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Discontinuous Legendre wavelet Galerkin method for optimal control of time delayed systems

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Abstract—Time-delay systems arise in many important applications in science and engineering and optimal control of delay differential equations are of theoretical and practical importance. This paper presents discontinuous Legendre wavelet Galerkin (DLWG) approach for solving optimal control problem of time-delayed systems. This new method demonstrates that operational matrices of derivative, delay and product are lower dimensions and sparse because of calculation only on each subinterval. The advantages are implemented to solve algebraic equations transformed from the time-delayed systems with less storage space and execution time. Finally, an experiment is included to illustrate the effectiveness and applicability of the proposed method.

Keywords—Time-delayed system, Optimal control, Legendre wavelet, Discontinuous Legendre wavelet Galerkin method.

I. INTRODUCTION

In general, delays occur frequently in biological, chemical, transportation and electronic systems, etc [1-9]. The optimal control problem of the time-delayed systems is very important to be investigated and has attracted the interest of many researchers [2-6]. Recently, orthogonal functions especially wavelet bases such as Walsh functions, block-pulse functions, Legendre polynomials, Haar wavelets and Legendre multiwavelets, etc. are adopted to do with the above optimal control of the dynamic systems [5-19]. Although the accuracy of computation can be improved by adding more orthogonal series, the more terms of orthogonal series enlarge the dimensions of the above operational matrices. This disadvantage increases the computational complexity when solving the systems.

The aim of the present paper is to design a new computational technique based on Legendre wavelet and discontinuous Galerkin (DG) method to overcome the above disadvantage [9]. Specifically, the operators of derivative, delay and product are calculated by Legendre wavelet on the subinterval. It is noted that the operational matrices obtained by this means are the same in each subinterval. Consequently, the operational matrices are lower dimensions and sparse because of the orthogonal property of Legendre wavelet. The advantages of the operational matrices are used to transform the time delayed systems into lower dimensions algebraic equations by the variational formulation.

The essential features and advantages of the new approach are briefly described as follows:

i. The time delayed systems are transformed into variational formulation by the DG method.

ii. The lower dimensional and sparse matrices of the derivative, delay and product are the same on each subinterval and utilized to elaborately evaluate each term of the variational formulation and the quadratic cost function, respectively.

iii. The discontinuity of Legendre wavelet at interface of element to element and the initial conditions are easier to be coped with through the numerical flux.

iv. The approach of the discontinuous element approximation is implemented to convert the optimal control of the delay systems to solve systems of algebraic equations on each subinterval.

v. The DLWG approach has the advantages of Legendre wavelet method and the DG method.

Finally, the DLWG method containing the above advantages is applied to solving optimal control of the systems. A numerical experiment illustrates the present approach is very effective.

II. Variational Formulation of Time-delayed Systems

In this section, we first briefly introduce Legendre wavelet bases and our notations. Second, we derive the variational formulation of the time-delayed systems.

2.1 Legendre Wavelet

For level $n = \ \dots$ of resolution and translation $l = \ \dots \ ^{n} -$, we define the subinterval $I_{nl} = \ ^{-n} l \ ^{-n} l +$. For $p = \ \dots$, we define a subspace as

$V_{p\,n} = \{ f \mid f|_{I_{nl}}$ is a polynomial of degree strictly less than p ;and f vanishes elsewhere}.

Let $\phi_k \ t$ denote Legendre wavelet bases at decomposition level $n =$ as

$$\phi_k \ t = \begin{cases} \sqrt{\ k+} \ L_k \ \ t- & t \in \\ & t \notin \end{cases} \tag{1}$$

where $L_k \ t$ is Legendre polynomial of degree k. Then $V_{p\,n}$ is spanned by $^{n} p$ functions which are obtained from $\phi \ \dots \ \phi_{p-}$ by dilation and translation, i.e.,

$$V_{p\,n} = \ \phi_{k\,nl} = \ ^{n} \ \phi_k \ ^{n} t - l \quad \le k \le p- \quad \le l \le \ ^{n} -$$

which satisfies $V_p \subset V_p \subset \dots \subset V_{p\,n} \subset \dots$ and forms an orthonormal basis for L. A function $x \ t \in L$ is approximated by Legendre wavelet bases as follows

$$\mathrm{P}_n x \ t = \sum_{k=}^{p-} \sum_{l=}^{\ ^{n}-} c_{k\,nl} \phi_{k\,nl} \ t = C^T \Phi . \tag{2}$$

where P_n is projection at the finest scale n, and $c_{k\,nl}$, $\phi_{k\,nl} \ t$ are denoted by vectors as

$$C = c_{\ n} \ \dots \ c_{p-\ n} \ c_{\ n} \ \dots \ c_{p-\ n} \ \dots \ c_{\ n\ ^{n}-} \ \dots \ c_{p-\ n\ ^{n}-} ,$$
$$\Phi = \phi_{\ n} \ \dots \ \phi_{p-\ n} \ \phi_{\ n} \ \dots \ \phi_{p-\ n} \ \dots \ \phi_{\ n\ ^{n}-} \ \dots \ \phi_{p-\ n\ ^{n}-} .$$

The aproximation of (4) is estimated as [19]

$$\left\| x \ t - \sum_{k=}^{p-} \sum_{l=}^{\ ^{n}-} c_{k\,nl} \phi_{k\,nl} \ t \right\| \quad O \ ^{-np} . \tag{3}$$

which is exponential convergence with respect to p and n.

2.2 Weak Formulation of The Delay Systems

Consider the linear systems with time delay in the state vector as follows:

$$\frac{dX \ t}{dt} = A \ t \ X \ t + A_d \ t \ X \ t - \tau + B \ t \ U \ t , \tag{4}$$

with the initial conditions

$$X \quad = X \qquad X \ t = \gamma \ t \qquad t \in -\tau \quad . \tag{5}$$

where $X \ t$ and $U \ t$ are q and q dimensional vectors of the state and control input, respectively. The $A \ t$, $A_d \ t$, $B \ t$ are continuous matrix functions with appropriate dimensions and $\tau >$ is the constant time delay. X, $\gamma \ t$ are constant specified vector and continuous vector function, respectively. Under the above assumptions, existence of unique solution of (4) with zero input is assured by the results given in [2]. The objective is to find the optimal control law $U \ t$ for $t \in$ which minimizes the quadratic cost function defined as

$$J = - X^T \ SX \ + - \int \left(X^T \ t \ Q \ t \ X \ t + U^T \ t \ R \ t \ U \ t \right) dt , \tag{6}$$

where S is a positive semi-definite symmetric matrix, $Q \ t$, $R \ t$ are positive semi-definite and positive-definite symmetric continuous an $q \times q$, $q \times q$ matrices functions, respectively.

First, we introduce some standard notations of the DLWG approach. Let X_l^+, X_l^- denote the values of the vector function X at $t_l = \ ^{-n} l$ from right and left, respectively,

$$X_l^+ = \lim_{\varepsilon \to +} X \ t_l + \varepsilon \qquad X_l^- = \lim_{\varepsilon \to +} X \ t_l - \varepsilon \ .$$

Let $\{X\} = X^+ + X^-$, $[X] = X^+ - X^-$ represent the mean and the jump of X at boundary of element, i.e., subinterval I_{nl}, respectively. Second, we multiply Eq.(4) by test functions $\phi_{k\,nl}$ and replace X by approximation solution X_h, and integrate over the element and use a simple formal integration by parts. Then we obtain the variational formulation as

$$\int_{I_{nl}} X_h \ t \ \phi'_{k\,nl} dt + \int_{I_{nl}} A \ t \ X_h \ t \ \phi_{k\,nl} dt + \int_{I_{nl}} A_d \ t \ X_h \ t - \tau \ \phi_{k\,nl} dt$$

$$= -\int_{I_{nl}} B\ t\ U\ t\ \phi_{k\,nl} dt + \overline{X_h}_{\ l+}\ \phi^{-}_{k\,nl\ \ l+} - \overline{X_h}_{\ l}\ \phi^{+}_{k\,nl\ \ l}, \quad (7)$$

where h is the size of the element, $\overline{X_h}$ is the flux and suitably chosen as

$$\overline{X_h} = \ \hat{}X_h\ \grave{}\ \grave{} - \sigma\ \ X_h\ , \qquad (8)$$

which is the local Lax–Friedrichs flux and $\sigma = \max |X'|$, refer to [9] for more details.

III. Computation of the Variational formulation and optimal control

In this section, we first calculate the lower dimensions operational matrices of the derivative, delay and product on subinterval. Second, we evaluate each term of the variational formulation of Eq.(7). Third, we consider the approximation of the quadratic cost function and the corresponding optimal control.

3.1 Computations of the operational matrices

Derivative operational matrix: Define the derivative operational matrix on the l *st* element as

$$\Phi'_l\ t\ = R\Phi_l\ t\ , \qquad (9)$$

where $\Phi_l = \phi_{\ nl}, \ldots \phi_{p-\ nl}{}^T$. The matrix R is calculated explicitly by

$$R_{\ k+\ \ k+} = \int_{-n_l}^{-n\ \ l+} \phi_{k\,nl}\ t\ \phi'_{k\ nl}\ t\ dt = {}^{n}\int \phi_k\ t\ \phi'_k\ t\ dt$$
$$= {}^{n+}\sqrt{\ k+}\ \sqrt{\ k+}\ v_{kk},$$

where $k\ k = \ \ldots\ p-$ .When $k-k$ is odd, $v_{kk} =$, otherwise $v_{kk} =$ .For example, let $p =$ and $n =$, respectively, we can obtain the derivative operational matrix as

$$R = \begin{pmatrix} & \sqrt{\ } & \\ & & \sqrt{\ \ } \\ & & \end{pmatrix}.$$

Delay operational matrix: Define the time-delayed operational matrix as

$$\Phi_l\ t - \tau\ = D\ \tau\ \Phi_l\ t\ , \qquad (10)$$

where the matrix $D\ \tau$ is computed as

$$D_{\ k+\ \ k+} = \int_{-n_l}^{-n\ \ l+} \phi_{k\,nl}\ t\ \phi_{k\,nl}\ t - \tau\ dt$$

$$= -\ \sqrt{\ k+}\ \sqrt{\ k+}\ \int_{-} L_k\ u\ L_k\ u - {}^{n+}\ \tau\ du,$$

where $L_k\ u$ is Legendre polynomial and $<\tau<\ ^{-n}$.For example , let $p =$, $n =$, respectively, we obtain the delay operational matrix as

$$D\ \tau\ = \begin{pmatrix} & - \sqrt{\ }\tau & \sqrt{\ }\tau \\ & & \\ & & \end{pmatrix}.$$

Product operational matrix: Define the product operational matrix as

$$\Phi_l\ t\ \Phi_l\ t\ {}^T P\ = P\Phi_l\ t \qquad (11)$$

where $P = g_{\ nl}, g_{\ nl} \ldots g_{p-\ nl}{}^T$ is a known vector and each element of the product operator P with the property of symmetry is computed clearly by

$$P_{\ k+\ \ k+} = \int_{-n_l}^{-n\ \ l+} \phi_{k\,nl}\ t\ \sum_{j=}^{p-} g_{j\,nl}\phi_{j\,nl}\ t\ \phi_{k\ nl}\ t\ dt,$$

where element of the P matrix is composed of linear combinations of elements of the vector P. For example , let $p =$, $n =$, $l =$, respectively [2]

$$P = \begin{pmatrix} \sqrt{\ }g & \sqrt{\ }g & \sqrt{\ }g \\ \sqrt{\ }g & \sqrt{\ }g\ \ + \frac{}{\sqrt{\ }}g & \frac{}{\sqrt{\ }}g \\ \sqrt{\ }g & \frac{}{\sqrt{\ }}g & \sqrt{\ }g\ \ + \frac{}{\sqrt{\ }}g \end{pmatrix}.$$

Comment 1: For each element, i.e., every l, the above operational matrices are the same, which are proved by using simple variable substitute.

Comment 2: If the level n of resolution is enough large, the lower $p \times p$ dimensional matrices compared with ${}^n p \times {}^n p$ dimensions [9] can simplify the storage and computational complexity when solving the systems.

3.2 Computation of variational formulation

We first introduce the Kroneker product: $I_{q\times q} \otimes \Phi_l$, where $I_{q\times q}$ is $q \times q$ identity matrix. Then the approximation solution of the delay system is represented as $X_h\ t\ = I_{q\times q} \otimes C^T_{rl}\Phi_l$, where r denotes the r *th* system of the delay systems (4) and $r = \ \ldots\ q$.

For all $k = \ \ldots\ p-$, we elaborately calculate the first term of Eq. (7) by utilizing the derivative operational matrix in (9) to obtain

$$\int_{I_{nl}} X_h\ t\ \phi'_{k\,nl}dt = \int_{I_{nl}} I_{q\times q}\ \otimes C_{rl}^T \Phi_l \phi'_{k\,nl}dt$$

$$= \int_{I_{nl}} I_{q\times q}\ \otimes C_{rl}^T \Phi_l \Phi_l'^T\ t\ \otimes I_{p\times}\ dt = I_{q\times q}\ \otimes R^T C_{rl}\ , \quad (12)$$

where C_{rl} denotes the Legendre wavelet approximation coefficients of the $r\,th$ system of the delay systems and $C_{rl} = c_r\ {}_{nl} \ldots c_{r\,k\,nl} \ldots c_{r\,p-\,nl}$.

For the second, third and fourth terms in (7), we let A , A_d and B denote the approximation coefficient matrices of the known functions $A\,t$, $A_d\ t$ and $B\,t$, respectively. Then, we compute the second term by using the product operational matrix (11) as

$$\int_{I_{nl}} A\ t\ X_h\ t\ \phi_{k\,nl}dt = \int_{I_{nl}} I_{q\times q}\ \otimes A_{\,r} P_A C_{rl}^T \Phi_l\ t\ \Phi_l\ t^{\ T} \otimes I_{p\times}\ dt$$

$$= I_{q\times q}\ \otimes\ A_{\,r} P_A C_{rl}\ , \quad (13)$$

where the element of $A_{\,r}$ is composed of linear combinations of elements of the matrix A . For the delay term (4), we need to compute the delay term by two parts according to the delay operator (10), which is defined as

$$X_h\ t-\tau\ = \begin{cases} \gamma\ t-\tau\ = I_{q\times q}\ \otimes\ K_{rl}^T \Phi_l\ t & \le t < \tau \\ I_{q\times q}\ \otimes\ D\ \tau\ C_{rl}^T \Phi_l\ t & \tau \le t \le \end{cases} \quad (14)$$

where K_{rl} denotes the approximation coefficients of the known function $\gamma\ t-\tau$. Similar to (13), we obtain the calculations of the third and fourth terms by using (10), respectively,

$$\int_{I_{nl}} A_d\ t\ X_h\ t-\tau\ \phi_{k\,nl}dt = \begin{cases} I_{q\times q}\ \otimes\ A_{dr} P_{Ad} K_{rl} & \le t < \tau \\ I_{q\times q}\ \otimes\ A_{dr} P_{Ad} D\ \tau\ C_{rl} & \tau \le t < \end{cases} \quad (15)$$

$$-\int_{I_{nl}} B\ t\ U\ t\ \phi_{k\,nl}\ t\ dt = -I_{q\times q}\ \otimes\ B_{\,r} P_B U_{rl}\ , \quad (16)$$

where U_{rl} is the approximation vector of the function $U\ t$.

Up to now, in order to effectively calculate the flux terms in (7), we should adequately evaluate the $\phi_{k\,nl}\ l\ \ {}^n$ and $\phi_{k\,nl}\ l\ \ {}^n$, respectively,

$$\phi_{k\,nl}\ l\ \ {}^n = {}^n \begin{cases} \sqrt{\ k+\ } & k \ \mathrm{R} \\ & k \ \mathrm{HY} \end{cases}$$

$$\phi_{k\,nl}\ l\ \ {}^n = \begin{cases} -\ \sqrt{\ k+\ } & k \ \mathrm{R} \\ & k \ \mathrm{HY} \end{cases} \quad (17)$$

Using (8) and (17), we obtain the calculations of the fifth and sixth terms as

$$\overline{X_h}\ {}_{l+}\ \phi^-_{k\,nl\ \ l+} - \overline{X_h}\ {}_{l}\ \phi^+_{k\,nl\ \ l}$$

$$= -I_{q\times q} \otimes \left({}^n I_{p\times} \otimes \begin{cases} +\ \sigma\ k+\ \sigma+ & k\ \mathrm{RGG} \\ & k\ \mathrm{HYHQ} \end{cases} \otimes I_{\times p} \right) C_{rl} = -I_{q\times q} \otimes C\ C_{rl}, \quad (18)$$

where

$$C\ = \left({}^n I_{p\times} \otimes \begin{cases} +\ \sigma\ k+\ \sigma+ & k\ \mathrm{RGG} \\ & k\ \mathrm{HYHQ} \end{cases} \otimes I_{\times p} \right) .$$

Note that when $l = \ \ {}^n -$, we compute Eq.(18) by using the initial condition $X\ \ = X$.

According to (12), (13), (15), (16) and (18), we convert the variational formulation (7) into the systems of algebraic equations which are solved for the approximate coefficients C_{rl} of $X\ t$ in terms of $U\ t$ as

$$C_{q\,p\times}\ = I_{q\times q} \otimes \left[R^T + A_{\,r} P_A + C\ \right]^- \left[-A_{dr} P_{Ad} K_{rl} - B_{\,r} P_B U_r\ \right], \quad (19)$$

$$C_{q\,p\times\ l} = I_{q\times q} \otimes \left[R^T + A_{\,r} P_A + A_{dr} P_{Ad} D\ \tau\ + C\ \right]^- \left[-B_{\,r} P_B U_{rl}\right], \quad (20)$$

where $C_{q\,p\times}$, $C_{q\,p\times\ l}$ denote the solutions defined on the intervals and τ , respectively.

3.3 Optimal control of the time-delayed systems

For the quadratic cost function in (6), the $Q\ t$, $R\ t$ are first expanded by Legendre wavelet as

$$Q\ t\ = Q_{q\times q\ p}\ I_q\ \otimes \Phi_l\ t\ ,$$

$$R\ t\ = R_{q\times q\ p}\ I_q\ \otimes \Phi_l\ t\ , \quad (21)$$

where $Q_{q\times q\ p}$, $Q_{q\times q\ p}$ denote the approximation coefficient matrices of the function matrices $Q\ t$, $R\ t$, respectively. Similarly, the term $X^T\ \ SX$ in (6) is denoted by $S_l = I_{q\times q} \otimes C_{rl}^T \Phi_l\ \ {}^T \otimes I_{q\times q} \otimes C_{rl}^T \Phi_l$. Then, using the product operator (11), we have

$$Q_{q\times q\ p}\ I_q\ \otimes \Phi_l\ t\ \ I_q\ \otimes \Phi_l\ t\ {}^T = I_{q\times q}\ \otimes\ C_{rl}^T \Phi_l\ {}^T Q_l,$$

$$R_{q\times q\ p}\ I_q\ \otimes \Phi_l\ t\ \ I_q\ \otimes \Phi_l\ t\ {}^T = I_{q\times q}\ \otimes\ C_{rl}^T \Phi_l\ {}^T R_l, \quad (22)$$

Let $C\ = C_{q\,p\times}$, $C_l = C_{q\,p\times\ l}$, $U_l = U_{q\,p\times\ l}$. Using (19), (20) and (22), we obtain the cost function as

$$J\ C\ \ C_l\ U_l\ = -\left[\ C\ \ {}^T S\ Q\ C\ + \sum_{l=}^{n-} C_l\ {}^T (Q_l + S_l) C_l + \sum_{l=}^{n-} U_l^T R_l U_l \right]. \quad (23)$$

Now, the essential process of the optimal control of the systems is to find the wavelet coefficient vectors C , C_l , U_l in (23) to minimize J subject to the constraints (19) and (20), which is solved by using Lagrange multiplier method as [2].

$$J = J\ C\quad C_l\ U_l\ + \lambda^T G\ C\quad C_l\ U_l\ + \lambda^T G\ C\quad C_l\ U_l\ , \quad (24)$$

where the vectors λ , λ denote the unknown Lagrange multipliers, G , G are functions constructed by the constraints (19) and (20), respectively. Then, the necessary conditions for minimizing (24) are represented as

$$\frac{\partial}{\partial C} J = \quad \frac{\partial}{\partial C_l} J = \quad \frac{\partial}{\partial U_l} J = \quad \frac{\partial}{\partial \lambda} J = \quad \frac{\partial}{\partial \lambda} J = \ . \quad (25)$$

Solving the algebraic systems (25), we obtain the explicit formulas of the wavelet coefficient vectors of approximate solution and the optimal control, respectively,

$$C = \left[d\left(-R - f^T bf \right)^{-} d^T a - I \right]^{-} v ,$$

$$U_l = \left(f^T bf - R_l \right)^{-} d^T aC \ . \quad (26)$$

$$C_l = -fU_l , \quad (27)$$

where $a = Q$, $b = Q_l + S_l$, $d = R^T + A_r P_A + C @ B_r P_B$,

$$v = R^T + A_r P_A + C @ A_{dr} P_{Ad} K_{rl} ,$$

$$f = \left[R^T + A_r P_A + A_{dr} P_{Ad} D\ \tau\ + C \right]^{-} B_r P_B .$$

Comment 3: For each subinterval, we need to solve smaller the $q\ p$ algebraic systems with higher efficiency rather than larger $q\ p^{\ n}$ systems compared with [2].

IV. Numerical Experiment

In this section, in order to verify our approach, a numerical experiment is implemented. Consider the time delayed systems described by [2, 17].

$$\frac{dX\ t}{dt} = \left[\quad \right] X\ t + \left[- \quad - \right] X\ t - \quad + \left[\ \right] u\ t \quad \le t < \ ,$$

$$X\ t = \quad {}^T @ \quad - \quad \le t \le \ . \quad (28)$$

The optimal control of this delay systems is to minimize the cost function

$$J = -\int \left(X^T\ t \left[\quad \right] X\ t + u\quad t \right) dt . \quad (29)$$

For this optimal control of time delayed systems, the coefficient matrices are $A = \left[\quad \right]$, $A_d = \left[- \quad - \right]$, $B = \left[\ \right]$, $S = \mathbf{0}$, $Q = \left[\quad \right]$, $R =$, respectively, which simplify the computations of solving the algebraic systems (25).

We let the order of Legendre wavelet $p =$ and the decomposition level $n =$, respectively, and the parameter in (8) be $\sigma =$. The optimal control solution $U\ t$ is approximated by

$$U\ t = \sum_{k=} \sum_{l=} c_{k\ l} \phi_{k\ l}\ t \ .$$

Using operational matrices of the derivative, delay and product and the optimal control solution (26), we obtain Legendre wavelet coefficient $c_{k\ l}$ of the optimal control solution u as

U=[0.7552, 0.1033, 0; 0.8897, -0.0256, 0; 0.7309, -0.0662, 0; 0.3102, -0.1797, 0],

which is illustrated in Figure.1.

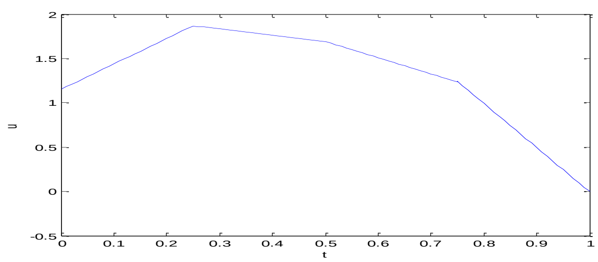

Figure.1. Optimal control solution by DLWG method

Similarly, the vector $X\ t$ is obtained. Then , we obtain the better optimal J = 2.0059 by substituting $X\ t$ and $U\ t$ into the cost function (6) compared with the results in [2, 17].

V. Conclusion

The DLWG approach is applied to the delay systems. The derivative, delay and product operational matrices with lower dimensions are implemented to calculate each term for the variational formulation. The optimal control of the delay systems is transformed to solve the algebraic systems. The method proposed in this paper decreases the computational complexity because of the lower dimensions and sparse operational matrices. Then the scheme and Lagrange multiplier method are applied to solving the example and the numerical solutions obtained show that this approach is effective. For high accuracy, the order p or the level n of resolution of Legendre wavelet should be increased.

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