

$$b_0 C_{uyy}(n, m) + b_1 C_{uyy}(n+1, m+1) + \dots + b_q C_{uyy}(n+q, m+q) \quad (12)$$

By stacking (12) for several values of n and m ranging from $-\psi$ to ψ where ψ denotes the range of the third order cumulants to be used, the system in (12) can be expressed in matrix format as follows

$$\underline{f} = -C_{3y} \underline{a}_p + C_{uyy} \underline{b}_q \quad (13)$$

The cumulants at 0 lag (i.e., $n = m = 0$), are contained in the vector \underline{f} . The coefficients of the ARMA (p, q) process are contained in the vectors \underline{a}_p and \underline{b}_q , in (1). The entries of the matrix C_{3y} represent the cumulants of the output sequence whereas the entries of the matrix C_{uyy} represent the cross-cumulants of the input and output data sequences.

Equation (13) can be written as

$$\underline{f} = C_{pq} \underline{\theta} \quad (14)$$

where C_{pq} is a composite data matrix contains the cumulants of the output sequence and the cross-cumulants of the input and output sequences

$$C_{pq} = [C_{3y} \quad C_{uyy}] \quad (15)$$

and $\underline{\theta}$ is the ARMA coefficients vector

$$\underline{\theta} = [-\underline{a} \quad \underline{b}]^T \quad (16)$$

Notice that the columns in the linear statistical model in (14) are linearly independent. Hence, we can multiply both sides of (14) by $(C_{pq})^T$, then

$$(C_{pq})^T \underline{f} = (C_{pq})^T C_{pq} \underline{\theta} \quad (17)$$

or

$$(C_{pq})^T C_{pq} \underline{\theta} = (C_{pq})^T \underline{f} \quad (18)$$

The Gram matrix for the data matrix C_{pq} is

$$G = (C_{pq})^T C_{pq} \quad (19)$$

Substituting (19) into (18) yields

$$G \underline{\theta} = (C_{pq})^T \underline{f} \quad (20)$$

or

$$G \underline{\theta} = \underline{f}_c \quad (21)$$

where $\underline{f}_c = (C_{pq})^T \underline{f}$.

Now, a general matrix $G (m \times n)$ can be decomposed as [15]

$$G = QR \quad (22)$$

where $Q (m \times m)$ is a unitary matrix and $R (m \times n)$ is an upper triangular matrix. If G is square, and the elements of Q being complex, then Q is unitary, ($Q^H = Q^{-1}$) [16], and

$$Q^H Q = I \quad (23)$$

The notation $(\cdot)^H$ denotes the complex conjugate transpose; i.e., $(Q^T)^*$ as $(Q)^H$, Q^{-1} is the matrix inverse, and I is the identity matrix. If Q has real elements and

$$Q^T Q = I \quad (24)$$

then Q is said to be an orthogonal matrix. Substituting (22) into (21), we obtain

$$(QR) \underline{\theta} = \underline{f}_c \quad (25)$$

Multiplying both sides of (25) by $(QR)^T$ yields,

$$(QR)^T (QR) \underline{\theta} = (QR)^T \underline{f}_c \quad (26)$$

Simplifying (26),

$$[R^T Q^T QR] \underline{\theta} = R^T Q^T \underline{f}_c \quad (27)$$

Using (24), i.e., ($Q^T Q = I$), then

$$[R^T R] \underline{\theta} = R^T Q^T \underline{f}_c \quad (28)$$

Multiply both sides of (28) by $[R^T R]^{-1}$,

$$[R^T R]^{-1} [R^T R] \underline{\theta} = [R^T R]^{-1} R^T Q^T \underline{f}_c \quad (29)$$

Simplifying (29),

$$\underline{\theta} = R^{-1} [R^{-T} R^T] Q^T \underline{f}_c$$

or

$$\underline{\theta} = R^{-1} Q^T \underline{f}_c \quad (30)$$

A. Computing the QR-Decomposition

The matrix G may be written as the product of an orthogonal matrix, Q , and an upper triangular matrix, R as follows:

$$G = [\underline{g}_1 \quad \underline{g}_2 \quad \dots \quad \underline{g}_n] =$$

$$[\underline{q}_1 \quad \underline{q}_2 \quad \dots \quad \underline{q}_n] \begin{bmatrix} r_{1,1} & r_{1,2} & \dots & r_{1,n} \\ 0 & r_{2,2} & \dots & r_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r_{n,n} \end{bmatrix} \quad (31)$$

The vectors $[\underline{g}_1 \quad \underline{g}_2 \quad \dots \quad \underline{g}_n]$ are linearly independent. The

vectors $[\underline{q}_1 \quad \underline{q}_2 \quad \dots \quad \underline{q}_n]$ are orthonormal vectors; that is,

$$\|\underline{q}_i\| = 1 ; \quad \underline{q}_i^T \underline{q}_j = 0 \quad \text{if } i \neq j \quad (32)$$

Since R has nonzero diagonal elements, then it is nonsingular matrix. This decomposition represents the n^{th} column of G by linear combination of orthogonal columns:

$$\underline{g}_n = \sum_{k=1}^n \underline{q}_k r_{kn} \quad (33)$$

One of the methods to compute the QR decomposition is the Gram-Schmidt and Householder algorithms [17].

The Gram-Schmidt algorithm begins as follows:

$$\begin{aligned} \underline{x}_1 &= \underline{g}_1; \\ \underline{q}_1 &= \frac{\underline{x}_1}{\|\underline{x}_1\|} \end{aligned} \quad (34)$$

Now, we construct \underline{x}_2 as follows:

$$\begin{aligned} \underline{x}_2 &= \underline{g}_2 - \frac{\underline{g}_2^T \underline{x}_1}{\|\underline{x}_1\|^2} \underline{x}_1 = \underline{g}_2 - (\underline{g}_2 \bullet \underline{q}_1) \underline{q}_1; \\ \underline{q}_2 &= \frac{\underline{x}_2}{\|\underline{x}_2\|} \end{aligned} \quad (35)$$

Continuing this procedure, each new vector \underline{x}_k is generated as follows:

$$\begin{aligned} \underline{x}_k &= \underline{g}_k - \sum_{i=1}^{k-1} (\underline{g}_k \bullet \underline{q}_i) \underline{q}_i; \\ \underline{q}_k &= \frac{\underline{x}_k}{\|\underline{x}_k\|} \end{aligned} \quad (36)$$

Note that $\|\cdot\|$ is the L_2 norm.

The elements for the R matrix are computed as follows:

$$\begin{aligned} r_{1,k} &= \underline{q}_1 \bullet \underline{g}_k \\ r_{2,k} &= \underline{q}_2 \bullet \underline{g}_k \\ &\vdots \\ r_{k-1,k} &= \underline{q}_{k-1} \bullet \underline{g}_k \end{aligned} \quad (37)$$

B. ARMA(p,q) Modeling

Now, the only observed data is the output sequence, $y(n)$. However, the excitation sequence, $u(n)$, is necessary to calculate the cross-cumulants, C_{uy} . Therefore, an estimate for the excitation sequence, $u(n)$, was obtained using the methods in [18, 19]. The method models the observed output data by a high order AR process. Thus, the system in (1) could be rewritten as

$$\hat{u}(n) = \sum_{i=0}^M \eta_i y(n-i) \quad (38)$$

where η_i are the coefficients of the high AR process and are estimated as follows.

$$\eta = \left[\frac{1}{N+1} \sum_{k=0}^N \underline{\Phi}(k) \underline{\Phi}(k)^T \right]^{-1} \left[\frac{1}{N+1} \sum_{k=0}^N \underline{\Phi}(k) y(k) \right] \quad (39)$$

with

$$\underline{\Phi}(k) = [-y(k-1) \ -y(k-2) \ - \dots \ -y(k-J)]^T \quad (40)$$

and J is the order of the high AR process. Using $\hat{u}(n)$ in the place of $u(n)$, the third order cross-cumulants can be calculated and ARMA coefficients can be estimated.

IV. RTESULTS AND DISCUSSIONS

Simulation studies have been performed to examine the proposed ARMA model coefficients estimation using the QRD-TOC algorithm. Several experiments were simulated at different signal-to-noise ratio (SNR) on the output sequence. A zero-mean, independent and identically distributed (i.i.d), and exponentially distributed non-Gaussian process was generated at the input. The results of the proposed method were compared with well-known procedures such as the Residual Time Series (RTS) and the Q-slice algorithm (QSA) methods at different levels of SNR on the output. The commands *armarts* and *armaqs* were used from the *Higher-Order Spectral Analysis Toolbox User's Guide* [20] to estimate the ARMA coefficients using the RTS and the QSA methods, respectively. All simulations were implemented and taken as the average of 100 Monte Carlo runs. The computations were performed in MATLAB tools. The data length of the sequence was taken to be $N=1500$ points for each experiment.

Example 1. Data were drawn using the ARMA(2, 2) model

$$y(n) + 0.3y(n-1) + 0.25y(n-2) = u(n) + 0.95u(n-1) + 0.65u(n-2) \quad (41)$$

This is an ARMA (2,2). It has two poles and two zeros. The poles are located at $-0.15 \pm j0.477$. The zeros are located at $-0.475 \pm j0.6514$. The sequence $y(n)$ is observed in additive Gaussian noise $s(n) = y(n) + w(n)$. A zero-mean, non-Gaussian distribution (namely exponential distribution) was used to generate the input sequence. Then, the input signal was driven through the system in (41). The output sequence was corrupted with additive Gaussian noise at SNR of 10dB on the output sequence. The, the cumulant matrix C_{pq} in (15) was built. The method in [18, 19] was used to estimate the input sequence. The experiment estimated the ARMA coefficients using the RTS, the QSA, and the proposed QRD-TOC methods. Table I presents the results of 100 Monte Carlo simulations for the RTS, QSA, and the proposed QRD-TOC algorithms at SNR of 10 dB.

Table 1. True and estimated ARMA (2, 2) model coefficients in Example 1, $a(0)=1, b(0)=1$.

	True	RTS	QSA	QRD-TOC
a(1)	0.3	0.2560	-0.5811	0.2984
a(2)	0.25	0.2608	0.2751	0.2456
b(1)	0.95	1.0012	0.7311	0.9505
b(2)	0.65	0.6196	1.3022	0.6464

Example 2. Data were drawn using the ARMA(6, 4) model

$$y(n) + 0.7907y(n - 1) + 0.042y(n - 2) - 0.556y(n - 3) - 0.0247y(n - 4) + 0.385y(n - 5) + 0.303y(n - 6) = u(n) + 0.345u(n - 1) + 0.53u(n - 2) + 0.399u(n - 3) + 0.814u(n - 4) \quad (42)$$

This model has six poles and four zeros, ARMA (6,4). The poles are located at $0.7102 \pm j0.041$, $-0.43 \pm j0.7448$, and $-0.6755 \pm j0.39$. The zeros are located at $0.485 \pm j0.84$, $-0.6576 \pm j0.6576$.

The data was generated as in Example 1. The same procedure was followed in estimating the ARMA coefficients. The experiment estimated the ARMA coefficients using the RTS, the QSA, and the proposed QRD-TOC methods. Table II presents the results of 100 Monte Carlo simulations for the RTS, QSA, and the proposed QRD-TOC algorithms at SNR of 20 dB.

Table 2. True and estimated ARMA (6,4) model coefficients in Example 2, $a(0)=1$, $b(0)=1$.

	True	RTS	QSA	QRD-TOC
a(1)	0.7907	0.7350	0.7023	0.7941
a(2)	0.0420	-0.0010	-0.0409	0.0548
a(3)	-0.5556	-0.5268	-0.5884	-0.5658
a(4)	-0.0247	-0.0220	0.0231	-0.0333
a(5)	0.3846	0.3793	0.3772	0.3955
a(6)	0.3026	0.3071	0.2822	0.3106
b(1)	0.3452	0.3257	0.1998	0.3588
b(2)	0.5300	0.5531	0.6127	0.5352
b(3)	0.3985	0.3917	0.5432	0.3654
b(4)	0.8138	0.7199	0.9914	0.8023

Example 3. (Colored Gaussian) Consider the ARMA (6, 4) process generated by the difference equation in (42). The colored Gaussian noise $w(n)$ was generated as a *sinc* function:

$$h(n) = 0.3\text{sinc}(0.01n) \quad (43)$$

The experiment estimated the ARMA coefficients using the RTS, the QSA, and the proposed QRD-TOC methods. Table III presents the results of 100 Monte Carlo simulations for the RTS, QSA, and the proposed QRD-TOC algorithms at SNR of 20 dB.

Table 3. True and estimated ARMA (6, 4) model coefficients in Example 3, $a(0)=1$, $b(0)=1$, colored.

	True	RTS	QSA	QRD-TOC
a(1)	0.7907	0.7282	0.7425	0.7840
a(2)	0.0420	-0.0012	-0.0124	0.0343
a(3)	-0.5556	-0.5167	-0.5688	-0.5608
a(4)	-0.0247	-0.0267	0.0016	-0.0296
a(5)	0.3846	0.3708	0.3687	0.3960

a(6)	0.3026	0.3030	0.2835	0.3142
b(1)	0.3452	0.3249	0.3681	0.3450
b(2)	0.5300	0.5419	0.5114	0.5115
b(3)	0.3985	0.3997	0.3055	0.3648
b(4)	0.8138	0.7034	0.5581	0.7639

V. CONCLUSION

The paper developed an efficient procedure for calculating ARMA model coefficients using QRD-TOC from the available output data. The results of the simulated experiments demonstrate the good performance of the developed technique when the observed sequence is contaminated by additive Gaussian noise. The proposed technique was compared with known procedures such as RTS, and the QSA. As it can be seen from Tables I, II, and III, the proposed QRD-TOC technique is more accurate than the RTS and QSA methods.

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