Discrete $(\frac{G'}{G})$-expansion: A method used to get exact solution of FDDE (fractional differential-difference equation) linked with NLTL (non-linear transmission line)

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Abstract—Here, we have used the discrete $(\frac{G'}{G})$-expansion procedure with the derivative operator MR-L (modified Riemann-Liouville) and FCT (fractional complex transform) to find the exact/analytical solution of an electrical transmission line which is non-linear. Results include solutions for integer and fractional DDE. We consider two special cases of solutions: hyperbolic and trigonometric. Hyperbolic solutions indicate propagation of singular wave on the transmission line. Trigonometric solutions show propagation of complex wave.

Keywords— Differential-difference equations (DDEs), Discrete $(\frac{G'}{G})$-expansion method, Fractional complex transform (FCT), Fractional differential-difference equation (FDDE), Fractional differential equation (FDE), Non-linear transmission line (NLTL), Ordinary differential equation (ODE), Partial fractional differential equation (PFDE).

I. Introduction

Fractional differential equations are the subset of fractional calculus in mathematics that expands regular distinction to a complicated or non-integer order [1]. This has major influence on the various subjects, such as quantum mechanics, physics, control systems, non-linear analysis in applied mathematics and other areas. In physics and engineering, many equations [2-3] have recently been expanded into non-integer orders in order to provide new models. They are given with some computational, numerical and some solution methods since solving an FDE is difficult. Similarly DDE finds applications in areas like engineering physics, applied mathematics. Different applications of DDE require specific effective method [4-7] for solutions. Some interesting DDE model can be found in literature [8-10].

Over the years, researchers are using different analysis of transmission lines [11-12] and different numerical methods to find exact/analytical solutions of DDEs linked with non-linear waves and solitons [13-15]. In [13], exact solutions for the DDEs linked with the NLTL network uses the $(\frac{G'}{G})$-expansion process where as [14] uses fractional analysis of the same network with discrete tanh method. Similarly [15] reports a different electrical transmission line circuit employing the discrete $(\frac{G'}{G})$-expansion process using fractional analysis.

This paper extends the application of the discrete $(\frac{G'}{G})$-expansion procedure to find an exact solution for local FDDE (fractional differential-difference equation) of Toda network. For simplicity, it will first transform the FDE to ODE using the fractional complex transform (FCT) method and then develop the solution method.

Organisation of this paper will be the following. Section II will provide the general expansion of the above mentioned method. Section III will discuss the fractional complex transform and the procedures for transforming the fractional differential equation (FDE) into an ordinary differential equation (ODE). Section IV will explain how to derive the solution from transmission line network using the solution process. It will be followed by the evolutionary profile in section V.

II. General $\left(\frac{G'(\xi)}{G(\xi)}\right)$ −expansion metod

Fractional complex transform (FCT) is used as a gentle way of transforming FDE to ODE. FCT can be used to rewrite the equations in form of ODE in order to get the solution of FDE.

For a function,

$$\xi = \xi(t, x, x_1, x_2, \dots x_n),$$

Let

$$U(\xi) = \sum_{i=0}^{n} a_i \left[\frac{G'(\xi)}{G(\xi)}\right]^i$$

be a solution of certain ODE, where $G(\xi)$ satisfies

$$G''(\xi) + \lambda\, G'(\xi) + \mu\, G(\xi) = 0. \quad (1)$$

Here, $G'(\xi) = \frac{dG(\xi)}{d\xi}$

and by using homogenous balancing principal n can be determined.

Now for equation (1), the auxiliary equation (AE) can be written as,

$$m^2 + \lambda\, m + \mu = 0$$

which is satisfied by $m = \frac{-\lambda \pm \sqrt{\lambda^2 - 4\mu}}{2}$.

Then the solutions of equation (1) are

$$G(\xi) = \begin{cases} e^{\frac{-\lambda\xi}{2}}\left(c_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\,\xi\right) + c_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\,\xi\right)\right), & \text{for } \lambda^2 - 4\mu > 0 \\ e^{\frac{-\lambda\xi}{2}}\left(c_1 \cos\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}\,\xi\right) + c_2 \sin\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}\,\xi\right)\right), & \text{for } \lambda^2 - 4\mu < 0 \\ e^{\frac{-\lambda\xi}{2}}\left(c_1 + \xi\, c_2\right), & \text{for } \lambda^2 - 4\mu = 0 \end{cases}$$

with arbitrary constants c_2 and c_1.

Hence $\frac{G'(\xi)}{G(\xi)}$ can be summarized as,

$$\begin{cases} \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\left[\frac{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right], & \text{for } \lambda^2 - 4\mu > 0 \\ \frac{-\lambda}{2} + \frac{\sqrt{-\lambda^2+4\mu}}{2}\left[\frac{-c_1\sin\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi\right) + c_2\cos\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi\right) + c_2\sin\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi\right)}\right], & \text{for } \lambda^2 - 4\mu < 0 \\ \frac{-\lambda}{2} + \frac{c_1}{c_1 + c_2\xi}, & \text{for } \lambda^2 - 4\mu = 0 \end{cases} \quad (2)$$

III. FRACTIONAL COMPLEX TRANSFORM (FCT)

The key steps to establish the exact PFDE solution by using FCT are provided below.
Let's assume the PFDE

$$P(u_1, \dots, u_k, D_t^\alpha u_1, \dots, D_t^\alpha u_k, D_{x_1}^\alpha u_1, \dots, D_{x_1}^\alpha u_k, \dots, D_{x_n}^\alpha u_1, \dots, D_{x_n}^\alpha u_k, D_t^{2\alpha} u_1, \dots, D_t^{2\alpha} u_k, D_{x_1}^{2\alpha} u_1, \dots, D_{x_1}^{2\alpha} u_k, \dots) = 0 \quad (3)$$

with variables $(t, x, x_1, x_2, \dots x_n)$.
Where P is the polynomial function in u_i, fractional derivatives are in terms of different partial derivatives.
And $u_i = u_i(t, x, x_1, x_2, \dots x_n)$ for $i = 1,2, \dots k$.
For $i = 1,2, \dots, k$, $j = 1,2, \dots, n$, $D_{x_j}^\alpha u_i$ as given in (3) is usually callled modified Riemann-Liouville derivative of fractional (α) order of $u_i(x_j)$ w.r.t x_j.
With α order, Jumarie's MR-L derivative is defined as follows:

$$D_\xi^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1+\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}\,(f(\xi) - f(0))\,d\xi, & 0 < \alpha < 1 \\ (f^{(m)}(t))^{(\alpha - m)}, & m \geq 1 \end{cases} \quad (4)$$

$\eta = \frac{\xi^\alpha}{\Gamma(1+\alpha)}$ is the modified FCT usually utilised for some FDEs that only applicable for wave solution.

As FCT follows chain rule, applying this to the function u (t, x):

$$\frac{\partial^\alpha u_i}{\partial t^\alpha} = \sigma_s \frac{\partial u_i}{\partial s}, \quad s = t^\alpha$$

$$\frac{\partial^\beta u_i}{\partial x^\beta} = \sigma_x \frac{\partial u_i}{\partial X}, \quad X = x^\beta. \quad (5)$$

Where σ_s, σ_x are the fractal indices.
To make this calculation easy, with $0 < \alpha \leq 1$ the properties of MR-L derivative is chosen.
Thus the MR-L property

$$D_t^s\, t^s = \frac{\Gamma(1+s)}{\Gamma(1+s-\alpha)} t^{s-\alpha} \quad (6)$$

can also be utilised in modified FCT, $\eta = \frac{\xi^\alpha}{\Gamma(1+\alpha)}$.

Procedures to adopt for FDE to ODE transformation is presented below in different steps.
Step 1: It is assumed

$$u_i(t, x, x_1, x_2, \dots x_n) = U_i(\xi) \text{ with}$$
$$\xi = c\frac{t^\alpha}{\alpha} + k_1\frac{x^\alpha}{\alpha} + k_2\frac{y^\alpha}{\alpha} + \dots + \xi_0 \quad (7)$$

where c, k_1, k_2 are unevaluated constants.
Using this above mentioned transform along with equation (5), equation (3) can be transformed into ODE as below,

$$\tilde{p}\,(U_1, \dots, U_k, cD_\xi^1 U_1, \dots, cD_\xi^1 U_k, k_1 D_\xi^1 U_1, \dots, k_1 D_\xi^1 U_k, \dots, c^2 D_\xi^2 U_1, \dots, c^2 D_\xi^2 U_k, k_1^2 D_\xi^2 U_1, \dots k_1^2 D_\xi^2 U_k, \dots, k_n^2 D_\xi^2 U_1, \dots, k_n^2 D_\xi^2 U_k, \dots) = 0 \quad (8)$$

The general representation of the transformation for $i = 1,2,\ldots,k$ and $r = 1,2,\ldots,N$ can be written as,

$$D_t^{r\alpha} U_i\ (t,x,x_1,x_2,\ldots x_n) = c^r D_\xi^r\ U_i(\xi),$$
$$D_{x_1}^{r\alpha} U_i\ (t,x,x_1,x_2,\ldots x_n) = k_1^r D_\xi^r\ U_i(\xi),$$
$$\vdots$$
$$D_{x_n}^{r\alpha} U_i\ (t,x,x_1,x_2,\ldots x_n) = k_n^r D_\xi^r\ U_i(\xi),$$

Step 2: To express the solution of equation (8) in terms of $\frac{G'(\xi)}{G(\xi)}$, the polynomial function can be assumed as:

$$U(\xi) = \sum_{i=0}^{m_j} a_{j,i} \left[\frac{G'(\xi)}{G(\xi)}\right]^i,$$
$$j = 1,2\ldots,k \tag{9}$$

where $G(\xi)$ is the root of equation (1).

And $a_{j,i}(i = 0,1,\ldots m_j \text{ and } j = 1,2\ldots,k)$ is a constant which can be evaluated by homogenous balance between non-linear terms and highest-order derivatives appearing in equation (8) with the condition $a_{j,i} \neq 0$.

Step 3: After substituting equation (9) into equation (8), and using equation (1), the co-efficient of the terms containing different powers of $\frac{G'(\xi)}{G(\xi)}$ equating to zero a set of algebraic equations in terms of λ, μ, $a_{j,i}$ are obtained.

Step 4: Substituting the solutions of the set of algebraic equations obtained in step 3 in equation (2), an interesting type of exact solution of equation (3) is expected to be established. Equation (3) should be turned into ODE for $\alpha = 1$ value.

IV. Fractional Non-Linear LC Ladder Transmission Line Equation

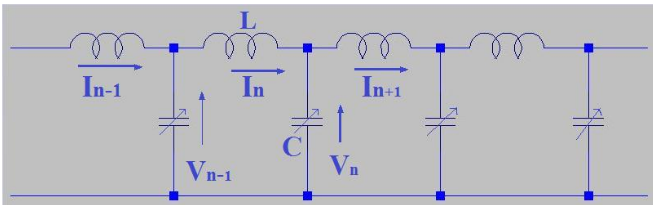


Figure-1. Non-linear TL network

The above figure is a non-linear transmission line with n identical unit cells. Each cell contains a linear inductors and non-linear capacitors which vary with voltage V_n applied to the nth cell. So it can be written as,

$$C(V_n) \approx C_0(1 - a V_n + bV_n^2) \tag{10}$$

Applying KCL in the above network, below DDE is obtained,

$$LC_0 \frac{d^2V_n}{dt^2} - aLC_0 \frac{d^2V_n^2}{dt^2} + b\ LC_0 \frac{d^2V_n^3}{dt^2} = V_{n+1} + V_{n-1} - 2\,V_n \tag{11}$$

The Taylor expansion for the right side of equation (11) can be written as,

$$\frac{d^2V_n}{dt^2} - a\frac{d^2V_n^2}{dt^2} + b\frac{d^2V_n^3}{dt^2} = \frac{2}{LC_0}\left(\frac{V_n''}{2!} + \frac{V_n''''}{4!} + \ldots\ldots\ldots\right)$$

where $V_n'' = \frac{d^2V_n}{dn^2}$.

Considering weakly dispersive limit of the discrete system of equation (11), the discrete index n may be assumed to be a continuous variable of x, especially if the voltage varies slowly from unit section to other. Considering δ is the dispersion between LC_0 pair, putting $x = n\delta$ and eliminating terms higher than δ^4, equation (11) can be written as,

$$\frac{d^2V_n}{dt^2} - a\frac{d^2V_n^2}{dt^2} + b\frac{d^2V_n^3}{dt^2} - \frac{1}{LC_0}\left(\frac{\delta^2 d^2V_n}{dx^2} + \frac{\delta^4 d^4V_n}{12\,dx^4}\right) = 0. \tag{12}$$

Transforming equation (12) into its continuum form and substituting the linear derivative operator by order α, equation (12) is reduced to,

$$D_t^{2\alpha}V_n - D_t^{2\alpha}aV_n^2 + D_t^{2\alpha}bV_n^3 - \frac{1}{LC_0}\left(\delta^2 D_x^{2\alpha}\,V_n + \frac{\delta^4}{12} D_x^{4\alpha}V_n\right) = 0 \tag{13}$$

Substituting $\xi = c\frac{t^\alpha}{\alpha} + k_1\frac{x^\alpha}{\alpha} + \xi_0$ into equation (13), we get,

$$c^2\frac{d^2V_n}{d\xi^2} - ac^2\frac{d^2V_n^2}{d\xi^2} + bc^2\frac{d^2V_n^3}{d\xi^2} - \frac{1}{LC_0}\left(\frac{\delta^2k_1^2d^2V_n}{d\xi^2} + \frac{\delta^4k_1^4d^4V_n}{12\,d\,\xi^4}\right) = 0$$

Writing $V_n(t,x) = U(\xi) = U$, the above equation can be written as,

$$c^2D_\xi^2U - ac^2D_\xi^2U^2 + bc^2D_\xi^2U^3 - \frac{1}{LC_0}\left(\delta^2k_1^2D_\xi^2\ U + \frac{\delta^4k_1^4}{12}D_\xi^4\ U\right) = 0 \tag{14}$$

Assuming integration constants to be zero, integrating twice the above equation (and considering $k_1 = k$) we get,

$$c^2(U - aU^2 + bU^3) - \frac{\delta^2k^2}{LC_0}U - \frac{\delta^4k^4}{12\,LC_0}U'' = 0. \tag{15}$$

Assume that the exact solution of above equation can be written as,

$$U(\xi) = \textstyle\sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i. \tag{16}$$

$m = 1$ is obtained by homogenous balance between the highest order derivative and non-linear terms.

Substituting $U = a_0 + a_1\left(\frac{G'}{G}\right)$ in equation (15) and equating the coefficients of different powers of $\left(\frac{G'}{G}\right)$ to zero, the following equations are obtained.

$$\left\{\begin{array}{l} \left(\frac{G'}{G}\right)^0 : c^2a_0 - ac^2a_0^2 + bc^2a_0^3 - \frac{\delta^2k^2}{\mathrm{LC}_0}a_0 - \frac{\delta^4k^4}{12\,\mathrm{LC}_0}a_1\lambda\mu = 0 \\ \left(\frac{G'}{G}\right)^1 : c^2a_1 - 2aa_0a_1c^2 + 3bc^2a_1a_0^2 - \frac{\delta^2k^2}{\mathrm{LC}_0}a_1 - \\ \qquad \frac{\delta^4k^4}{12\,\mathrm{LC}_0}(a_1\lambda^2 + 2a_1\mu = 0 \\ \left(\frac{G'}{G}\right)^2 : -aa_1^2c^2 + 3bc^2a_0a_1^2 - \frac{\delta^4k^4}{12\,\mathrm{LC}_0}3a_1\lambda = 0 \\ (17) \\ \left(\frac{G'}{G}\right)^3 : bc^2a_1^3 - \frac{\delta^4k^4}{12\,\mathrm{LC}_0}2a_1 = 0 \end{array}\right. \tag{17}$$

The constants a_0, a_1, c, k are obtained by solving the equations (17).

Case-1:

For $\lambda^2 - 4\mu > 0$, $\xi = c\frac{t^\alpha}{\alpha} + k_1\frac{x^\alpha}{\alpha} + \xi_0$ and $0 < \propto \leq 1$

$$\left.\begin{array}{ll} a_0 = \frac{a(\lambda+\sqrt{\lambda^2-4\mu})}{3b\sqrt{\lambda^2-4\mu}}, & a_1 = \frac{2a}{3b\sqrt{\lambda^2-4\mu}} \\ k = \pm\frac{2a\sqrt{6}}{\delta\sqrt{-(2a^2-9b)(\lambda^2-4\mu)}}, & c = \frac{6a\sqrt{\frac{6b}{LC_0}}}{(2a^2-9b)\sqrt{\lambda^2-4\mu}} \end{array}\right\}$$

For which the solution will be,

$$V_n(t,x) = U =$$

$$a_0 + a_1\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\left[\frac{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right]\right) \tag{18}$$

Case-2:

For $\lambda^2 - 4\mu < 0$, $\xi = c\frac{t^\alpha}{\alpha} + k_1\frac{x^\alpha}{\alpha} + \xi_0$ and $0 < \propto \leq 1$

$$\left.\begin{array}{ll} a_0 = \frac{-a(\lambda-\sqrt{\lambda^2-4\mu}}{3b\sqrt{-\lambda^2+4\mu}}, & a_1 = \frac{-2a}{3b\sqrt{-\lambda^2+4\mu}} \\ k = \pm\frac{2a\sqrt{6}}{\delta\sqrt{-(2a^2-9b)(-\lambda^2+4\mu)}}, & c = \frac{6a\sqrt{\frac{6b}{LC_0}}}{(2a^2-9b)\sqrt{-\lambda^2+4\mu}} \end{array}\right\}$$

For which the solution will be,

$$V_n(t,x) = U =$$

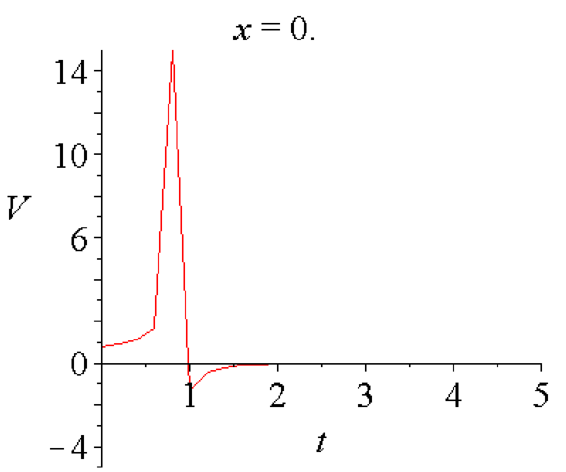
$$a_0 + a_1\left(\frac{-\lambda}{2} + \frac{\sqrt{-\lambda^2+4\mu}}{2}\left[\frac{-c_1\sin\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi\right) + c_2\cos\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi\right) + c_2\sin\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi\right)}\right]\right) \tag{19}$$

Case-3: As the coefficients a_0 and a_1 are undefined, $\lambda^2 - 4\mu = 0$ does not have any solutions.

V. Results and Discussion

To understand the implications of the equations derived above we consider hyperbolic and trigonometric solutions only. For the third rational case, a_0 and a_1 become undetermined because their denominator contains $(\lambda^2 - 4\mu)$, which is zero. This makes the solution undetermined and hence we exclude this. For both these (hyperbolic and trigonometric) cases we use $C_1 = 0$, $C_2 = 1$, $a = 0.21$, $b = 0.197$, $\frac{1}{LC_0} = 3$, $\delta = 1$, $\mu = 3$. But we take λ as 4 in hyperbolic and and 1 in trigonometric case.

(a)



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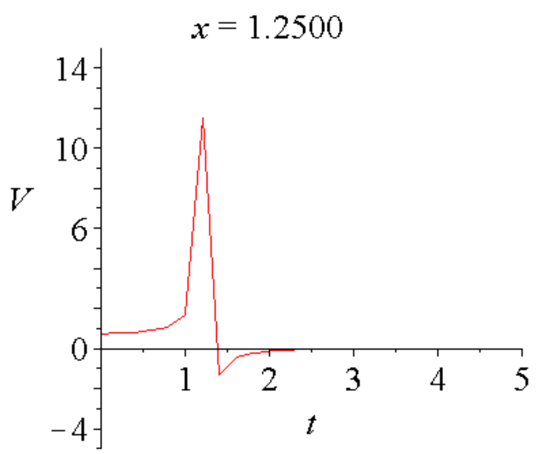

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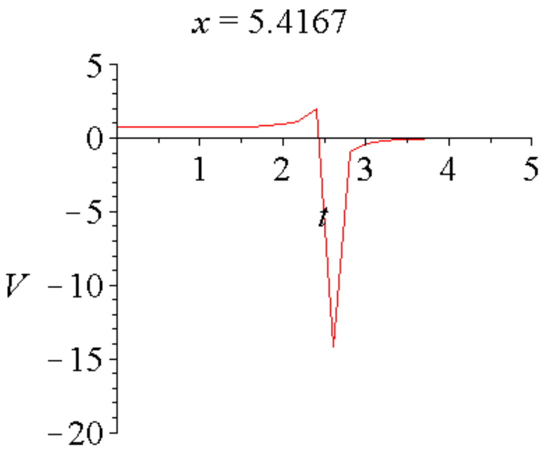

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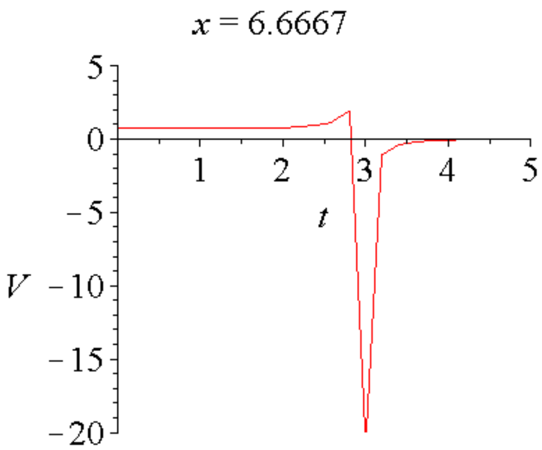
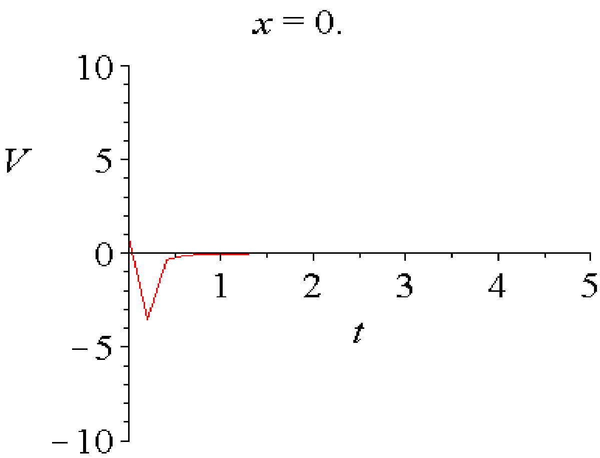


Figure-2. Evolution of voltage with time $V_n(t,x)$ for case-1 (hyperbolic) at different locations on TL for $\alpha = 1$

(a)



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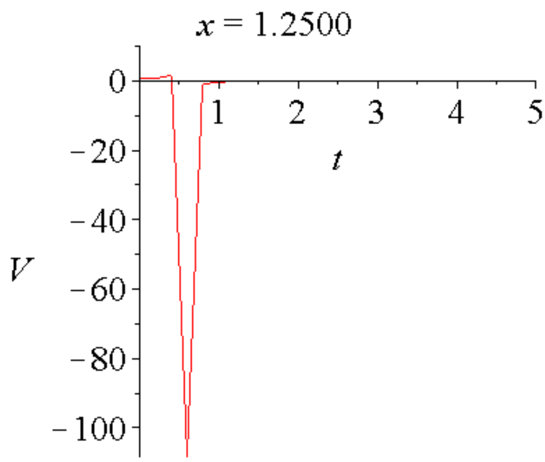

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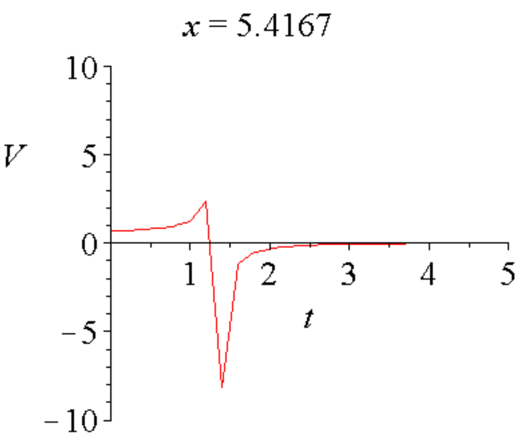

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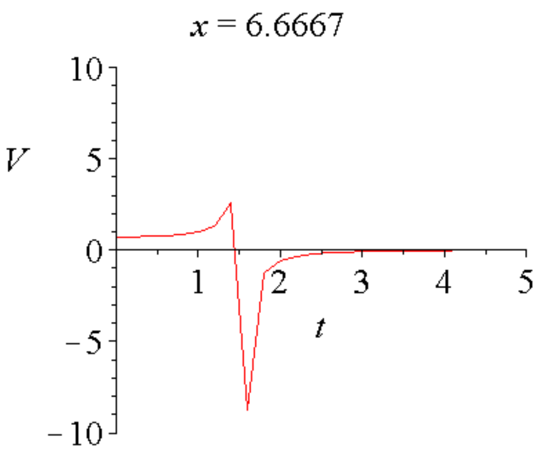

Figure-3. Evolution of voltage with time $V_n(t,x)$ for case-1 (hyperbolic) at different locations on TL for $\alpha = 0.5$

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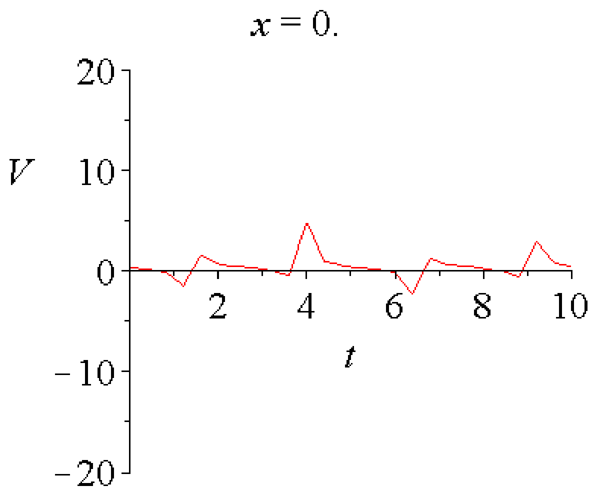
(a)



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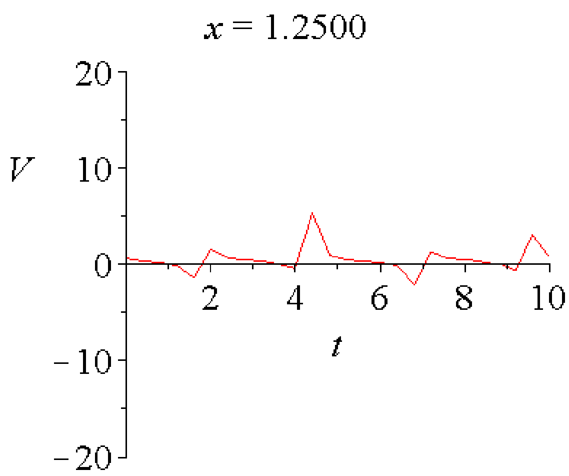

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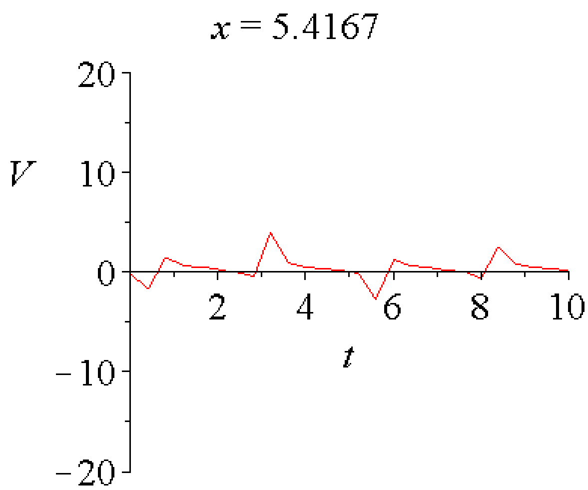

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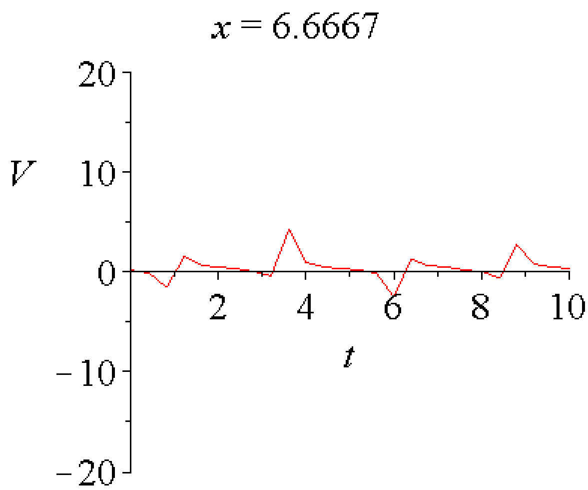
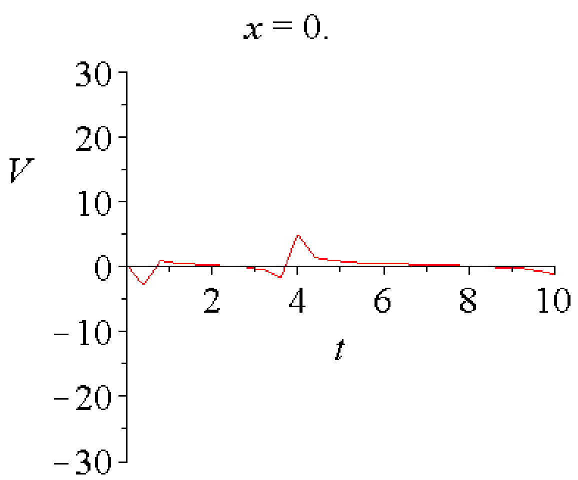


Figure-4. Evolution of voltage with time $V_n(t,x)$ for case-2 (trigonometric) at different locations on TL for $\alpha = 1$

(a)



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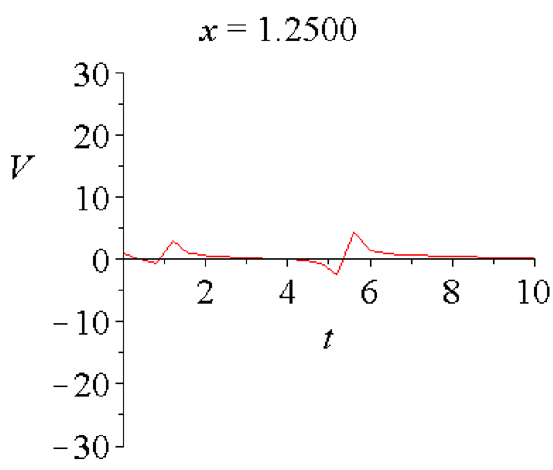

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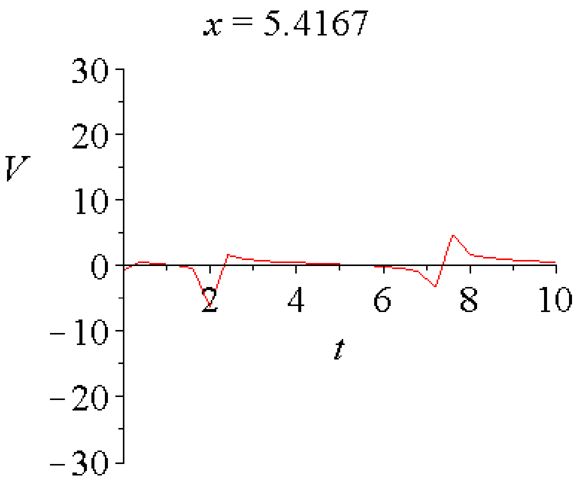

(d)

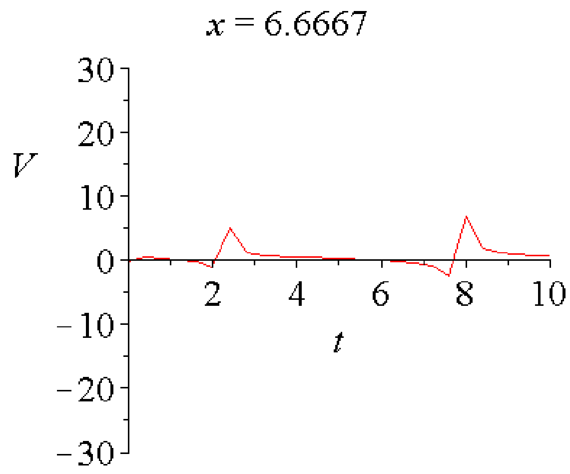

Figure-5. Evolution of voltage with time $V_n(t,x)$ for case-2 (trigonometric) at different locations on TL for $\alpha = 0.5$

Figure 2(a), (b), (c), (d) shows the time evolution of voltage for $\alpha = 1$ at different positions on the transmission line. It shows evolution of singular voltage indicating bright ((a), (b)) and dark ((c), (d)) waves. Figure 3(a), (b), (c), (d) repeats the same process for $\alpha = 0.5$. In this case voltage generation is accelerated and in all locations under consideration the nature of the generated wave is dark.

For the trigonometric case with $\alpha = 1$ the evolution of wave is shown in figure 4(a), (b), (c), (d) at same locations as for the hyperbolic case. Unlike in the hyperbolic case this indicates the evolution of a complex wave with time which oscillates aperiodically between positive and negative voltages. The figure 5(a), (b), (c), (d) repeats it for $\alpha = 0.5$. In figure 4 the voltage generation is random in nature. In figure 5 the voltage generation is more ordered and accelerate as one moves along the transmission line.

VI. Conclusion

Using appropriate parameters, we were able to obtain three exact solutions as we can see in case 1,2,3 are hyperbolic, trigonometric, and rational respectively, in order to achieve our goal of establishing exact solution of FDDE of the given network. Finally we conclude by stating that, for problem modelling and the resolution of un-differentiable numerical in fractional order time-space, the fractional calculus is an useful tool. Using this tool is has been changed how the voltage evolves with time along a non-linear transmission line under different modelling conditions. This will find application in non-linear discrete systems like all electrical switches and periodic non-linear structures. In future this model can be expanded to consider coupled non-linear transmission line and planar non-linear TL.

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