It follows that the functional (54) on the function $\Omega_{j'}$ equals zero. Thus for the proof of formula (55) it is sufficient to discuss the cases $j = j'$ and $j = j'+1$. By formulas (49) -- (50) and (54) for $u = \Omega_j$ we obtain

$$G_j(\Omega_j) = \mathbf{D}\cdot(\mathbf{c}\frac{d}{dt}F_j^{\pm}\Omega_j - \mathbf{c}'F_j^{\pm}\Omega_j) =$$

$$\mathbf{D}\cdot(\mathbf{cB}\cdot\mathbf{c}' - \mathbf{c}'\mathbf{B}\cdot\mathbf{c})[\det(\mathbf{B},\mathbf{C},\mathbf{D})]^{-1}. \quad (56)$$

Taking into account Lemma 5 and equality $\mathbf{C} = \mathbf{c}\times\mathbf{c}'$ we see that $\mathbf{D}\cdot(\mathbf{cB}\cdot\mathbf{c}' - \mathbf{c}'\mathbf{B}\cdot\mathbf{c}) = det(B,C,D)$. By (118) we obtain $G_j(\Omega_j) = 1$. *If* $u = \Omega_{j-1}$ we have

$$G_j(\Omega_j) = \mathbf{D}\cdot(\mathbf{c}\frac{d}{dt}F_j^{\pm}\Omega_j - \mathbf{c}'F_j^{\pm}\Omega_j). \quad (57)$$

By formulas (51) and (52) we deduce

$$F_j^{\pm}\Omega_{j-1} = [\det(\mathbf{B},\mathbf{C},\mathbf{D})]^{-1}\mathbf{D}\cdot\mathbf{c}, \quad (58)$$

$$\frac{d}{dt}F_j^{\pm}\Omega_{j-1} = [\det(\mathbf{B},\mathbf{C},\mathbf{D})]^{-1}\mathbf{D}\cdot\mathbf{c}'. \quad (59)$$

Using (58) and (59) in the relation (119) we obtain

$$G_j(\Omega_{j-1}) = \mathbf{D}\cdot(\mathbf{c}\frac{d}{dt}F_j^{\pm}\Omega_{j-1} - \mathbf{c}'F_j^{\pm}\Omega_{j-1}) =$$

$$\mathbf{D}\cdot(\mathbf{cD}\cdot\mathbf{c}' - \mathbf{c}'\mathbf{D}\cdot\mathbf{c})[\det(\mathbf{B},\mathbf{C},\mathbf{D})]^{-1} = 0. \quad (60)$$

This concludes the proof.

Corollary 7. *The system of the linear functionals* $\{G_j\}_{j\in Z}$ *can be represented in the form*

$$G_j(u) = det(\mathbf{\Phi}_{j+1}, \mathbf{\Phi}'_{j+1}, \mathbf{\Phi}_j\,(F^{\pm})'_j u - \mathbf{\Phi}'_j F_j^{\pm}u). \quad (61)$$

VII. WAVELET DECOMPOSITION

Let $\mathbf{B}_\varphi$(X) be the linear span of the coordinate $\mathbf{B}_\varphi$-splines on the grid X. Thus

$$\mathbf{B}_\varphi(X) = \{\, u \mid u = \textstyle\sum_{j\epsilon Z} c_j\Omega_j \;\; \forall c_j \in R^1 \,\}$$

is the linear space. The last one is named *the space of* B_φ-*splines of second order on the grid* X.

We introduce a grid

$$Y: \ldots < y_{-2} < y_{-1} < y_0 < y_1 < y_2 < \ldots,$$

$$\lim_{j\to-\infty} y_j = a, \;\; \lim_{j\to+\infty} y_j = b.$$

We suppose that X⊂Y. By definition we put $\tilde{F}_j^{\pm} = \tilde{F}^{\pm}(y_j)$ $\tilde{\mathbf{\Phi}}_j = \mathbf{\Phi}(y_j)$ $\tilde{\mathbf{\Phi}}'_j = \mathbf{\Phi}'(y_j)$, $\tilde{\mathbf{N}}_j = \tilde{\mathbf{\Phi}}_j \times \tilde{\mathbf{\Phi}}'_j$, $\tilde{\mathbf{b}}_j = \tilde{\mathbf{N}}_j \times \tilde{\mathbf{N}}_{j+1}$. Using the formulas which are analogous to formulas (3)–(8) with replacing the vectors $\mathbf{b}_j$ ~~by~~ with the vectors $\tilde{\mathbf{b}}_j$, we obtain the new coordinate splines ω_j, *j*∈*Z*. Let $\mathbf{B}_\varphi$(Y) be a space of B_φ-splines of the second order on the grid Y,

$$\mathbf{B}_\varphi(Y) = \{\tilde{u} \mid \tilde{u} = \textstyle\sum_{j\epsilon Z}\hat{c}_j\omega_j \;\; \forall \hat{c}_j \in R^1\}.$$

We can see that embedding $\mathbf{B}_\varphi(X) \subset \mathbf{B}_\varphi(Y)$ is true.

We discuss the operator *P*, which projects the space $\mathbf{B}_\varphi$(Y) on the space $\mathbf{B}_\varphi$(X). More precisely we suppose that projective operator *P* is defined by the formula

$$P\tilde{u} = \textstyle\sum_j G_j(\tilde{u})\Omega_j \quad \forall\tilde{u}\in\mathbf{B}_\varphi(X). \quad (62)$$

We also introduce the operator $Q = I - P$, where *I* is the identity operator. The space W = $Q\mathbf{B}_\varphi(Y)$ is named *the wavelet space*. The direct decomposition

$$\mathbf{B}_\varphi(Y) = \mathbf{B}_\varphi(X) + W, \quad (63)$$

is called *the spline-wavelet decomposition* of the space $\mathbf{B}_\varphi$(Y).

In what follows we discuss the infinite series in the form $\sum_j c_j\omega_j$, $c_j \in R^1$, where the summation is extended on all integer numbers *j* ∈ Z. Under each fixed $t \in (a,b)$ the series contains no more than three nonzero terms. Therefore the mentioned series converge (in the sense of point-wise convergence) for arbitrary sequence of coefficients $\{c_j\}_{j\in Z}$, $c_j \in R^1$.

Consider the case when the grid *Y* is obtained from the grid *X* by adding one knot *x*, $x\in(x_k,x_{k+1})$. By definition, we put **X**=**N**(*x*).

Let $\tilde{u}$ be an element of the space $\mathbf{B}_\varphi(Y)$. Discuss the decomposition of the mentioned element with the projective operator *P*, $P\tilde{u}=\sum_i a_i\,\Omega_i$, $Q\tilde{u}=\sum_{i'} b_{i'}\omega_{i'}$, where $a_i=G_i(\tilde{u})$, $b_{i'}=\hat{G}(Q\tilde{u})$. Suppose that coefficients a_i and $b_{i'}$ are done.

Theorem 8. *The next relations are fulfilled:*

$$\Omega_i = \textstyle\sum_j d_{i,j}\omega_j, \quad (64)$$

where

$$d_{i,j} = \delta_{i,j} \;\; for \;\; j \le k\text{-}1,$$
$$d_{i,j} = \delta_{i+1,j} \;\; \text{for} \;\; j \ge k+2, \;\; \forall i\in Z, \quad (65)$$

$$d_{k-1,k} = \frac{\det(\mathbf{N}_k,\mathbf{X},\mathbf{N}_{k+1})}{\det(\mathbf{N}_{k-1},\mathbf{N}_k,\mathbf{N}_{k+1})},\ d_{k,k} = \frac{\det(\mathbf{N}_{k-1},\mathbf{N}_k,\mathbf{X})}{\det(\mathbf{N}_{k-1},\mathbf{N}_k,\mathbf{N}_{k+1})}, \quad (66)$$

$$d_{i,k} = 0 \ for \ \forall i \in Z \setminus \{k-1,k\},\ d_{i,k+1} = 0 \ for \ \forall i \in Z \setminus \{k,k+1\},$$

$$d_{k,k+1} = \frac{\det(\mathbf{X},\mathbf{N}_{k+1},\mathbf{N}_{k+2})}{\det(\mathbf{N}_k,\mathbf{N}_{k+1},\mathbf{N}_{k+2})}, \quad (67)$$

$$d_{k+1,k+1} = \frac{\det(\mathbf{N}_k,\mathbf{X},\mathbf{N}_{k+1})}{\det(\mathbf{N}_k,\mathbf{N}_{k+1},\mathbf{N}_{k+2})}. \quad (68)$$

Proof. As it is shown in [20], we are in conditions of maximum pseudo-smoothness, and therefore the spaces $\mathbf{B}_\varphi(X)$ and $\mathbf{B}_\varphi(Y)$ are embedded, $\mathbf{B}_\varphi(X) \subset \mathbf{B}_\varphi(Y)$. Thus representation (64) exists. Consider the system of linear functionals defined by formula

$$g_j(u) = det(\tilde{\mathbf{\Phi}}_j, \tilde{\mathbf{\Phi}}'_{j+1}, \tilde{\mathbf{\Phi}}_j\,(\tilde{F}^\pm)'_j u - \tilde{\mathbf{\Phi}}'_j \tilde{F}^\pm_j u). \quad (69)$$

Similarly to functionals (61), we establish that the system $\{g_j\}_{j\in Z}$ is biorthogonal to the system $\{\omega_j\}_{j\in Z}$,

$$g_j(\omega_i) = \delta_{i,j} \qquad \forall i,j \epsilon Z.$$

We apply these functionals to relation (64). Taking into account formulas (5) - (8) и (10), we obtain relations (65) - (68). This completes the proof.

Formulas (65) – (68) are called by *calibration relations*, the functions ω_j are called *calibrating functions*, and the functions $\mathbf{\Omega}_j$ are called *calibrated functions*.

By formulas (69) we see that

$$g_j = G_j \text{ for } j \le k-1,\ g_j = G_{j-1} \text{ for } j \ge k+2. \quad (70)$$

According to formulas (63) and (64) we have

$$\tilde{u} = \sum_i a_i \mathbf{\Omega}_i + \sum_{i'} b_{i'} \omega_{i'} = \sum_{i'} (\sum_i a_i d_{i,i'} + b_{i'}) \omega_{i'}.$$

Hence for values $c_j = f_j(\tilde{u})$ we obtain *the reconstruction formulas* $c_j = \sum_i a_i d_{i,j} + b_j$, $j \in Z$.

Theorem 9. *Reconstruction formulas for spline-wavelet decomposition (63) can be written in the form*

$$c_j = a_j + b_j \ \text{ for } j \le k-1;\ c_j = a_{j-1} + b_j \ \text{ for } j \ge k+2, \quad (71)$$

$$c_{k+1} = a_k\, d_{k,k+1} + a_{k+1}\, d_{k+1,k+1} + b_{k+1},$$

$$c_k = a_{k-1}\, d_{k-1,k} + a_k\, d_{k,k} + b_k, \quad (72)$$

Proof. If j does not equal k or k+1 then by (65) we obtain the relations (71). If $j = k$ or if $j = k + 1$ then by formulas (66) – (68) we have (72).

VIII. DECOMPOSITION FORMULAS

Let $\tilde{u}$ be an element of the space $\boldsymbol{B}_\varphi$(Y). Suppose that the coefficients c_k in the decomposition $\tilde{u} = \sum_k c_k \omega_k$ are done. Applying the equality $a_i = G_i(\tilde{u})$, by (72) we consequently obtain

$$b_j = c_j - \sum_i d_{i,j} a_i = c_j - \sum_i d_{i,j} \sum_k c_k G_i(\omega_k).$$

The formulas

$$a_i = \sum_k c_k G_i(\omega_k), \quad (73)$$

$$b_j = c_j - \sum_i d_{i,j} \sum_k c_k G_i(\omega_k) \quad (74)$$

are named by *the decomposition formulas*.

Lemma 8. *The next relations are right,*

$$G_j(\omega_j) = 1 \quad for \quad j \le k-1, \quad (75)$$

$$G_j(\omega_{j+1}) = 1 \ \ for \ \ j \ge k+1, \quad (76)$$

$$G_k(\omega_{k-1}) = \frac{\det(\mathbf{X},\mathbf{N}_k,\mathbf{N}_{k+1})}{\det(\mathbf{N}_{k-1},\mathbf{N}_k,\mathbf{X})}, \quad (77)$$

$$G_k(\omega_k) = \frac{\det(\mathbf{N}_{k-1},\mathbf{N}_k,\mathbf{N}_{k+1})}{\det(\mathbf{N}_{k-1},\mathbf{N}_k,\mathbf{X})}. \quad (78)$$

$$G_j(\omega_i) = 0 \quad (79)$$

where

$$(i,j) \notin \{(i',i') \mid i' \le k-1\} \subset \{(i',i'+1) \mid i' \ge k+1\} \subset \{(k,k-1),(k,k)\}. \quad (80)$$

Proof. We use the relations (70). If $j \le k-1$ then we have $G_j(\omega_i) = \hat{G}_j(\omega_i) = \delta_{j,i}$, $i \in Z$. If $j \ge k+1$ then we obtain $G_j(\omega_i) = \hat{G}_{j+1}(\omega_i) = \delta_{j+1,i}\ \forall i \in Z$. It remains to discuss the values $G_k(\omega_i)$, $i \in Z$. Taking into account the formula supp $\omega_i = [\xi_{i-1},\xi_{i+2}]$, we see that $F_k(\omega_i) = 0$ for $i \notin \{k-1,k\}$. We apply formulas (5)–(8) and (10) on segment $[x_k,x]$. As a result we have $\omega_{k-1}(t) = \mathbf{X}\cdot\varphi(t)[\det(\mathbf{N}_{k-1},\mathbf{N}_k,\mathbf{X})]^{-1}$ for $t \in [x_k,x]$.

By definition, we put

$$D = \det(\mathbf{\Phi}_{k+1}, \mathbf{\Phi}'_{k+1}, \mathbf{\Phi}_k (F^\pm)'_k (\mathbf{X}\cdot\varphi) - \mathbf{\Phi}'_k F^\pm_k (\mathbf{X}\cdot\varphi))$$

so that

$$G_k(\omega_{k-1}) = D/\det(\mathbf{N}_{k-1},\mathbf{N}_k,\mathbf{X}). \quad (81)$$

Taking into account the relations $F^\pm_k \varphi = \mathbf{\Phi}_k$, $(F^\pm)'_k \varphi = \mathbf{\Phi}'_k$, we obtain $D = \det(\mathbf{\Phi}_{k+1}, \mathbf{\Phi}'_{k+1}, \mathbf{\Phi}_k(\mathbf{X}\cdot\mathbf{\Phi}'_k) - \mathbf{\Phi}'_k(\mathbf{X}\cdot\mathbf{\Phi}_k))$. Using the formula of mixed product, we get

$$D = \det(\mathbf{\Phi}_{k+1}, \mathbf{\Phi}'_{k+1}, \mathbf{\Phi}_k)(\mathbf{X}\cdot\mathbf{\Phi}'_k) - \det(\mathbf{\Phi}_{k+1}, \mathbf{\Phi}'_{k+1}, \mathbf{\Phi}'_k)(\mathbf{X}\cdot\mathbf{\Phi}_k) = (\mathbf{\Phi}_{k+1}\times\mathbf{\Phi}'_{k+1})\cdot\mathbf{\Phi}_k(\mathbf{X}\cdot\mathbf{\Phi}'_k) - (\mathbf{\Phi}_{k+1}\times\mathbf{\Phi}'_{k+1})\cdot\mathbf{\Phi}'_k(\mathbf{X}\cdot\mathbf{\Phi}_k).$$

By the formula $\mathbf{\Phi}_{k+1}\times\mathbf{\Phi}'_{k+1} = \mathbf{N}_{k+1}$ we have

$$\mathcal{D} = \det\begin{pmatrix} \mathbf{N}_{k+1}\cdot\mathbf{\Phi}_k & \mathbf{N}_{k+1}\cdot\mathbf{\Phi}'_k \\ \mathbf{X}\cdot\mathbf{\Phi}_k & \mathbf{X}\cdot\mathbf{\Phi}'_k \end{pmatrix}.$$

If we apply Lemma 5 to the last expression then we obtain

$$\mathcal{D} = \det(\mathbf{N}_{k+1}, \mathbf{\Phi}'_k \times \mathbf{\Phi}_k, \mathbf{X}).$$

Hence, by (81) we deduce relation (77). Taking into account equality supp $\omega_{k+1} = [x_k,x_{k+1}]$, for $t \in [x_{k-1},x_k]$ we have $\omega_k(t) = \det(\mathbf{N}_{k-1},\mathbf{N}_k,\mathbf{N}_{k+1})[\det(\mathbf{N}_{k-1},\mathbf{N}_k,\mathbf{X})]^{-1}\mathbf{\Omega}_k(t)\ \forall t \in [x_{k-1},x_k]$.We apply the functional G_k to the last identity, and as a result we obtain formula (78). Thus formulas (75)–(80) have been proved. This concludes the proof.

Theorem 10. *The decomposition formulas of representation (37) can be written in the next form*

$$a_i = c_i \ \ for\ i \le k-1, \qquad a_i = c_{i+1} \ \ for\ i \ge k+1, \qquad (56)$$

$$a_k = \big(\det(\mathbf{X},\mathbf{N}_k,\mathbf{N}_{k+1})c_{k-1} + \det(\mathbf{N}_{k-1},\mathbf{N}_k,\mathbf{X})c_k\big)$$

$$/\det(\mathbf{N}_{k-1},\mathbf{N}_k,\mathbf{X}), \qquad (83)$$

$$b_{k+1} = -\frac{\det(\mathbf{X},\mathbf{N}_{k+1},\mathbf{N}_{k+2})}{\det(\mathbf{N}_k,\mathbf{N}_{k+1},\mathbf{N}_{k+2})\det(\mathbf{N}_{k-1},\mathbf{N}_k,\mathbf{X})}\times$$

$$\times\, [\det(\mathbf{X},\mathbf{N}_k,\mathbf{N}_{k+1})c_{k-1} + \det(\mathbf{N}_{k-1},\mathbf{N}_k,\mathbf{N}_{k+1})c_k + c_{k+1}$$

$$-\det(\mathbf{N}_k,\boldsymbol{X},\mathbf{N}_{k+1})c_{k+2}/\det(\mathbf{N}_k,\mathbf{N}_{k+1}\mathbf{N}_{k+2}) \qquad (84)$$

$$b_j = 0 \ \ for\ j \ne k-1.$$

Proof. Formulas (82) follows from (73), (75)–(76). For $i = k$ by Lemma 7 we see that only two terms remain in the sum (72), $a_k = c_{k-1}F_k(\omega_{k-1})+c_kF_k(\omega_k)$. Applying formulas (77) – (78), we obtain relation (83). Thus formulas (82) – (83) have been proved.

Using the equivalent writing of equalities (71) – (72) and formula (73), we have

$$b_j = c_j - a_j \ \text{ for } j \le k-1, \qquad (85)$$

$$b_j = c_j - a_{j-1} \quad \text{for } \ j \ge k+2, \qquad (86)$$

$$b_k = c_k - a_{k-1}d_{k-1,k} - a_kd_{k,k},$$

$$b_{k+1} = c_{k+1} - a_kd_{k,k+1} - a_{k+1}d_{k+1,k+1}, \qquad (87)$$

By (56) and (60) we obtain

$$b_j = 0 \quad \text{for} \quad j \le k-1 \ \text{ or } \ j \ge k+2. \qquad (88)$$

Substituting the values a_{k-1} and a_k from expressions (82) – (83) in formula (87), we deduce the next relations

$$b_k = c_k - a_{k-1}\det(\mathbf{N}_k,\mathbf{X},\mathbf{N}_{k+1})[\det(\mathbf{N}_{k-1},\mathbf{N}_k,\mathbf{X})]^{-1}$$

$$-\ a_k \det(\mathbf{N}_{k-1},\mathbf{N}_k,\mathbf{X})[\det(\mathbf{N}_{k-1},\mathbf{N}_k,\mathbf{N}_{k+1})]^{-1}.$$

By (82)–(83) we have

$$b_k = 0. \qquad (89)$$

Formulas (88)–(89) show that the equalities (84) are true. Similarly from (87) with the help of relations (68) and (83) we have (84). This concludes the proof.

IX. CONCLUSION

There are two basic techniques for approximating functions with a singularity. The first is to extract a singularity (additive or multiplicative) so that the remainder of such a selection turns out to be a smooth function. As a result, the matter is reduced to the approximation of a smooth remainder. This technique can be effective in cases where the type of feature and its exact location are known. In the case where the function of interest is a solution to the initial boundary value problem, it is difficult to count on the availability of accurate information. In the absence of accurate information, a critical situation can arise. Instead of a smooth remainder, one can obtain a remainder with doubled singularities, which can lead to a halt in the computational process. Note that even with accurate information, the result may be unacceptable due to round-off errors that accompany the computational process when using floating point calculations.

The second technique is that the expected feature is introduced into the approximation apparatus. In the case considered in this paper, this is the apparatus of singular splines. In the absence of accurate information about the location and type of a feature, the situation is less critical: as a result, the quality of the approximation will be violated, but not the entire computational process. In particular, if the type of a feature is known, but the coefficient of its occurrence in the function of interest is not known, then this coefficient is found from one or another optimization relationship (the minimum of the energy norm in the finite element method, etc.). With inaccurate information, you can hope for less damage.

In this paper, the second technique is considered: the singularity is introduced into the spline approximation apparatus. When constructing splines, it has long been customary to use approximation relations, since they allow one to construct spaces of splines with the required approximation properties (see [2], [12] – [16], [20] – [27]). To refine the approximation of the required functions (for example, in the finite element method) and in the compression of information flows (for example, using wavelet decompositions), a chain of embedded spline spaces on embedded grids is required.

Unfortunately, approximation relations do not guarantee the embedding of the spline spaces corresponding to the embedded grids. It is necessary to additionally require the maximum smoothness of these spaces. It turned out that it is enough to consider one or another generalized smoothness. With respect to the spaces of singular splines, it is possible in a number of cases to introduce generalized smoothness in such a way that the existing singularity turns out to be “smooth” from the new point of view. Thus, the introduction of generalized smoothness leads to the corresponding calibration relations and to embedded spline spaces on embedded grids.

This paper proposes a general approach to constructing spaces of singular splines for approximating functions that have a singularity in the function itself or in its derivatives. This approach is to introduce the concept of generalized smoothness. Due to this, a function with a singularity can be considered as a generalized smooth function. This approach leads to the construction of a system of embedded spaces on embedded grids. The sequence of embedded spaces can be used for approximating singular functions, for implementing the finite element method in the degenerate problems, for constructing multi-grid methods, and also for the wavelet decomposition of the singular numerical flows. The linear shells of the coordinate splines are spaces with the property of embedding on embedding grids.

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