Experimental Analysis of Pattern Similarity between Bessel Kernel and Born-Jordan Kernel

Ming Li [1], Xue-Kang Gu [2], and Wei Zhao [3]

***Abstract*—** Kernels play a role in time-frequency (TF) analysis of signals. Various types of kernels have been introduced in TF analysis. Usually, different types of kernels (i.e., kernels in different function form) correspond different types of TF distributions (TFDs). From a view of pattern matching, however, different TFDs may achieve the similar TFD result for a same signal if the used kernels are arranged such that they are similar in pattern under a certain condition. Essential issues in this regard are 1) which kernels may be similar in pattern and 2) under what conditions their patterns are similar. The answers to those issues are meaningful in TF analysis.

As a stage work, this paper gives an experimental analysis of the pattern similarity between two types of kernels, the typical Born-Jordan kernel (i.e., Sinc kernel) and the Bessel one. Correlation coefficient is used to measure the pattern similarity. We present the correlation curve between two and propose the quantitative conditions that both kernels are similar and dissimilar. The analysis shows that the maximum similarity between them may reach 0.987 when the value of a scaling factor of Bessel kernel equals to 0.18. On the other hand, the minimum of the correlation between them is less than or equal to 0.55 when the scaling factor is less than or equal to 0.01. Hence, this paper suggests that Bessel kernel is more flexible than the typical Born-Jordan's in TF analysis. A case study is demonstrated.

***Keywords*—** Time-frequency analysis, Born-Jordan distribution, Bessel distribution, pattern matching.

I. Introduction

Time-frequency (TF) analysis is a powerful tool in analyzing non-stationary signals [1-5, 22, 23]. The general representation of TF distribution (TFD) of a signal in the Cohen's class is given by

$$GTFD(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x+\frac{\tau}{2})f^*(x-\frac{\tau}{2})\Phi(u,\tau)e^{j(ux-\tau\omega-ut)}dxd\tau du, \quad (1)$$

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where $f(t)$ is a signal to be analyzed, $\Phi(u, \tau)$ is a transform kernel, and $*$ the complex conjugate. The following is called ambiguity function of $f(t)$,

$$A(u,\tau) = \int_{-\infty}^{+\infty} f(x+\frac{\tau}{2})f^*(x-\frac{\tau}{2})e^{jux}\,dx. \quad (2)$$

From (1) and (2), one may obtain $GTFD(t, \omega)$ given by

$$GTFD(t,\omega) = \mathcal{F}[A(u,\tau)\Phi(u,\tau)], \quad (3)$$

where $\mathcal{F}$ stands for the operator of the Fourier transform. Suppose $f(t)$ is real. Then, expanding it into a Fourier series yields

$$f(t) = \Sigma A_n\cos(n\omega_1 t + \varphi_n), \quad (4)$$

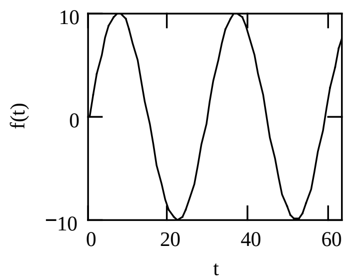
where A_n and φ_n denote the amplitude and the initial phase of the nth harmonic, respectively. Because

$$f(t+\tau/2)f(t-\tau/2) = \Sigma A_n\cos[n\omega_1(t+\tau/2)+\varphi_n]\Sigma A_n\cos[n\omega_1(t-\tau/2)+\varphi_n], \quad (5)$$

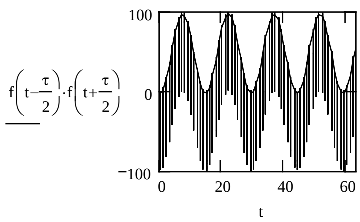
$A(u, \tau)$ contains both auto-terms being useful information we expect and cross-terms being harmful as they may obscure the auto-terms. For instance, if $f(t) = 10\sin(3\pi t/44)$ as shown in Fig. 1 (a), one will observe some cross-terms contained in the product

$$f(t+\tau/2)f(t-\tau/2),$$

see Fig. 1 (b).



(a)



(b)

Fig. 1. Observation of cross-terms. (a) $f(t) = 10\cos(3\pi t/22)$. (b) $f(t+\tau/2)f(t-\tau/2)$.

To suppress cross-terms as many as possible, a kernel is utilized. However, when a kernel is used, auto-terms may probably be sacrificed as a side effect of cross-terms suppressing. For that reason, great efforts have been made in studying different types of kernels, see e.g., [3, 7-16]. Utilizing different types of kernels yields different types of TFDs. For instance,

$$\Phi_W(u,\ \tau) = 1$$

corresponds to the Wigner's distribution [1, 5]. The Choi-Williams' distribution takes the following Gaussian function as its kernel (CWK)

$$\Phi_{CW}(u,\ \tau) = e^{-\frac{u^2\tau^2}{\sigma}} \quad (\sigma > 0),$$

where σ is a scaling factor [7]. For the typical Born-Jordan distribution (Born-Jordan distribution or BJD for short) [1, 8], the kernel (BJK) is given by

$$\Phi_{BJ}(u,\ \tau) = \mathrm{Sinc}(u\tau), \tag{6}$$

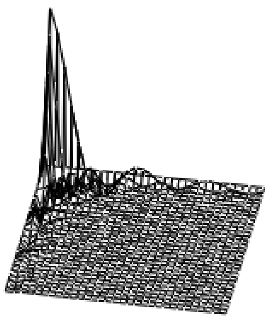
where

$$\mathrm{Sinc}(x) = \frac{\sin(x)}{x}.$$

For the Bessel distribution (BD), its kernel (BK) is written by

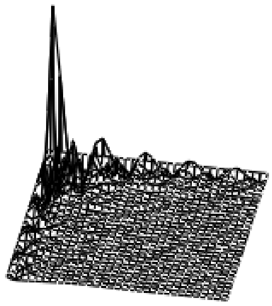
$$\Phi_B(u,\ \tau) = \frac{J_1(2\pi\alpha u\tau)}{\pi\alpha u\tau}, \tag{7}$$

where J_1 is the first kind Bessel function of order one and $\alpha > 0$ is a scaling factor [3]. When $0 < \alpha \leq 0.5$, BD preserves the TF support properties [3]. Figs. 2 (a) ~ (b) indicates BK and BJK, respectively.



BK

(a)



BJK

(b)

Fig. 2. Illustrations of BK and BJK. (a) Bessel kernel for $\alpha = 0.10$. (b) Born-Jordan kernel.

As known, a TFD is signal-dependent [11, 15]. On the other hand, it is also kernel-dependent. Consequently, each type of kernel is considered to be suitable for a specific class of signals. The patterns of two different kernels in function form may be significantly different from each other (e.g., WK and CWK) or similar under a certain condition as discussed in this paper. Though various types kernels have been studied, the report about pattern similarity of kernels that are different in function form is rarely seen. Radical issues in this regard are 1) which kernels may be similar and 2) under what conditions the selected kernels that are different in function form are similar or dissimilar in pattern. Obviously, the answers to those issues are meaningful in TF analysis.

This paper, a stage work though, is a substantial extension of our early work [6]. It shows that BD and BJD for a same signal may be similar if BK is arranged such that it is similar to BJK in pattern though two kernels are different in function form. It also shows that BD and BJD for a same signal may be dissimilar if BK is arranged such that it significantly differs from BJK in pattern. The experimental analysis presented in this paper suggests that BK is more flexible than BJK in TF analysis. The analysis method discussed in this paper can be extended to study pattern similarities among other kernels.

The rest of paper is organized as follows. Section 2 introduces the measure of pattern similarity. Section 3 discusses the pattern similarity between BK and BJK. A case study is demonstrated in section 4 and conclusions are given in Section 5.

II. A Similarity Measure

Denote Φ_1 a kernel. Define its inner product by L_2 norm as

$$<\Phi_1,\ \Phi_1> = \|\Phi_1\|_2. \tag{8}$$

Let H_1 be the space containing all kernels. Then, H_1 is a pre-Hilbert space [19]. According to the properties of kernels, for a kernel Φ, there is a sequence Φ_n such that $\Phi_n \to \Phi$ for $n \to \infty$. Thus, combining all limit points of kernels with H_1 yields a

Hilbert space and we still denote it as H_1 without confusion causing.

Let $\Phi_2 \in H_1$ be another kernel differing from $\Phi_1 \in H_1$. Then, there is a distance between Φ_1 and Φ_2. The distance is denoted as $\|\Phi_1 - \Phi_2\|$. Theoretically speaking, there are many measures for characterizing the distance, see e.g., [17, 21, 22]. However, correlation is a commonly used technique in pattern matching, see e.g., [17] and our early work [18, 19, 24]. According to the Cauchy-Schwarz inequality, one has

$$\left| \|\Phi_1 \cdot \Phi_2\|_2 \right| \le \sqrt{\|\Phi_1\|_2} \cdot \sqrt{\|\Phi_2\|_2}. \tag{9}$$

Denote

$$\rho_{\Phi_1,\Phi_2} = \frac{\left| \|\Phi_1 \cdot \Phi_2\|_2 \right|}{\sqrt{\|\Phi_1\|_2} \cdot \sqrt{\|\Phi_2\|_2}}. \tag{10}$$

Then, $0 \le \rho_{\Phi_1,\Phi_2} \le 1$. According to the correlation theory, if $\Phi_1 = \lambda\Phi_2$, where λ is a real, one has $\rho_{\Phi_1,\Phi_2} = 1$, implying that the pattern of Φ_1 is exactly similar to that of Φ_2 [17]. As far as pattern analysis of kernels is concerned, we normalize kernels such that $\Phi_{max} = 1$. Thus, here and below, we consider $\lambda = 1$ without the generality losing for studying ρ_{Φ_1,Φ_2}.

In practical terms, it is unnecessary to require $\rho_{\Phi_1,\Phi_2} = 1$ due to errors (e.g., errors in digital signal processing). In the engineering sense, the pattern of Φ_1 is quite accurately similar to that of Φ_2 if

$$\rho_{\Phi_1,\Phi_2} \ge 0.9,$$

see e.g. [18, 21]. Therefore, we take

$$\Phi_1 \approx \Phi_2 \text{ for } \rho_{\Phi_1,\Phi_2} \ge 0.9.$$

Accordingly, $\|\Phi_1 - \Phi_2\| \approx 0$ in this case.

III. EXPERIMENTAL SIMILARITY ANALYSIS OF BK AND BJK

A BJK has no scaling factor, see (1.6), but there is a scaling factor in BK, which can be used to control its shape, see Fig. 2 (a) and Fig. 3. Since the case of $\alpha \to 0$ is trivial in our study, we consider $\alpha \in [0.01, 0.5]$ in what follows. Compute the correlation coefficient between two kernels yields a correlation curve as shown in Fig. 4, where $\rho(\alpha)$ denotes that the correlation coefficient varies with α.

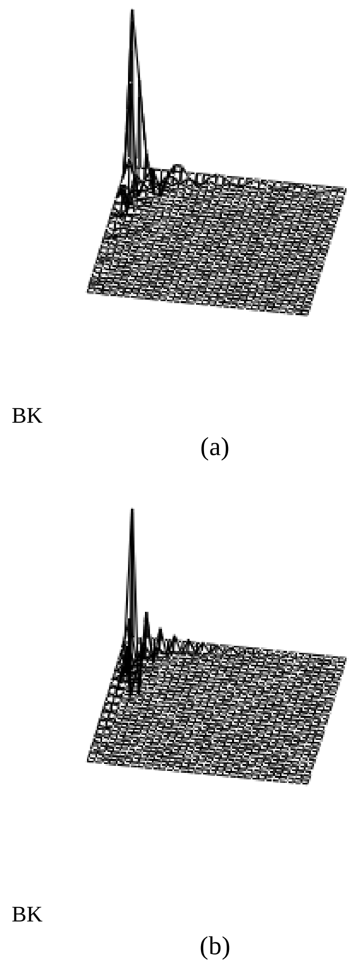


Fig. 3. Bessel kernel. (a) BK with $\alpha = 0.18$. (b) BK with $\alpha = 0.49$.

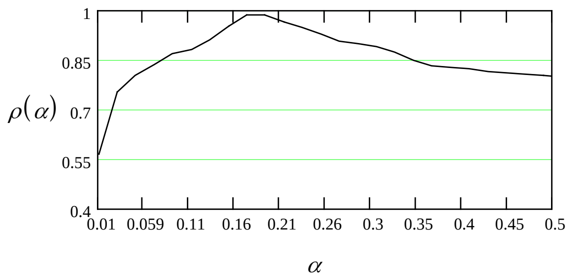


Fig. 4. Correlation curve between BJK and BK.

From Fig. 4, we obtain the maximm of the correlation coefficient

$$\rho(\alpha)|_{max} = 0.987.$$

It occurs at $\alpha = 0.18$. The condition for

$$\rho(\alpha) \ge 0.8$$

is

$$\alpha \in [0.05, 0.5]$$

while the condition for

$$\rho(\alpha) < 0.8$$

is

$$\alpha \in [0.01, 0.05].$$

The minimum is

$\rho(\alpha)|_{\min} = \rho(0.01) = 0.55.$

Thus, from a view of pattern matching, BK has the functionality of BJK but not vice versa.

IV. A Case Study

Suppose a signal to be analyzed is

$$f(t) = \cos(3\pi t/22) + \cos(14\pi t/11)$$

see Fig. 5.

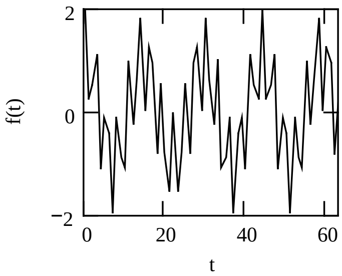


Fig. 5. Waveform of a signal.

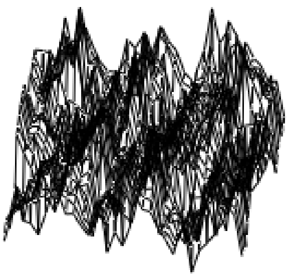
Denote

$$\text{BD} = \mathcal{F}[A(u,\tau)\Phi_{\text{B}}(u,\tau)]$$

and

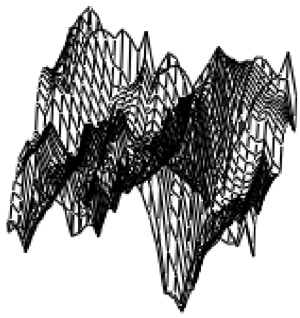
$$\text{BJD} = \mathcal{F}[A(u,\tau)\Phi_{\text{BJ}}(u,\tau)]$$

as the Bessel distribution and Born-Jordan distribution of $f(t)$, respectively. Fig. 6 indicates the BJD of $f(t)$ and Fig. 7 the BD for $\alpha = 0.18$. The correlation coefficient between Fig. 6 and Fig. 7 is 0.961, which is slightly less than 0.987. This may be caused by errors in digital computations. As a comparison, we gives the BD for $\alpha = 0.49$ in Fig. 8. By eye, one may see the obvious difference between Fig. 8 ($\alpha = 0.49$) and Fig. 6 (BJD). The correlation coefficient between Fig. 6 and Fig. 8 is 0.80.



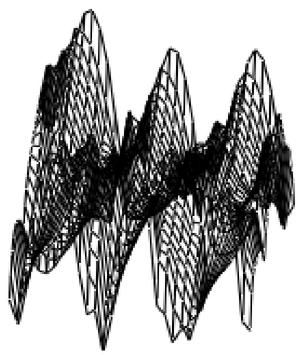
BJD

Fig. 6. Born-Jordan distribution of $f(t) = \cos(3\pi t/22) + \cos(14\pi t/11)$.



BD

Fig. 7. Bessel distribution of $f(t) = \cos(3\pi t/22) + \cos(14\pi t/11)$ for $\alpha = 0.18$.



BD

Fig. 8. Bessel distribution of $f(t) = \cos(3\pi t/22) + \cos(14\pi t/11)$ for $\alpha = 0.49$.

V. Conclusion

This paper has experimentally analyzed the pattern similarity between two kernels used in TF analysis. One is BK and the other BJK. The analysis method has been discussed and demonstrated. The analysis result shows that $\rho(\alpha)|_{\max} = \rho(0.18) = 0.987$ and $\rho_{\min} \le 0.55$ for $\alpha \le 0.01$. In addition, $\rho(\alpha) \ge 0.8$ for $\alpha \in [0.05, 0.5]$. Thus, BK is more flexible than BJK in TF analysis from a view of pattern matching in general. BK has the functionality of BJK but not vice versa.

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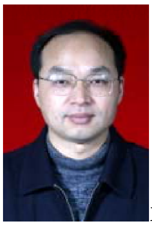

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