## Properties and Applications of a Tan–G Family of "Adaptive Functions"

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Abstract— In this note we study some properties of an new TAN–G class of trigonometric cumulative distribution functions proposed by Souza, O. Junior, de Brito, Chesneau, Fernandes and Ferreira [1]. We consider also modified families of "adaptive functions" with "polynomial variable transfer" with applications to the Antenna–feeder Analysis. We study the "saturation" - d in the Hausdorff sense for some special cases of this family. The article discusses only model aspects of the possible application that this new family finds in the above area. The models are very sensitive to the coefficients of the polynomial (respectively the number and location of its zeros). This makes these models attractive for simulations. Numerical examples, illustrating our results using CAS MATHEMATICA are given.

Keywords— Tan–G generalized "adaptive function", Hausdorff distance, upper and lower bounds, modified families of functions with "polynomial variable transfer", radiate pattern.

## I. INTRODUCTION

In the last few years, there have been serious studies in the literature related to the proposed general classes of

trigonometric distributions. In particular, the SIN-G and COS-G families (see, [6]-[7]). We will note that the "sine potential correction" can be used to construct families of adaptive functions for modeling processes in the field of Growth Theory, Antenna-feeder Analysis, Signal Theory and Signal Processing.

In [2] the authors proposed the following new Log–Logistic– Tan generalized families of cumulative distributions functions:

$$M_{1}(t) = 1 - \left(1 + s^{-c} \left(\tan\left(\frac{\pi}{2} \left(1 - e^{-t^{b}}\right)^{\alpha}\right)\right)^{t}\right)^{-1}$$
(1)

for  $t > 0, s > 0, c > 0, b > 0, \alpha > 0$ ;

$$M_{2}(t) = 1 - \left(1 + s^{-c} \left(\tan\left(\frac{\pi}{2} \left(1 - e^{-at^{2}}\right)^{\alpha}\right)\right)^{c}\right)^{-1}$$
(2)  
for  $t > 0, s > 0, c > 0, a > 0, \alpha > 0;$ 

$$M_{3}(t) = 1 - \left(1 + s^{-c} \left(\tan\left(\frac{\pi}{2}(1 - e^{-bt})^{\alpha}\right)\right)^{c}\right)^{-1}$$
(3)  
for  $t > 0, s > 0, c > 0, b > 0, \alpha > 0.$ 

Various modifications of this "powerful" class of functions have been proposed and studied by a number of researchers.

Questions related to the synthesis and analysis of transfer functions, radiation diagrams and filters characteristics are elaborated in detail in [3]–[4].

In [1] the authors proposed a new TAN–G class of trigonometric cumulative distribution functions of the form:

$$G(t) = \tan\left(\frac{\pi}{4}F_1(t)\right)$$
where  $F_1(t)$  is the baseline distribution. (4)

Let us consider the special case from the TAN–G class (4) [5]:

$$M_{4}(t) = \tan\left(\frac{\pi}{4}(1 - e^{-bt})\right)$$
(5)  
for  $t > 0$ ;  $b > 0$ .

**Definition** [8], [9]. The Hausdorff distance (the H-distance) [8]  $\rho(f, g)$  between two interval functions f, g on  $\Omega \subseteq \mathbb{R}$ , is the distance between their completed graphs F(f) and F(g) considered as closed subsets of  $\Omega \times \mathbb{R}$ . More precisely,

$$\rho(f,g) = \max\{\sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B||\}, \quad (6)$$

wherein ||. || is any norm in  $\mathbb{R}^2$ , e.g. the maximum norm ||(t, x)|| = max{|t|, |x|}; hence the distance between the points  $A = (t_A, x_A), B = (t_B, x_B)$  in  $\mathbb{R}^2$  is ||A - B|| =max ( $|t_A - t_B|, |x_A - x_B|$ ). **Definition**. *The Heaviside step–function is defined by:* 

$$h_{t_0}(t) = \begin{cases} 0, & \text{if } t < t_0, \\ [0,1], & \text{if } t = t_0, \\ 1, & \text{if } t > t_0. \end{cases}$$

The basic approaches for approximation of functions and point set of the plane by algebraic and trigonometric polynomials in respect to Hausdorff distance are connected to the work and achievements of Bl. Sendov [9] who established a Bulgarian school in Approximation theory.

In this article we study the Hausdorff approximation of the  $h_{t_0}(t)$  by special family  $M_4(t)$  (5).

The detailed study of the new family (5) in terms of properties, saturation, etc. is the subject of the proposed article. Model aspects of the possible application, which it finds in the above directions, are considered.

Particular attention is paid to the possibility of generating generalized adaptive functions with the so-called "polynomial transfer" and the possibilities that are found for simulating radiation diagrams and filter characteristics.

For some modelling and approximation problems, see [10]–[19].

### II. MAIN RESULTS

A. Hausdorff approximation of the Heaviside step function by  $M_4(t)$ 

Let  $t_0$  is the solution of the nonlinear equation

$$M_4(t_0) - \frac{1}{2} = 0.$$

Evidently, for the "median level" we have

$$t_0 = -\frac{1}{b} \ln\left(1 - \frac{4}{\pi}\arctan(0.5)\right).$$

For the "saturation" - d in the Hausdorff sense we find

$$F(d) := M_4(t_0 + d) - 1 + d = 0,$$
(7)

We examine the following approximation of F(d) as we use the function

$$H(d): = -\frac{1}{2} + \left(1 + \frac{5\pi b}{16} \left(1 - \frac{4}{\pi} \arctan(0.5)\right)\right) d$$
  
$$\approx -\frac{1}{2} + (1 + 0.4021886b) d$$
(8)

Indeed from Taylor expansion, we get  $F(d) - H(d) = O(d^2)$ . The functions H(d) and F(d) are increasing.

This means that H(d) approximates F(d) with  $d \to 0$  as  $O(d^2)$  (see Figure 1.).

Let A = 2.1(1 + 0.4021886b) and  $d_l = \frac{1}{A}$ ;  $d_r = \frac{\ln A}{A}$ 

then we see that for b > 1:  $H(d_l) < 0$ ;  $H(d_r) > 0$ .



Fig. 1. The functions *F* and *H* for b = 60.



Fig. 2. The function  $M_4$  for: a) b = 60;  $t_0 = 0.0148736$ ; Hausdorff distance d = 0.0442484;  $d_l = 0.0189481$ ;  $d_r = 0.0751492$ ; b) b = 100;  $t_0 = 0.00892414$ ; Hausdorff distance d = 0.0303793;  $d_l = 0.0115527$ ;  $d_r = 0.0515349$ .

Thus, we prove the following theorem for upper and lower estimates for the Hausdorff approximation d.

**Theorem 1.** For the Hausdorff distance d between shifted Heaviside function  $h_{t_0}(t)$  and the function  $M_4(t)$  the following inequalities hold true:

$$d_l := \frac{1}{A} < d < \frac{\ln A}{A} := d_r. \tag{9}$$

B. Numerical experiments

The family  $M_4(t)$  for

a) b = 60;  $t_0 = 0.0148736$ ; Hausdorff distance d = 0.0442484;  $d_l = 0.0189481$ ;  $d_r = 0.0751492$ ; b) b = 100;  $t_0 = 0.00892414$ ; Hausdorff distance d = 0.0303793;  $d_l = 0.0115527$ ;  $d_r = 0.0515349$ is plotted on Fig. 2.

The investigation of the characteristic "saturation" - d is important.

We note that the cumulative distribution function  $M_4(t)$  has been widely used in survival and reliability analysis.

## C. A family of recurrence generated functions

We construct a family of recurrence generated functions by

$$\gamma_{i+1}(t) = \tan\left(\frac{\pi}{4} \left(1 - e^{-b(t+\gamma_i(t))}\right)\right)$$
  
 $i = 0, 1, 2, ...$ 
(10)

with

 $\gamma_0(t) = M_4(t); \ \gamma_0(0) = 0.$ 

The recurrence generated:  $\gamma_0(t)$ ,  $\gamma_1(t)$ ,  $\gamma_2(t)$ .  $\gamma_3(t)$  and  $\gamma_4(t)$  from family (10) for fixed b = 3.4 are visualized on Fig. 3. This demonstrates relatively fast saturation (in the Hausdorff sense), and this is important if consumers use a fixed element of the specified range as a model to minimize side emission in the context of the discussions in Section D. The new family (10) may find application in the field of Neural Networks.



Fig. 3. The graphics of recurrence generated adaptive functions:  $\gamma_0$  (dashed),  $\gamma_1$  (red),  $\gamma_2$  (green),  $\gamma_3$  (orange) and  $\gamma_4$  (thick) for fixed b = 3.4.

## *D.* A new class of "adaptive function" with "polynomial variable transfer"

Formally, we define the following "adaptive function" with "polynomial variable transfer":

$$M_4^*(t) = \tan\left(\frac{\pi}{4} \left(1 - e^{-b|f(t)|}\right)\right),$$
  

$$f(t) = \sum_{i=0}^n a_i t^i, \ a_0 = 0.$$
(11)

**Example 1**. Consider the adaptive function

$$M_4^*(t) = \tan\left(\frac{\pi}{4} \left(1 - e^{-b|t(1-t)(2-t)(3-t)(4-t)|}\right)\right)$$
  
For  $b = 0.015 M_4^*(t)$  is depicted on Fig. 4.



Fig. 4. The adaptive function  $M_4^*(t)$  for b = 0.015 (A typical filter characteristic).

**Example 2**. Let  $t = r\cos\theta + c$ , where  $\theta$  is the azimuthal angle. Then, typical emitting charts using  $M_4^*(\theta)$  for

a)  $n = 3, b = 1.05, a_0 = 0, a_1 = -0.9, a_2 = 0.3, a_3 = 2.7, r = 2, c = -0.8;$ 

b)  $n = 5, b = 1.05, a_0 = 0, a_1 = -1.9, a_2 = -0.3, a_3 = 3.7, a_4 = -0.1, a_5 = -0.85, r = -1.09, c = -0.7$ 

are plotted on Fig. 5-Fig. 6.



Fig. 5. A typical radiation pattern using  $M_4^*(\theta)$  for  $n = 3, b = 1.05, a_0 = 0, a_1 = -0.9, a_2 = 0.3, a_3 = 2.7, r = 2, c = -0.8$ in  $(-\frac{\pi}{2}, \frac{\pi}{2})$ .



Fig. 6. A typical radiation pattern using  $M_4^*(\theta)$  for  $n = 5, b = 1.05, a_0 = 0, a_1 = -1.9, a_2 = -0.3, a_3 = 3.7, a_4 = -0.1, a_5 = -0.85, r = -1.09, c = -0.7$  in  $(-\pi, \pi)$ ).

From the attached animations (figures 5-6, see also figure 7 in Appendix) it can be seen that the diagrams are very sensitive to the coefficients of the polynomial (respectively the number and location of its zeros). This makes these models attractive for simulations.

### III. CONCLUDING REMARKS

The reader may formulate the corresponding approximation problem - investigation on the "saturation" in the Hausdorff sense by using the new models  $M_5(t)$  and  $M_6(t)$  following the ideas given in this article.

The basic problems considered in [10]–[11] are - approximation of functions and point sets by algebraic and trigonometric polynomials in Hausdorff metric as well as their applications in the field of antenna-feeder technique, analysis and synthesis of antenna patterns and filters, noise minimization by suitable approximation of impulse functions.

We will explicitly note that the new generalized families of adaptive functions with polynomial transfer considered at this stage can be used successfully only for simulations of antenna factors, due to the presence of many free parameters - the coefficients  $a_i$  of the polynomial f(t).

It is well known that not always such adaptive functions can be realized in practice as antenna factors.

In this sense, the specialists working in this scientific field have a say.

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## IV. APPENDIX

We will explicitly note that the reader may consider other classes of the TAN–G family (4).

For example, the TAN–G family of cumulative functions with a Rayleigh correction can be defined as follows:

$$M_{5}(t) = \tan\left(\frac{\pi}{4}\left(1 - e^{-bt^{2}}\right)\right)$$
(12)  
for  $t > 0$ :  $h > 0$ 

Following the ideas presented in this article, we can define the following new adaptive function with "polynomial variable transfer":

$$M_5^*(t) = \tan\left(\frac{\pi}{4}\left(1 - e^{-bf^2(t)}\right)\right),$$
  

$$f(t) = \sum_{i=0}^n a_i t^i, \ a_0 = 0.$$
(13)

Another example. A possible extension of the TAN–G family is the following class:

$$M_6(t) = \tan\left(\frac{\pi}{4}\left(1 - e^{-bt^a}\right)^c\right)$$
(14)  
for  $t > 0$ ;  $b > 0$ ;  $a > 0$ ;  $c > 0$ .

The corresponding adaptive function with "polynomial variable transfer" is of the type:

$$M_6^*(t) = \tan\left(\frac{\pi}{4} \left(1 - e^{-bf^a(t)}\right)^c\right),$$
  

$$f(t) = \sum_{i=0}^n a_i t^i, \ a_0 = 0.$$
(15)

**Example 3**. A typical radiation pattern using  $M_6^*(\theta)$  for  $n = 3, b = 1.9, a = 2, c = 1, a_0 = 0, a_1 = -0.99, a_2 = 0.4, a_3 = 2.5, r = 1.6, c = -0.7$ ; is plotted on Fig. 7.



Fig. 7. A typical radiation pattern using  $M_6^*(\theta)$  for  $n = 3, b = 1.9, a = 2, c = 1, a_0 = 0, a_1 = -0.99, a_2 = 0.4, a_3 = 2.5, r = 1.6, c = -0.7.$ 

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## Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Nikolay Kyurkchiev carried out the Theorem 1, Appendix and Concluding remarks.

Anton Iliev conducted the Example 1 and carried out Introduction.

Asen Rahnev has organized and executed the Examples 2 and carried out Introduction.

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